SHEAR BANDS IN BIAXIAL TESTS ON DRY COARSE SAND

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ABSTRACT

A series of plane-strain biaxial compression tests on dry coarse sand is described in which the state of shear-band formation, shear-band orientation, and shear-band growth are investigated. The results show that both the shear strain at shear banding and the shear-band orientation depend on the confining pressure. The experimental results are compared with the theoretical predictions of equilibrium bifurcation theory for a number of local and non-local incremental constitutive equations. Significant deviation between the experimental results and theoretical predictions for local constitutive equations involving the Mohr-Coulomb yield condition is observed at higher confining pressures. Much better agreement is obtained if a yield function possessing a corner is introduced. The analysis of the shear-band growth beyond the onset of localization reveals that simple purely kinematic models cannot describe adequately the observed evolution in the shear-band thickness.

Key words: deformation, failure, laboratory tests, sand (IGC: D6)

INTRODUCTION

This paper presents results and analysis of a biaxial compression test program on dry sand specimens tested at various confining pressures (Han, 1991). The objective of this program was threefold: 1) to investigate the state of shear-band formation in a poorly-graded coarse sand, 2) to compare the results with predictions of equilibrium bifurcation theory for a number of local incremental constitutive equations for sand, and 3) to investigate the progressive growth of shear bands.

While several results from biaxial tests on sands and model granular materials have been reported in the literature (cf. Arthur et al.,

1977; Desrues et al., 1985; Duthilleul, 1982; Tatsuoka et al., 1990; Vardoulakis, 1980; Vardoulakis and Graf, 1985; Vardoulakis et al., 1985), the amount of data is very limited. In these tests, various sands and various types of laboratory devices with varying abilities to allow unobstructed formation of shear bands have been used, making comparison somewhat difficult. The tests reported in this paper were performed with a new device developed at the University of Minnesota (Vardoulakis and Drescher, 1988). This device, called a biaxial apparatus, allows for kinematically-free formation of shear bands, and is characterized by an improved axial load guidance system, low wall friction, and enhanced load and displacement recording instrumentation.

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Preliminary biaxial test results on dry specimens are reported in the work of Drescher et al. (1991), and on water-saturated specimens in the work of Han and Vardoulakis (1991).

The majority of the experimental works on shear bands concentrates on evaluating their orientation. Depending on the apparatus type, grading and density of sand, it has been found that the shear-band inclination angle θ with respect to the minor compressive principal stress (compression taken positive) varies between limits $\theta_R \leq \theta \leq \theta_C$, where θ_R and θ_C are given by the well-known Coulomb and Roscoe formulae

$$\theta_C = \frac{\pi}{4} + \frac{\phi}{2} \quad \theta_R = \frac{\pi}{4} + \frac{\psi}{2} \quad (1)$$

where ϕ is the friction angle and ψ is the dilatancy angle. These limits are usually derived from either static or kinematic considerations for sand modelled as a rigid-perfectly plastic material. This crude approximation of the actual hardening/softening type sand behavior gives no direct information on the magnitude of strains at which shear bands occur.

Significant help in understanding the formation of shear bands is offered by equilibrium bifurcation theory. This theory explains why shear bands form in tests on specimens in spite of "ideal" boundary conditions. In short, the formation of shear bands is identified with the existence of a non-uniform deformation field that is admissible even in ideal loading conditions for a material described by an incremental constitutive equation. Various constitutive equations incorporating the hardening/softening response of sands can be analyzed and the results compared with experimental data. Bifurcation theory also provides information on strains at shear banding, shear-band orientation, and the thickness of shear bands.

The next section describes the biaxial tests whose results are subsequently used and compared with equilibrium bifurcation analysis. This is followed by analysis of the growth of shear bands.

EXPERIMENTS

The biaxial apparatus used in these tests is described in detail by Drescher et al., (1991). Also the test procedures and data evaluation algorithm are described in that paper. Below, only the main components of the apparatus, and the characteristics of the sand and specimens, are presented.

Apparatus

The operational principle of the plane-strain biaxial apparatus is based on the concept initially suggested by Vardoulakis and Goldscheider (1981). A prismatic specimen of 40 mm width, 80 mm length, and 140 mm height, is enclosed between two rigid walls 80 mm apart and placed on a platen which rests on a linear bearing. The linear bearing provides kinematic freedom for the formation of planar shear bands with the lower specimen portion sliding horizontally (Fig. 1a). The apparatus is placed in a pressure chamber, and the specimen is subjected to a confining

Fig. 1. Biaxial apparatus, a) schematic, b) incremental deformation in a shear band

pressure and kinematically or statically-controlled axial load. As the stress perpendicular to the rigid walls is not measured, and the analysis is performed in only two dimensions, the axial stresses and strains are denoted as σ_1 and ε_1 , and the lateral stresses and strains as σ_2 and ε_2 . Prior to shear banding, the specimen deforms uniformly, and accurate measurements of its load-displacement response can be obtained by means of load cells placed below and above the specimen and by several LVDTs. Uniform deformation is enhanced by enlarged loading platens, glass lining of parts in contact with the specimen, and lubrication between the glass lining and the rubber bag surrounding the specimen and rubber sheets placed at the bottom and top of the specimen.

Sand and Specimens

The biaxial tests were performed on coarse, poorly-graded Ottawa sand (ASTM designation C-190), with rounded particles and the mean grain diameter $D_{50}=0.72$ mm. Homogeneous dense specimens, with an initial porosity $n_0=0.32-0.33$, were prepared by raining the sand into a prismatic former from a constant free-fall height of 300 mm. Homogeneous specimens were used to determine the stress-strain sand response in the pre-localization regime. The bedding error was evaluated on a dummy aluminum specimen covered with a thin layer of tested sand, rubber sheet, and lubricant. The shear-band orientation was measured upon completion of tests directly on the specimens; a distorted grid imprinted on the sides of the rubber bag surrounding the specimen allowed for accurate shear-band orientation measurements. In addition, shear-band orientation was measured on radiographs taken at the end of tests. However, in most tests the shear band crossed the specimen corner and intersected with the upper loading platen. Therefore, additional tests were performed on specimens containing a centrally located small lens of looser material (a density imperfection) thereby causing the shear band to cross the free boundaries (Vardoulakis and Graf, 1985).

The specimens with imperfections were also used to evaluate the thickness of shear bands

slightly beyond their onset and at large deformation, because no platen induced kinematic constraints affected their progressive growth. In this way, the shear band could be deformed to relative tangential displacements of about 10 mm before significant bending occurred.

Test Program and Results

All tests were performed with displacement-controlled axial loading with the displacement-rate of 0.2 mm/min. The specimens were subjected to a constant confining pressure of 50, 100, 200, or 400 kPa. Fig. 2(a) shows a typical axial load-axial displacement curve obtained from monotonous loading tests on homogeneous specimens at confining pressure of 200 kPa; a similar curve from tests on specimens with imperfections is shown in Fig. 2(b). In homogeneous specimens, the interaction of the shear band with the upper loading platen results in an increase of the axial load at large displacements. This is not observed in tests on specimens with imperfections. The solid line depicts the load

Fig. 2. Axial load and lower platen displacement vs. axial displacement, a) homogeneous specimen, b) inhomogeneous specimen

measured above the specimen, and the broken line that measured below the specimen. The difference between these curves is small, which means that the glass lining and lubrication significantly reduced wall friction; the friction coefficient for the interface was found to be 0.012. The lower curves in Fig. 2 depict the lateral displacement of the lower loading platen as a function of the axial displacement. Point S marks the onset of continuous platen movement, and corresponds to the formation of two parallel folds visually observed on the opposite free surfaces of the specimen. This point was identified with the state of a fully developed shear band; on the axial load-axial displacement curve this point is located close to the peak. Beyond point S the latter curve no longer represents the material response but rather the system (specimen with a shear band) response. For this reason, only pre-shear banding data were used to construct the shear stress-shear strain, and volumetric strain-shear strain curves. The following definitions were introduced: shear stress $t = (\sigma_1 - \sigma_2)/2$, mean stress $s = (\sigma_1 + \sigma_2)/2$, stress ratio $\mu = t/s$, shear strain (twice) $\epsilon_t = \epsilon_1 - \epsilon_2$, and volumetric strain $\epsilon_s = \epsilon_1 + \epsilon_2$. The $\mu = \mu(\epsilon_t)$ and $\epsilon_s = \epsilon_s(\epsilon_t)$ curves are shown in Fig. 3(a) and (b), respectively. Other tests involving unloading before localization also were performed to determine the material parameters and functions in the constitutive equations discussed in the next section. Table 1 summarizes the main test results. The first column describes the type of specimen: *h* stands for homogeneous, and *i* for one containing an imperfection. The next columns show the confining pressure σ_2 , shear strain ϵ_t , strain-increment ratio $d\epsilon_s/d\epsilon_t$ and stress ratio t/s , all at point S, and, the shear-band inclination angle θ measured with respect to the direction of the lateral stress σ_2 ; it is seen that the angle θ was practically identical in both types of specimens. The last two columns list the thickness of shear bands at point U of the axial load-axial displacement curve, δ_U , and at point L corresponding to large deformation in the shear band, δ_L (see Fig. 2). The results presented in Table 1 indicate that with increasing confining pressure the shear strain at which a well-defined shear

Fig. 3. Stress-strain curves, a) stress ratio vs. shear strain, b) volumetric strain vs. shear strain

Table 1. Test results

type	σ_2 [kPa]	ϵ_t [%]	$d\epsilon_s/d\epsilon_t$	t/s	θ [°]	δ_U [mm]	δ_L [mm]
<i>h</i>	50.1	6.51	-0.377	0.677	59		
<i>h</i>	55.1	5.71	-0.370	0.674	59		
<i>h</i>	100.6	6.34	-0.376	0.680	58		
<i>h</i>	100.9	5.98	-0.363	0.678	58		
<i>h</i>	201.7	7.92	-0.375	0.680	56		
<i>h</i>	205.0	7.49	-0.368	0.674	56		
<i>h</i>	400.6	7.22	-0.324	0.668	55		
<i>h</i>	401.6	7.38	-0.307	0.650	55		
<i>i</i>	201.7				56	5.8	
<i>i</i>	202.4				56	5.8	
<i>i</i>	205.0				56	5.8	
<i>i</i>	194.7				57		10.0
<i>i</i>	196.8				56		10.4
<i>i</i>	200.9				56		10.0
<i>i</i>	202.8				56		10.2
<i>i</i>	204.9				56		10.0
<i>i</i>	797.0				55		9.8

band forms increases, whereas the corresponding stress ratio and shear-band inclination angle decrease. A similar influence of confining pressure was detected by Desrues et al. (1985) in biaxial tests on a dense, poorly-graded fine sand.

STATE OF A FULLY DEVELOPED SHEAR BAND

Equilibrium Bifurcation Theory

The fundamentals of equilibrium bifurcation theory as applied to sand in plane-strain conditions have been presented in a number of papers (cf. Vardoulakis, 1980; Vardoulakis and Graf, 1985; Wan et al., 1989). Therefore, only the main elements of this theory are presented below. The theory considers the shear band as a thin material layer across which the gradient in velocity (displacement increment) v_i undergoes a discontinuous change (jump). This can be expressed mathematically as

$$[v_{i,j}] = g_i n_j \quad [v_i] = 0 \quad (2)$$

where $[\cdot]$ denotes the jump between two tensors or vectors, n_i are the directional cosines of the normal to the shear-band, and g_i are amplitudes of jump components. The incremental equilibrium in the direction normal to the shear-band boundaries requires that

$$[\dot{t}_{ij}] n_i = 0 \quad [\sigma_{ij}] = 0 \quad (3)$$

where σ_{ij} is the Cauchy stress tensor, and $\dot{t}_{ij}$ is the time derivative of the nominal (Piola-Kirchoff) stress tensor. The first of conditions (3) can be rewritten as

$$[\dot{\sigma}_{ij} + A_{ij}] n_i = 0 \quad (4)$$

where

$$A_{ij} = \sigma_{ij} v_{k,k} - \sigma_{kj} v_{i,k} + \frac{1}{2} (\sigma_{ik} v_{j,k} - \sigma_{ik} v_{k,j} + \sigma_{jk} v_{i,k} - \sigma_{jk} v_{k,i})$$

and $\dot{\sigma}_{ij}$ is the Jaumann stress-rate. With the help of the first of conditions (2), and an incremental constitutive equation relating stress-rate and strain-rate $\dot{\epsilon}_{ij}$, say

$$\dot{\sigma}_{ij} = D_{ijkl} \dot{\epsilon}_{kl} \quad (5)$$

Eq. (4) becomes a set of equations for the magnitude of g_i

$$Q_{ij} g_j = 0 \quad (6)$$

where

$$Q_{ij} = n_k D_{iklj} n_l + n_k A_{ijkl} n_l$$

$$A_{ijkl} = \sigma_{ij} \delta_{kl} - \frac{1}{2} (\sigma_{ik} \delta_{jl} + \sigma_{kj} \delta_{il} + \sigma_{il} \delta_{jk} - \sigma_{ij} \delta_{ik})$$

A non-trivial solution to this system is then sought by equating the determinant of the system to zero

$$\det(Q_{ij}) = 0 \quad (7)$$

This yields the characteristic quartic equation for the shear-band inclination angle, with coefficients dependent upon the loading state. The loading state at which the roots of the quartic equation become real is identified with the existence of shear band. This state is termed bifurcation or the localization state; the corresponding algebraic equation is termed the localization condition. The localization condition is a function of the components of the incremental stiffness matrix D_{ijkl} which depend on the type of the constitutive equation selected. For example, for the constitutive Eq. (5) the localization condition is

$$(D - a_1 D_{xxxx} - a_2 D_{yyyy})^2 - 4a_1 a_2 D_{xxxx} D_{yyyy} = 0 \quad (8)$$

where

$$D = D_{xxxx} D_{yyyy} - D_{xxyy} D_{yyxx}$$

$$a_1, \quad a_2 = D_{xyxy} \pm \sqrt{\frac{1}{2} s_{ij} s_{ij}}$$

and s_{ij} is the stress deviator. The real roots of the characteristic quartic equation, represent the shear-band inclination angle θ , and can be expressed as

$$\tan \theta = \pm \sqrt[4]{\frac{a_2 D_{yyyy}}{a_1 D_{xxxx}}} \quad (9)$$

For an incremental constitutive equation the components of D_{ijkl} vary as a function of the loading state or, equivalently, of some measure of induced strains. A strain state thus can be identified at which condition (8) is met,

and compared with that at which shear banding occurs in a test. Likewise, the shear-band inclination angle given by Eq. (9) can be compared with that observed in the specimens. It should be mentioned that the condition defined by Eq. (8) is satisfied before the peak in the shear stress-shear strain curve if the dilatancy angle is smaller than the friction angle; so-called non-associativity.

Constitutive Equations

Since the bifurcation state and shear-band inclination angle depend on the form of the incremental stiffness matrix which, in turn, depends on the type of the constitutive equation, several local rigid-plastic or elasto-plastic constitutive equations for sand were considered. These equations were formulated two-dimensionally, for no intermediate principal stress perpendicular to the plane of deformation is measured in the biaxial apparatus. However, plane-strain bifurcation analysis makes use of only in-plane stresses and strains, and the assumptions that are necessary to predict plane-strain behavior from, for example, triaxial compression test results are avoided in two-dimensional formulation. Both the deformation theory and flow theory framework were employed. To account for possible inherent anisotropy of specimens due to the raining of sand (cf. Tatsuoka et al., 1990), plastic anisotropy was also incorporated in one model. All models describe plastic hardening of the sand, originating from a zero-stress state, defined by the stress ratio function $\mu = \mu(\varepsilon_t^p)$, where ε_t^p is the plastic part of ε_t . For a linear Mohr-Coulomb yield condition $F(\mu) = 0$, $\mu = \sin \phi$, where ϕ is the mobilized friction angle being a function of ε_t^p . Plastic dilatancy is incorporated through the dilatancy function $\beta = \beta(\varepsilon_t^p)$, where $\beta = \dot{\varepsilon}_s^p / \dot{\varepsilon}_t^p = -\sin \psi$, and ψ is the dilatancy angle again dependent upon ε_t^p . Two nondimensional parameters representing the secant modulus $h_s = \mu / \varepsilon_t^p$, and the tangent modulus $h_t = \dot{\mu} / \dot{\varepsilon}_t^p$, of a $\mu = \mu(\varepsilon_t^p)$ curve are introduced. Also, in view of the observed small dependency of the stress ratio t/s on the magnitude of the confining pressure, a non-linear Mohr-Coulomb yield condition $F(\mu, s)$

$= 0$ in addition to a linear approximation was considered. The elastic properties are defined by the shear modulus G and bulk modulus K ; for compactness in writing equations, a parameter $\alpha = G/K = (1 + \nu)/(1 - \nu)$ is used, where ν is Poisson's ratio.

In the following, only the general form of the considered constitutive equations is presented; more information can be found in Han (1991). Also, explicit forms of the pertinent localization condition and shear-band orientation equation are given. The non-linear yield condition was adopted in only the first two models, and in model 3 the Jaumann stress-rate was replaced by the time stress-rate, $\dot{\sigma}_{ij} = \dot{\sigma}_{ij}$, to simplify the derivations.

model 1: rigid plastic, deformation theory
Constitutive equation

$$\dot{\sigma}_{ij} = D_{ijkl} \dot{\varepsilon}_{kl} + D_{ijkl}^s \delta_{kl} \dot{s} \quad (10)$$

where $\dot{s}$ is related to the strain-rates through the dilatancy constraint, i.e., the dilatancy function β (Vardoulakis and Graf, 1985).

Localization condition

$$[2h_t - h_s(1 + \mu\beta) - \mu(\mu + \beta)]^2 - (h_s^2 - \mu^2) \times (1 - \mu^2)(1 - \beta^2) = 0 \quad (11)$$

Shear-band inclination angle

$$\tan \theta = \pm \sqrt{\frac{(1 + \mu)(1 - \beta)(h_s - \mu)}{(1 - \mu)(1 + \beta)(h_s + \mu)}} \quad (12)$$

where μ and β in Eq. (12) correspond to the bifurcation state. For the linear yield condition $\mu = \mu(\varepsilon_t^p)$, and for the non-linear one $\mu = \mu(\varepsilon_t^p s_0^n / s^n)$ where s_0 is an arbitrary reference state and n is a constant (Vermeer, 1978).

model 2: elasto-plastic, deformation theory and flow theory

Constitutive equation

$$\dot{\sigma}_{ij} = D_{ijkl} \dot{\varepsilon}_{kl} \quad (13)$$

Localization condition

$$\begin{aligned} & \frac{8h_1}{1 + \nu} - (\mu + \beta)^2 + 4h_1^2 \zeta^2 + 2h_1 \zeta h_2 \\ & - 2(\mu + \beta)(2h_1 \zeta + h_2) \zeta \\ & + \left(h_2^2 + \frac{4h_1}{1 + \nu} \right) \zeta^2 = 0 \end{aligned} \quad (14)$$

Shear-band inclination angle

$$\tan \theta = \pm \sqrt[4]{\frac{\left[(1+\beta)(1-\mu) + \frac{2h_1}{1+\nu}\right](1-\xi)}{\left[(1-\beta)(1+\mu) + \frac{2h_1}{1+\nu}\right](1+\xi)}} \quad (15)$$

where

$$h_1 = -\frac{1}{G} \frac{\partial F}{\partial \mu} h_t \quad h_2 = \frac{2h_1}{1+\nu} - (1+\mu\beta)$$

$$\zeta = \frac{1}{\chi} - 1 \quad \xi = \frac{\varepsilon_t^e}{\chi}$$

$$x = \frac{\varepsilon_t^e}{\varepsilon_t} - \text{deformation theory}$$

$x=1$ - flow theory

Function μ depends on the yield condition as in Model 1. Model 2 for the flow theory was originally suggested by Vardoulakis (1988).

model 3: elasto-plastic, plastically anisotropic, flow theory

Constitutive equation

$$\dot{\sigma}_{ij} = D_{ijkl} \dot{\varepsilon}_{kl} \quad (16)$$

Localization condition

$$h_1 [16a_t(1+\alpha) + 16\alpha(1-a_s)(\mu-\beta) - a_t\alpha] \times (1+a_s)^2(\mu+\beta) - \frac{8a_t a_s(1-a_s)}{1-\nu} = 0 \quad (17)$$

Shear-band inclination angle

$$\tan \theta = \pm \sqrt[4]{\frac{h_1(1+\alpha) + \alpha(a_t + b_1\mu)(a_t - b_1\beta) + (1-a_s)(b_3\mu - b_4\mu\beta - b_3\beta)}{h_1(1+\alpha) + \alpha(a_t - b_2\mu)(a_t + b_2\mu) + (1-a_s)(b_3\mu - b_5\mu\beta - b_3\beta)}} \quad (18)$$

where

$$b_1 = \frac{1+a_s\nu}{1+\nu} \quad b_2 = \frac{a_s+\nu}{1+\nu} \quad b_3 = \frac{1}{2}(1+\alpha)a_t$$

$$b_4 = \frac{2a_s(1+\nu)-1}{(1-\nu)^2(1-\nu)} \quad b_5 = \frac{2a_s+\nu(1-2\nu)+1}{(1-\nu)^2(1+\nu)}$$

and constants a_s and a_t describe the inherent anisotropy of the yield condition (Chen and Baladi, 1985).

Material Parameters and Functions

Numerical values of parameters G , K , and ν , and functions μ and β , were obtained by least square approximations of the appropriate stress-strain curves in the prelocalization regime, e.g., Fig. 3. The shear modulus G was determined from the averaged unloading $t=t(\varepsilon_t)$ curves, whereas Poisson's ratio was determined from the averaged initial slope of $\varepsilon_2=\varepsilon_2(\varepsilon_1)$ curves. The function μ for model 1, where $\varepsilon_t^p=\varepsilon_t$, was derived directly from the averaged $\mu=\mu(\varepsilon_t)$ curves. These were approximated for the linear and non-linear yield condition, respectively, by

$$\mu = \frac{\varepsilon_t}{C_1 + C_2 \varepsilon_t} \quad \mu = \frac{\frac{s_0''}{s''} \varepsilon_t}{C_1 + C_2 \frac{s_0''}{s''} \varepsilon_t} \quad (19)$$

For the remaining models, the function μ was obtained from the expressions above, strain-rates decomposition $\dot{\varepsilon}_t = \dot{\varepsilon}_t^e + \dot{\varepsilon}_t^p$, and Hooke's law (Schaeffer, 1990). The function β was derived from the following approximation to the $\varepsilon_s = \varepsilon_s(\varepsilon_t)$ curve

$$\varepsilon_s = -C_3 \varepsilon_t + C_4 \ln(1 + C_5 \varepsilon_t) \quad (20)$$

and a procedure similar to that for determining μ . As it is well-known for sands, the results showed the mobilized dilatancy angle to be significantly smaller than the mobilized friction angle; non-associativity. The constants a_s and a_t in model 3 could not be determined from tests on specimens only loaded in the direction of raining of the sand. A range for these constants was taken from cuboidal test results of Arthur and Menzies (1972). Table 2 lists the numerical values of the parameters and constants; the value of s_0 (initial confining pressure) was taken as 40 kPa.

Comparison between Experiments and Predictions

The shear strains at which the localization conditions (11), (14) and (17) are satisfied were calculated numerically using the parameters and functions presented above. The shear-

Table 2. Material parameters

model	G [MPa]	ν	C_1	C_2	C_3	C_4	C_5	n	a_s	a_t
1) $F(\mu)$			0.0028	1.48	0.34	0.0038	453			
1) $F(\mu, s)$			0.0014	1.46	0.34	0.0038	453	0.31		
2) $F(\mu)$	75	0.2	0.0028	1.48	0.34	0.0038	453			
2) $F(\mu, s)$	75	0.2	0.0014	1.46	0.34	0.0038	453	0.31		
3) $F(\mu)$	75	0.2	0.0028	1.48	0.34	0.0038	453		1.25 1.75	0.85 0.95
n.loc $F(\mu)$	75	0.2	0.0028	1.48	0.34	0.0038	453			

band inclination angle was calculated from Eqs. (12), (15), and (18). The results for each model are presented in Figs. 4, 5, 6, and 7 as a function of confining pressure σ_2 . In Figs. 4, 5, and 6, the solid lines correspond to the linear yield condition and the dashed lines to

the non-linear one. For model 3, several values of the constants a_s and a_t were selected (Fig. 7). The black dots represent the experimental results (see Table 1).

All figures show significant deviation between the theoretical predictions and ex-

Fig. 4. Test results and predictions; model 1, a) shear strain at shear banding vs. confining pressure, b) shear-band inclination angle vs. confining pressure

Fig. 5. Test results and predictions; model 2, deformation theory, a) shear strain at shear banding vs. confining pressure, b) shear-band inclination angle vs. confining pressure

Fig. 6. Test results and predictions; model 2, flow theory, a) shear strain at shear banding vs. confining pressure, b) shear-band inclination angle vs. confining pressure

perimental results. The measured shear strains at shear banding are much larger than predicted. The smallest discrepancy is given by model 2, deformation theory with a non-linear yield condition. Also, all models predict that the shear-band inclination angle increases with increasing confining pressure; the experimental results show the opposite trend. However, at particular values of confining pressure the agreement is good.

The discrepancy between the predicted and observed shear strains at shear banding can be attributed to several reasons. As pointed out by Vardoulakis (1988), shear bands form gradually and become visible externally only after some shearing takes place within the band. Thus, the strains at which the shear bands are observed in tests may be greater than predicted by the localization condition. Secondly, the computations were performed for one set of material parameters and functions whose values were obtained from averag-

Fig. 7. Test results and predictions; model 3, flow theory, a) shear strain at shear banding vs. confining pressure, b) shear-band inclination angle vs. confining pressure

ed test results of a limited number of tests. A sensitivity analysis was performed, therefore, for different values of G and β , in reality both being dependent upon the confining pressure. Also, the following equation derived by Lade and Nelson (1987) for G was used

$$\frac{G}{s_0} = k \left(\frac{\sigma_2}{s_0} \right)^\lambda \quad (21)$$

where λ and k are constants found to have value of 0.53 and 1457, respectively, and $s_0 = 40$ kPa. Even though different theoretical predictions were obtained, an example of which for model 2 (linear yield condition and flow theory) is shown in Fig. 8, the discrepancies remain substantial. Fig. 9 depicts the influence of β (deformation theory) taken as the average of: β_1 -from the tests at a confining pressure of 50 kPa, β_2 -from all tests, and β_3 -from all tests but assumed to be constant rather than a function of $\dot{\epsilon}_t^p$; G was taken as the average from all tests. It is seen that the β_3

Fig. 8. Test results and predictions; model 2, flow theory, various G , a) shear strain at shear banding vs. confining pressure, b) shear-band inclination angle vs. confining pressure

Fig. 9. Test results and predictions; model 2, deformation theory, various β , a) shear strain at shear banding vs. confining pressure, b) shear-band inclination angle vs. confining pressure

approximation gives reasonable agreement at moderate confining pressures.

The Mohr-Coulomb yield condition has the most profound effect on shear strains at bifurcation. This condition in plane-strain is represented in the $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$ space by a smooth, conical surface. An increment in stresses tangential to this surface yields a purely elastic response. This means that the D_{xyxy} component of the incremental stiffness matrix which appears in the localization condition is proportional to G , and thus very large. A much smaller value of D_{xyxy} , and larger shear strains at bifurcation, are obtained if a vertex at the yield surface is postulated (Rudnicki and Rice, 1975). Experimentally, this can only be detected from tests where the principal directions of stress rotate (cf. Sture et al., 1985; Matsuoka et al., 1989). The sensitivity of the effect of the vertex can be examined by introducing in D_{xyxy} an additional parameter C_a quantifying the deviation angle between the

stress and stress-increment vectors (Christoffersen and Hutchinson, 1979)

$$D_{xyxy} = (1 - C_a \epsilon_t) \frac{\epsilon_t^e}{\epsilon_t} G \quad (22)$$

This is demonstrated in Fig. 10(a) for model 2, linear yield condition, deformation theory, and averaged over all tests G and β ; a much better agreement between the prediction and experiments is obtained. The introduction of a vertex in the yield surface also leads to a decrease in the shear-band inclination angle with increasing confining pressure, Fig. 10(b); only modifying G and β for the smooth yield surface does not produce this result.

Finally, it may be of interest to compare the measured shear-band orientation with that predicted by Eqs. (1) and expression due to Arthur et al. (1977) and Vardoulakis (1980)

$$\theta = \frac{\pi}{4} + \frac{1}{4} (\phi + \psi) \quad (23)$$

Fig. 10. Test results and predictions; model 2) with a yield vertex, deformation theory, various C_a , a) shear strain at shear banding vs. confining pressure, b) shear-band inclination angle vs. confining pressure

Fig. 11. Test results and classical predictions for the shear band inclination angle

The values of ϕ and ψ were taken at the localization state, and the results are presented in Fig. 11. Clearly, the experimental results

locate far away from the Coulomb prediction; at higher confining pressure the results conform to the Roscoe prediction. This result supports the theoretical finding of Vermeer (1990), who showed that if Coulomb's shear band forms the normal stress inside the band that is parallel to its boundaries is greater than outside the band. This stress can only be supported by the rubber bag surrounding the specimen. In coarse sand, the shear band is relatively thick, and the rubber bag may not provide enough support.

SHEAR BAND GROWTH

The bifurcation theory for a local constitutive equation only predicts the bifurcation state and orientation of shear band, and no information is obtained on its thickness; this is because the system of equations for g_i at shear banding becomes ill-posed and no longer is elliptic. Non-local constitutive equations have been postulated in the literature to remedy this drawback (cf. Vardoulakis and Aifantis, 1991). These authors incorporated the second order gradient of strain-rate in a flow theory constitutive equation (model 2 type), which leads to an additional term in the elasto-plastic constitutive equation containing a characteristic length l

$$\dot{\sigma}_{ij} = D_{ijkl} \dot{\epsilon}_{kl} + D_{ijkl}^n l^2 \nabla^2 \dot{\epsilon}_{kl} \quad (24)$$

The presence of the second right-hand term prevents Eq. (4) from becoming ill-posed at the bifurcation state. This means that solutions can be sought in the post-bifurcation regime where the nominal stress-rate tensor fulfills the equilibrium equations

$$\dot{t}_{ij,i} = 0 \quad (25)$$

Using Eq. (24), and the definitions for the strain-rate and spin-rate tensors, Eq. (25) can be written as

$$D_{ijkl}(v_{k,li} + v_{l,ki}) + D_{ijkl}^n l^2 \nabla^2 (v_{k,li} + v_{l,ki}) + \sigma_{ik}(v_{j,ki} - v_{k,ji}) + \sigma_{jk}(v_{i,ki} - v_{k,ii}) = 0 \quad (26)$$

As Eq. (26) is elliptic, periodic solution of v_i of the form

$$v_i = c_i \cos(qn_k x_k) \quad (27)$$

is postulated, where c_i is the amplitude, x_k the coordinate, $q = \pi/\delta$, and δ is the thickness of shear band. The substitution of the derivatives of v_i with respect to coordinates into Eq. (26) gives

$$Q_{ij}^n c_j = 0 \quad (28)$$

where

$$\delta^2 = \frac{-[\alpha\mu\beta(\zeta \sin^2 2\theta + 1) - \beta \cos 2\theta - \xi\beta(1 - \alpha\mu \cos 2\theta)]\pi^2 l^2}{h_3 + \cos^2 2\theta + (\mu + \beta) \cos 2\theta + \zeta h_1 \alpha \sin^2 2\theta - \xi[\mu + (h_3 + 1) \cos 2\theta + \beta \cos^2 2\theta]} \quad (30)$$

where

$$h_3 = \frac{2h_1}{1+\nu} + \mu\beta$$

The characteristic length l is related to the size R of a representative elementary area by

$$l = \frac{R}{\sqrt{8}} \quad (31)$$

The elementary area can be taken as the Voronoi cell (Chang, 1990) for which R varies between D_{50} and $2D_{50}$ for sand packing; the value $R = \sqrt{2}D_{50}$ was assumed, and the corresponding material parameters are listed in Table 2. Fig. 12(a) and (b) show the results of calculations, where the approximation $\dot{\sigma}_{ij} = \dot{\sigma}_{ij}$ was made for the deformation theory. At the bifurcation state, the non-local theory leads to that $\delta = \infty$. As the deformation progresses, the thickness δ rapidly decreases, which can be interpreted as the localization process. The minimum of the thickness δ can then be regarded as that of a fully developed shear band and compared with the thickness δ_U . It is seen from Fig. 12 that the minimum thickness for the flow theory closely corresponds to thickness δ_U even though the shear strain at bifurcation is much smaller than observed. The deformation theory predicts a much larger shear-band thickness than δ_U . It should be mentioned that a linear Mohr-Coulomb yield condition was assumed in the constitutive models, and averaged values of G and β were taken in the computations.

The tests on specimens with imperfections also provided data on the dilatancy angle ψ_s within the shear band. In fact, knowing the

$$Q_{ij}^n = (n_k D_{kij}^n n_l + n_k D_{kil}^n n_j)(ql)^2 - (n_k D_{kij}^n n_l + n_k D_{kil}^n n_j + 2\sigma_{ij} n_k n_k)$$

Non-trivial solutions require that

$$\det(Q_{ij}^n) = 0 \quad (29)$$

which leads to the following expression for the thickness of shear band

Fig. 12. Test results and predictions for the thickness of shear band, a) flow theory, b) deformation theory

shear-band orientation and assuming that outside the band the material is rigid, it is possible to express the dilatancy angle as a function of the axial and lower platen displacement-increments (Fig. 1(b))

$$\psi_s = \theta - \arctan\left(\frac{\Delta u_a}{\Delta u_h}\right) \quad (32)$$

Fig. 13 shows the dilatancy angle as a function

Fig. 13. Shear-band dilatancy angle vs. shear-band relative displacement

of the relative displacement u_s across the shear band. Initially there is a negative dilatancy (contraction), also reported by Vardoulakis and Golscheider (1981). At large displacements the dilatancy angle becomes nearly constant.

The dilatancy angle ψ_s can be used to evaluate the growth of the shear-band thickness if a shear band model is assumed. For a material layer model of shear band, the increment in its thickness can be calculated from

$$\Delta\delta = \Delta u_s \tan \psi_s \quad (33)$$

and, for an initial thickness of, say δ_U its increase with u_s can be calculated. The solid lines in Fig. 14 show the results, where three different initial thicknesses of 5.2, 5.7, and 6.2 mm were assumed. The computational results locate below the measured value of δ_L .

Fig. 14. Shear-band thickness vs. shear-band relative displacement

It is also possible, however, to employ a different shear-band model. Namely, to assume that the material within the shear band ceases to dilate, and its growth in thickness is produced by engaging particles initially outside the band by an abrasion-like process. In contrast, it is assumed in the first model that the shear band does not grow by absorbing new material but only by expansion of material already within the shear band. The abrasion-like process can be modelled approximately within the concept of the density shock (Drescher and Michalowski, 1984). Then, from the continuity for the flux of matter across a moving shock, the following expression for the increment in the shear-band thickness is obtained

$$\Delta\delta = \Delta u_s \tan \psi_s \frac{\rho_2}{\rho_2 - \rho_1} \quad (34)$$

where ρ_1 and ρ_2 are the densities inside and outside the shear band, respectively. The broken lines in Fig. 14 show the results of calculations for three densities within the shear band of 1.55, 1.65, and 1.75 Mg/m^3 . The density outside the band was taken as that just before shear banding, $\rho_2 = 1.77 \text{ Mg/m}^3$. Clearly, the shock model overestimates the shear-band thickness at large strains. The results above may indicate that the shear-band thickness growth is a combination of dilation and abrasion or, possibly, involves the formation of subsequent bands. An indirect support for the latter is provided by Fig. 15 which shows the

Fig. 15. Variation of ratio of voids and grains across shear band

distribution of the ratio of voids and grains area, $n_a = A_v/A_g$, across the shear band at point L . This distribution was obtained from the image analysis of a thin section of specimen hardened with a resin. The two peak-areas can be interpreted as the existence of two parallel shear bands.

CONCLUSIONS

The biaxial experiments show that both the shear strain at shear banding and the shear-band orientation in dry, poorly-graded coarse sand depend on the magnitude of confining pressure. The shear strain increases, and the shear-band inclination angle measured with respect to the direction of the minor principal stress decreases when the confining pressure increases. The shear-band orientation is much below that given by the Coulomb formula; for lower confining pressures it is about 2° greater, and for higher confining pressures it is equal to that predicted by the Roscoe formula. The bifurcation analysis for local incremental constitutive equations and Mohr-Coulomb type yield condition predicts much smaller shear strains at shear banding than those observed in tests. Also, neither the orientation angle at higher confining pressure nor the experimentally observed trend in shear-band orientation in relation to the magnitude of confining pressure is satisfactorily described by the theory. It is doubtful if the use of other local constitutive equations without modifying the yield condition would diminish the discrepancies noted. In fact, a much closer agreement between the theory and experiments is obtained if the existence of a vertex in the yield surface is incorporated in the analysis. This finding conforms to previous works on other types of sand (cf. Vardoulakis, 1980), and necessitates experiments with principal stress axes rotating for determining the shape of the yield surface in plane-strain conditions.

The shear-band growth analysis indicates that a non-local constitutive equation and flow theory give a shear-band thickness at localization close to that observed in tests. The subsequent increase in the thickness can be at-

tributed to the combination of material dilation and abrasion, the latter resulting in the formation of parallel shear bands. This would imply periodicity, a phenomenon observed in tests on water-saturated sand (Han and Vardoulakis, 1991).

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REFERENCES

- 1) Arthur, J. F. R. and Menzies, B. K. (1972): "Inherent anisotropy in a sand," *Géotechnique*, Vol. 22, No. 1, pp. 115-128.
- 2) Arthur, J. F. R., Dunstan, T., Al-Ani, Q. A. J. and Assadi, A. (1977): "Plastic deformation and failure in granular media," *Géotechnique*, Vol. 27, No. 1, pp. 53-74.
- 3) Chang, C. S. (1990): "Strain tensor and deformation for granular material," *J. Eng. Mech.*, Vol. 116, No. 4, pp. 790-804.
- 4) Chen, W. F. and Baladi, G. Y. (1985): "Soil plasticity: theory and implementation," *Developments in Geotechnical Engineering*, Vol. 38, pp. 790-804.
- 5) Christoffersen, J. and Hutchinson, (1979): "A class of phenomenological corner theories of plasticity," *J. Mech. Phys. Solids*, Vol 27, No. 5/6, pp. 465-487.
- 6) Desrues, J., Lanier, J. and Stutz, P. (1985): "Localization of the deformation in tests on sand samples," *Eng. Fract. Mech.*, Vol. 21, No. 4, pp. 909-921.
- 7) Drescher, A. and Michalowski, R. L. (1984): "Density variation in pseudo-steady plastic flow of granular media," *Géotechnique*, Vol. 34, No. 1, pp. 1-10.
- 8) Drescher, A., Vardoulakis, I. and Han, C. (1991): "A biaxial apparatus for testing soils," *Geotech. Test. J. ASTM*, Vol 13, No. 2, pp. 226-234.
- 9) Duthilleul, B. (1982): "Rupture progressive: simulation physique et numerique," Ph.D Thesis, Institut de Mecanique de Grenoble.
- 10) Han, C. (1991): "Localization of Deformation in Sand," Ph.D Thesis, University of Minnesota.
- 11) Han, C. and Vardoulakis, I. (1991): "Plane-strain

- compression experiments on water-saturated fine-grained sand," *Géotechnique*, Vol. 41, No. 1, pp. 49-78.
- 12) Lade, P. V. and Nelson, R. B. (1987): "Modeling the elastic behavior of granular materials," *Int. J. Num. Anal. Meth. Geomech.*, Vol. 11, No. 5, pp. 521-542.
 - 13) Matsuoka, H, Suzuki, Y. and Murata, T. (1989): "A constitutive model for soils evaluating principal stress rotation and its application to finite element analysis," *Proc. NUMOG III*, Niagara Falls, pp. 155-162.
 - 14) Rudnicki, J. W. and Rice, J. R. (1975): "Conditions for the localization of deformation in pressure-sensitive dilatant materials," *J. Mech. Phys. Solids*, Vol. 23, No. 6, pp. 371-394.
 - 15) Schaeffer, D. G. (1990): "Instability and ill-posedness in the deformation of granular materials," *Int. J. Num. Anal. Meth. Geomech.*, Vol 14, No. 4, pp. 253-278.
 - 16) Sture, S., Ko, H. Y., Budiman, J. and Ontuna, A. (1985): "Development and application of a directional shear test," *Proc. 11th ICSMFE*, San Francisco, Vol. 2, pp. 1061-1064.
 - 17) Tatsuoka, F., Nakamura, S., Huang, C. and Tani, K. (1990): "Strength anisotropy and shear direction in plane strain tests of sand," *Soils and Foundations*, Vol. 30, No. 1, pp. 35-54.
 - 18) Vardoulakis, I. (1980): "Shear band inclination and shear modulus of sand in biaxial tests," *Int. J. Num. Anal. Meth. Geomech.*, Vol. 4, No. 2, pp. 103-119.
 - 19) Vardoulakis, I. (1988): "Theoretical and experimental bounds for shear-band bifurcation strain in biaxial tests on dry sand," *Res Mechanica*, Vol. 23, pp. 239-259.
 - 20) Vardoulakis, I. and Aifantis, E. C. (1992): "A gradient flow theory of plasticity for granular materials," *Acta Mech.*, in print.
 - 21) Vardoulakis, I. and Drescher, A. (1988): "Development of biaxial apparatus for testing frictional and cohesive granular media," NSF Report, Washington, D.C.
 - 22) Vardoulakis, I. and Goldscheider, M. (1981): "Biaxial apparatus for testing shear bands in soils," *Proc. 10th ICSMFE*, Stockholm, Vol. 4, pp. 819-824.
 - 23) Vardoulakis, I. and Graf, B. (1985): "Calibration of constitutive models for granular materials using data from biaxial experiments," *Géotechnique*, Vol. 35, No. 3, pp. 299-317.
 - 24) Vardoulakis, I., Graf, B. and Hettler, A. (1985): "Shear-band formation in a fine-grained sand," *Proc. 5th Int. Conf. Num. Meth. Geomech.*, Nagoya, pp. 517-522.
 - 25) Vermeer, P. A. (1978): "A double hardening model for sand," *Géotechnique*, Vol. 28, No. 4, pp. 413-433.
 - 26) Vermeer, P. A. (1990): "The orientation of shear bands in biaxial tests," *Géotechnique*, Vol. 40, No. 2, pp. 223-236.
 - 27) Wan, R. G., Chan, D. H. and Morgenstern, N. R. (1989): "The numerical modelling of the development of shear bands in geomechanics," *Proc. NUMOG III*, Niagara Falls, pp. 319-329.