A NEW LOOK AT THE STRESS DILATANCY RELATION IN CAM-CLAY MODEL

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ABSTRACT

A new look is offered on the energy based stress dilatancy relation in Cam-clay model (Roscoe et al., 1963). In the present study, the volume change due to dilatancy is regarded as the volume change in the void. From this view point, the hypotheses made in deriving the stress dilatancy relation in Cam-clay model will be replaced by four hypotheses as follows; (1) the void consists of contractive and dilative voids, (2) the increment of the volumetric strain in the contractive void is proportional to the increment of the plastic deviator strain of soil through the constant \( M \), (3) the energy dissipated by the void skeleton of the dilative void is none, and (4) the tangential stiffness of soil is continuous.

The four hypotheses make it possible to uncover a greater potential in Cam-clay model. In particular, it is suggested that the phase transformation line, which divides the stress space into dilative and contractive zones, should be distinguished from the critical state line.

Key words: constitutive equation of soil, clay, deformation, dilatancy, sand, strain (IGC: E-13/D-6)

INTRODUCTION

The stress dilatancy relation of soil has been studied as one of the central issues in the soil mechanics. The proposals made by Newland and Alley (1957), Rowe (1962), Shibata (1963) and Roscoe et al. (1963) among others are well known. In particular, the energy based stress dilatancy relation (Roscoe et al., 1963; Schofield and Wroth, 1968) has been one of the most frequently quoted relations in the literatures. The relation is written by

\[
p' d\psi + q d\psi = M p' d\gamma_p
\]

in which \( p' \): effective mean stress \((=(\sigma' + 2\sigma_o) / 3)\) (compression positive), \( q \): deviator stress \((=\sigma'_o - \sigma'_p)\), \( \psi_p \): plastic volumetric strain \((=(\epsilon_{\psi_p} + 2(\epsilon_{\psi_p}))\) (compression positive), and \( \gamma_p \): plastic deviator strain \((=(2/3)(\epsilon_{\gamma_p} - (\epsilon_{\gamma_p}))\), with the subscripts \( a \) and \( r \) implying axial and radial components, respectively.

In the practice of constitutive modeling of soil, the volume change due to dilatancy has been interpreted as the volume change in the soil skeleton; the incremental stress strain relation does not involve the extra term representing the volume change in the void due to the dilatancy. In this practice, the strain of the soil, which consists of the soil skeleton and the void, has been regarded the same as the strain of the soil skeleton. Though this was not the case when the energy based stress dilatancy relation in Cam-clay model shown in Eq. (1) was proposed and discussed with respect to the change in the void ratio, the concept of the volume change in the void due to the dilatancy became implicit when the yield surface was introduced in the Cam-clay model. This still remains the current practice of constitutive modeling.

In the present study, a new look at the relation in Eq. (1) will be offered from the point of view in which the volume change due to the dilatancy is explicitly treated as the volume change in the void. It will be shown that the new look will not only give us a new insight into the stress dilatancy relation in concern but also indicate a greater flexibility of this well known relation in the constitutive modeling of soil. A micromechanical approach will play a central role to arrive at some of the fundamental hypotheses made in the present study.

CONCEPT OF EFFECTIVE STRAIN

Effective stress of granular materials is given as a certain average of contact forces between particles. In a tensor expression, this is written (e.g. Satake, 1989) by

\[
\sigma'_l = \frac{1}{V} \Sigma ln(P_l)
\]

in which \( V \): a volume of a representative element; \( l \): a length of a branch (a line segment connecting centers of

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particles being in contact); \( n_i \): unit direction vector of the branch; and \( P_i \): a contact force, as shown in Fig. 1. The summation \( \Sigma \) is taken for all the contacts in the representative element. The bracket appearing in the subscripts in Eq. (2) represents symmetric part of a tensor.

Let us partition the contact force into components parallel to the branch and perpendicular to it as

\[
P_i = N n_i + T_i
\]

in which

\[
n_i n_i = 0
\]

From Eqs. (2) and (3), effective stress is given by

\[
s_{ij}' = \frac{1}{V} \Sigma I (N n_i n_j + T_{ij} n_i n_j)
\]

This equation represents that effective stress is given as a summation of normal stress contributions associated with the normal contact forces \( N \) and the tangential contact forces \( T_i \). In particular, a mean effective stress is given as a summation of isotropically distributed normal contact forces \( N \). This will be easily understood from Eqs. (4) and (5) as

\[
(1/3) s_{ij}' = (1/3V) \Sigma I (N n_i n_j + T_{ij} n_i n_j)
\]

It should be noted that the tangential component of contact forces \( T_i \) never contributes to the mean effective stress.

Analogous discussion applies to effective stress increment \( d\sigma_{ij}' \) and contact force increment, such that

\[
d\sigma_{ij}' = (1/V) \Sigma I (dN n_i n_j + dT_{ij} n_i n_j)
\]

Let us consider that \( dN \) is determined by the change in a distance of particle centers \( dA \) by

\[
dN = k_n dA
\]

It may be noted that, if the particles are rigid, \( dA \) is the same as the normal component of the relative displacement of contacts.

How the relative displacements of particle centers in granular materials relate to the strain of ordinary defini-

tion seems to be one of less established subject in the mechanics of granular materials (e.g. Satake, 1989). The author also made a tentative effort in this subject (1993b) but obviously more efforts are called for. In the present study, the author follows the original line of discussion presented in Iai (1993a) and presents it below in a more tangible manner.

For simplicity, let us begin with an imaginary case where no dilatancy is associated with deformation of a granular material. Since soil particles are very rigid, change in the normalized distance of particle centers \( dA/\ell \) of an individual branch may be very small compared to the normal strain component in the direction of the branch \( dA/\ell \). However, number of contacts along this direction will change in accordance with \( dA/\ell \). This is equivalent to the change in the number of branches along this direction and hence the effective stress will change as understood from the summation in Eq. (7).

Consider a virtual granular material in which no change occurs in the number of contacts or branches. If the same stress change should be induced in this virtual granular material as those induced in the actual granular material with the same initial branches before deformation, the change in the distance of particle centers in the virtual granular material should be given as a summation of all those changes in the existing as well as newly generated or disappeared branches in the actual granular material because the stress change is given by the summation in Eq. (7). Let us define \( dA_v \) as the change in the distance of particle centers in the virtual granular material. Then, \( dA_v \) will be closely related with \( dA/\ell \); the relation may be expressed as an incremental relation.

Finally, consider the actual case where dilatancy is associated with deformation of a granular material. In order to maintain the mean effective stress constant during deformation, \( d\sigma_{ij}' \) should not be affected by the dilatancy because, as mentioned earlier, the mean effective stress is given as a summation of isotropically distributed normal forces. In consequence, \( dA_v \) in the virtual granular material is given as

\[
dA_v = D n_i (d\varepsilon_{ij} - d\varepsilon_{ij})
\]

in which \( \varepsilon_{ij} \) is volumetric strain due to dilatancy and \( D \) denotes a coefficient in the incremental relationship and generally depends on the current state and the history of the granular material.

The quantity appearing in Eq. (9) as

\[
\varepsilon_{ij} = \varepsilon_{ij} - \varepsilon_{ij}
\]

is considered as a net strain contributing to the change in the distance of particle centers. A certain reciprocity may be noted in this quantity to the effective stress; effective stress is given as an average of contact forces whereas effective strain is an average field to specify the net change in the particle distance. Thus this quantity may well be called "effective strain." Since the effective strain represents the net strain of soil skeleton, the rest of the strain, i.e. the volumetric strain due to dilatancy, can only be regarded as the strain in something else. In the
In the present study, this "something else" is called "void." It should be noted that the void defined here is an imaginary concept and it may be different from the void of ordinary definition to be specified by the volume ratios such as void ratio or porosity.

The effective strain can be redefined in the notation of strain appearing in Eq. (1) as

\[ \varepsilon' = \varepsilon - \varepsilon_0 \]  \hspace{1cm} (11)

in which \( \varepsilon, \varepsilon' \) and \( \varepsilon_0 \) denote the volumetric strain, the effective volumetric strain and the volumetric strain due to dilatancy, respectively. The effective deviator strain, which represents the deviator strain in the soil skeleton, is defined the same as the deviator strain \( y \) in the soil.

The contrast between the current practice and the present study with respect to the dilatancy may be understood through the following examples. Under the drained condition with a constant effective mean stress, the current practice is to regard the volume decrease due to the dilatancy as the volume decrease in the soil skeleton as shown in Fig. 2(a). The present study is to regard the volume decrease due to the dilatancy as the volume decrease in the void as shown in Fig. 2(b). In the present study, the soil skeleton maintains its original volume if there is no change in the effective mean stress. In this regard, it may well be said that the volumetric strain of the soil skeleton defined in the present study represents the volumetric strain which "properly" reflects the effective stress conditions. The reciprocity between the concept of the effective strain and that of the effective stress may be noted again here.

Under the constant volume (or undrained) condition, the current practice is to regard the volume decrease due to the dilatancy as the volume decrease in the soil skeleton, which is forced to expand under the constraint on the volume, resulting in the decrease in the effective mean stress. The present study is to regard the volume decrease due to the dilatancy as the volume decrease in the void, which forces the soil skeleton to expand under the constraint on the volume of the soil, resulting in the decrease in the effective mean stress. An illustrative example is shown in Fig. 3.

In addition to the concept of the effective strain, a concept of void skeleton will be introduced in the present study. The concept of void skeleton is a reciprocal concept of soil skeleton. The volumetric and deviator components of the strain in the void skeleton are defined by the volumetric strain of the void \( \varepsilon_0 \) and the deviator strain of the soil \( y \), respectively. If only the volume of the void skeleton is considered, it is simply called as the void as seen earlier. It should be mentioned again that the void defined here is an imaginary concept. Similarly the void

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**Fig. 2.** Comparison between the current practice and the present study in the understanding of the volume changes of soil skeleton and void due to dilatancy under the drained condition with a constant effective mean stress.
skeleton defined here is an imaginary concept and it may be different from the quantity which may be directly specified from the deformation of actual void.

Since the void skeleton is an imaginary concept and its volumetric strain is defined different from that of the soil skeleton, it would be possible to develop a more generalized theory by choosing a deviator strain of void skeleton different from that of soil. However, this degree of generality is not necessary in the present study because no additional deviator strain is required in Eq. (9) and the preceding discussions.

The concepts of the void and the void skeleton are introduced in the present study to provide a tangible idea on the partitioning of strain. However, some readers might find these concepts nuisance and misleading because the strain of the void and the void skeleton are actually a part of the strain of soil and may be different from the actual deformation of void. If this is the case, the words “the effective strain,” “the volumetric strain of the void,” and “the strain of the void skeleton” can readily be replaced by the symbols \((v', \gamma)\), \((v_0, 0)\), and \((v_0, \gamma)\), respectively, each representing a part of actual strain of soil.

**BASIC ENERGY RELATIONS**

Some of the basic energy relations relevant to the present study may be summarized as follows. First of all, the increment of the total plastic energy transmitted across the boundaries to the unit volume of the soil element is given by

\[
dE_p = p'dv_p + qd\gamma_p
\]

The energy dissipated within the unit volume of the soil is also given by

\[
dW_p = p'dv_p + qd\gamma_p
\]

Similarly, the energy dissipated within the unit volume of the soil skeleton is given by

\[
dW_p' = p'dv_p' + qd\gamma_p
\]

Since the increment of the plastic effective volumetric strain is given from Eq. (11) as

\[
dv_p' = dv_p + dv_0
\]

the total plastic energy dissipated in the soil is given from Eqs. (13) through (15) as
This relation implies that the plastic energy dissipated in the soil is partly dissipated in the soil skeleton and partly dissipated in the void, the latter of which causes the volume change due to the dilatancy. This relation is more explicitly written by substituting Eq. (14) into Eq. (16) as

$$dW_p = p'dv_0 + qdy_p + p'dv_0$$

Since the energies considered here are plastic (i.e. irrecoverable), the energy balance requires

$$dE_p = dW_p$$

It should be noted that Eq. (18) is given because of the energy balance of the transmitted and the dissipated energies. It is not because of the relations given in Eqs. (12) and (13). This distinction will become important in the later chapter when the non-uniformity of the deformation within the soil specimen is considered.

This distinction may be more easily seen through the following observations. In a more rigorous discussion, the right hand side of Eq. (12) should be, at first, written by the energy due to the tractions and the displacements acting on the boundaries of the soil element. Then, since the unit volume of the soil element with uniform stress and strain is considered here, the energy given in terms of the tractions and the displacements reduces to that appearing in the right hand side of Eq. (12). Similarly, the right hand side of Eq. (13) should be, at first, written as an integration of the energy density $p'dv_0 + qdy_p$ over the soil element. Then, since the unit volume of the soil element with uniform stress and strain is considered here, the energy given in terms of the integration reduces to that appearing in the right hand side of Eq. (13). As mentioned earlier, the energy balance in Eq. (18) applies whether the stress and the strain are uniform or not.

As mentioned earlier, the effective strain represents the net strain contributing to the change in the distance of particle centers. Thus it may be meaningful to partition the strain energy into that due to the effective strain and the rest of the energy as seen in the above discussion.

**BEHAVIOR OF VOID**

In order to discuss the stress dilatancy relation in Cam-de model, three hypotheses will be made on the behavior of void and void skeleton in the present study on behalf of the hypotheses made in the original studies by Roscoe et al. (1963) and Schofield and Wroth (1968).

**HYPOTHESIS 1**: the void consists of contractive and dilative voids; i.e. the increment of the volumetric strain in the void due to the dilatancy is given by

$$dv_0 = dv_{0c} + dv_{0d}$$

in which the subscripts $c$ and $d$ imply the contractive and the dilative voids, respectively.

**HYPOTHESIS 2**: the increment of the volumetric strain in the contractive void is given by

$$dv_{0c} = Md\gamma_p$$

**HYPOTHESIS 3**: the energy dissipated by the void skeleton of the dilative void is none, such that

$$p'dv_{0d} + qdy_p = 0$$

These hypotheses are made by referring to a series of micromechanical studies by Oda and Konishi (1974) among others. A background to these hypotheses will be described below.

**Dilative Void**

First of all, let us go back to the assemblage of soil particles discussed earlier. Deformation of the assemblage of soil particles involves deformation, generation and disappearance of branches. If this process is followed step by step at infinitesimal intervals, at one instance the deformation occurs without generation or disappearance of branches, at the next instance with it. This process continues throughout the deformation of granular materials.

Let us consider a step of deformation without generation or disappearance of branches. In this process, energy increment in the assemblage of rigid soil particles will be given by the contact force $P$ and the displacement increment $du$ at the contact as

$$dW = \Sigma P_0 d\gamma$$

If $du$ is perpendicular to $P_0$, there will be no work. Either will there be no work if relative displacement is not induced at the contact because the work done by one contact force will be canceled by the work done by the other contact force in the opposite particle.

A rigid frictional relation is often assumed between the contact force and the relative displacement. This is mainly for the sake of simplicity. In the present study, a linear relation will be assumed with respect to their normal component. With respect to their tangential component, a hysteresis relation of strain hardening type will be assumed.

According to the results obtained by Oda and Konishi (1974), most of the contact forces are directed along the branches. For these contacts, no work will be done except for the work associated with the change in the relative distance of particle centers. This work will be close to zero when there is no change in the effective mean stress because the positive work done by the branches in one direction will be canceled by the negative work done by the branches perpendicular to them.

Some work may be done with the rest of the contact forces which are not directed along the branches. As shown in Fig. 4, equilibrium of contact forces is satisfied within a certain range of direction, of which upper and lower limits may be given by a friction angle of particles. For a given relative displacement increment at the contact, direction of the contact force can be classified into two groups according to its direction relative to that of the relative displacement increment; one is to do the positive work, the other is to do the negative work.
Behaviour of the former group, shown in Fig. 4 (b), was discussed by Rowe (1962) among others; this group will deform along with the overall deformation of granular material but the contribution to the deviator stress of granular materials will decrease as seen from the discussion by Rowe (1962). Thus the work done by this group becomes very small relative to the overall work done by the granular materials in which the deviator stress is increasing.

Behavior of the latter group, shown in Fig. 4(c), has been less frequently discussed to the knowledge of the present author. With the assumption of hysteretic relation between the contact force and the relative displacement, deformation of this group can naturally be induced just like the deformation of soil at the stress reversal. This group will likely to deform to form a more stable structure to resist the increase in the overall deviator stress of the granular material. This is consistent with the formation of relatively stable structure called “columns” by Oda and Konishi (1974). The contact force of this group will increase with deformation to resist the overall stress and thereby the negative work will increase in its magnitude. However, the deformation will eventually decrease when this group becomes a part of the stable structure. Thus, negative work done by this group will at first increase but eventually decrease to none at the final stable state, at which the contact force will be directed closer to the direction of the branch. It should be noted that deformation of this group can be induced while satisfying the Drucker’s stability postulate (Drucker, 1951) in the sense that, with the notation of displacements shown in Fig. 4, \(dP_i/d\epsilon_i(A) - d\sigma_i(B) > 0\) to change the direction of \(P_i\).

Altogether the positive work done by the first group will be canceled by the negative work done by the second group, resulting in very small amount of work. Under the constant mean effective stress condition, this work may be close to zero.

The present view of the behavior of particles differs from the conventional view in one aspect. In the conventional view such as seen in Rowe (1962) or its updated versions assuming multiple sliding mechanism including the effect of particle rolling, relative displacement of particle centers is implicitly (or explicitly) assumed to be consistent with at least one shear strain component along certain direction, which may or may not be parallel to the sliding plane(s). Thus the sliding or its equivalence including the effect of rolling continues to consume the energy as long as the aforementioned strain component continues to increase. In the present view, this is not the case; the sliding or its equivalence of individual contact contributes to the work only for a certain duration but then stops contributing the work, to be replaced by other contacts to begin the contribution to the work in accordance with the change in the contact force and the particle displacement. It might be said that the conventional view imposes an artificial constraint on the displacement of the particles, requiring more energy than the reality.

The above discussions all points to a single fact that
the work done by the deformation without generation or dissipation of branches under a constant mean effective stress is very small compared to those which will be generated by the deformation of granular material involving generation and disappearance of branches. In the notation of strains in Eq. (13), this may be approximated by the following equation.

\[ p' \delta v_p + q \delta y_p = 0 \]  

(23)

Under a constant mean effective stress, the volumetric strain change is only due to the volumetric strain change in the void and hence \( \delta v_0 = \delta v_v \). In order to distinguish the contribution to \( \delta v_0 \) by the deformation without involving the generation and dissipation of branches, this will be denoted by \( \delta v_{v0} \). This leads to Eq. (21) in HYPOTHESIS 3. As understood from Eq. (21) or (23), \( \delta v_{v0}/\delta y_p = -(q/p') \) is always negative, indicating a dilatant nature of \( v_{v0} \).

**Contractive Void**

When the deformation of granular material involves generation and disappearance of branches, let us begin the discussion from Eq. (9) together with the relevant discussions made on it. First of all, it should be noted that the tangential component of contact forces never contributes to the mean effective stress as mentioned earlier.

Secondly, Eqs. (8) and (9) yield

\[ dN^* = k_n Dn_n (\delta v_{ij} - \delta v_0 \delta_{ij}) \]  

(24)

in which the superscript * indicates quantities in the virtual granular material discussed earlier. Difference in the tangential bulk moduli for isotropic compression and rebound of soils suggests that larger normal effective strain is required in compression than in extension to achieve the increase or decrease of the same number of branches in the normal direction.

During shear under a constant mean effective stress, the increase of the number of branches in one direction should be almost the same as the decrease of the number of branches perpendicular to it. Otherwise the normal contact force increment given by Eq. (24) in one direction will be different in its magnitude from the normal contact force decrease in the perpendicular direction and hence a pair of these contact force increments will change the mean effective stress. Consequently, during shear under a constant mean effective stress, normal effective strain is larger in compression than in extension.

This demands that shearing under a constant mean effective stress involves compressive volumetric strain increment in association with deviator strain increment. This may be written by

\[ \delta v_p = M \delta y_p \]  

(25)

At the current progress of the study conducted by the author, it is not yet possible to find out how the coefficient \( M \) is related to more fundamental constants parameters of soils nor to determine whether the coefficient \( M \) is constant or not. Within the scope of the present study, it will be assumed for the sake of simplici-

ty that \( M \) is constant. Under a constant mean effective stress, \( \delta v_{v0} = \delta v_v \) as mentioned earlier. By denoting the volumetric strain increment in the void associated with the generation and dissipation of branches by \( \delta v_{v0c} \), Eq. (25) leads to Eq. (20) in the HYPOTHESIS 2.

As mentioned earlier, the deformation of soil is due to the deformation without generation or disappearance of branches at one instance, that with it at next instance. In the limit in which these steps occur continuously, the volumetric strain is given as a summation of these contributions. This leads to Eq. (19) in the HYPOTHESIS 1.

**A NEW LOOK AT THE DILATANCY IN CAM-CLAY**

From the HYPOTHESES 1 through 3, the stress dilatancy equation in Cam-clay model shown in Eq. (1) will be derived as follows. First of all, the volume change in the void is related to the plastic deviator strain due to Eqs. (19) through (21) as

\[ \delta v_0 = (M - q/p') \delta y_p \]  

(26)

The increment of the effective plastic strain is given from Eqs. (15) and (26) as

\[ \delta v_p' = \delta v_p - (M - q/p') \delta y_p \]  

(27)

This is rewritten as

\[ \delta v_p'/\delta y_p = M - q/p' + \delta v_p'/\delta y_p \]  

(28)

This may be called the generalized stress dilatancy relation.

During this loading process, the energy dissipated within the soil is given from Eqs. (17) and (26) as

\[ dW_p = p' \delta v_p' + p' M \delta y_p \]  

(29)

It may be noted in Eq. (29) that the energy dissipated within the soil is only due to the energy dissipated in the plastic volume change of the soil skeleton and the energy dissipated in the contractive void given by Eq. (20) as

\[ p' \delta v_{v0} = p' M \delta y_p \]  

(30)

This is because the energy “dissipated” in the dilative void is given by Eq. (21) as

\[ p' \delta v_{v0} = -q \delta y_p \]  

(31)

and this cancels out the energy dissipated in the deviatoric deformation of the soil skeleton when the summation is taken in Eq. (17).

When the loading process does not involve the plastic volume change in the soil skeleton, such that

\[ \delta v_p' = 0 \]  

(32)

the stress dilatancy relation in Eq. (28) and the energy dissipated within the soil in Eq. (29) will be given as

\[ \delta v_p'/\delta y_p = M - q/p' \]  

(33)

\[ dW_p = p' M \delta y_p \]  

(34)

Thus, the relation given in Eq. (1) is obtained as shown in Eq. (33). The result given in Eq. (34) was originally a
hypothesis made in the studies by Roscoe et al. (1963) and Schofield and Wroth (1968). The results of the present chapter show that the original hypothesis of Eq. (34) is replaced in the present study by the hypotheses with respect to the volume change of the void as stated at the beginning of the previous chapter.

**YIELD SURFACES FOR SOIL SKELETON**

As partly discussed in the previous study (Iai, 1993b), the concept of the effective strain may be a promising tool in the constitutive modeling of soils. In the present study, if the loading process does not involve the plastic volume change in the soil skeleton, the flow rule for the soil skeleton is given by Eq. (32). With the assumption of normality condition with respect to the effective strain, the direction of the incremental plastic effective strain vector $(d\psi_p, dy_p)$ should be at a right angle with that of the incremental stress vector $(dp', dq)$ representing the yield surface. This demands

$$\frac{dq}{dp'} = -\frac{d\psi_p}{dy_p}$$ (35)

The differential equation for specifying the yield surface is obtained from Eqs. (32) and (35) as

$$\frac{dq}{dp'} = 0$$ (36)

Thus, the yield surface of the soil skeleton is given by integrating Eq. (36) as

$$q = \text{const.}$$ (37)

In the present approach, the link between the constant of integration in Eq. (37) and the isotropic normal consolidation pressure $p'$ becomes obscure. This may be a certain drawback of the present approach. This, on the other hand, will give us a significant flexibility in the constitutive modeling. The integration constant appearing in Eq. (37) can be regarded as a function of plastic strain. Let us write this as

$$q - f(\psi_p, y_p) = 0$$ (38)

Since the current stress point $(p', q)$ should always be on the yield surface during plastic deformation associated with the stress increment $(dp', dq)$, Eq. (38) demands

$$dq - \left( -\frac{\partial f}{\partial p} dp' + \frac{\partial f}{\partial y} dy_p \right) = 0$$ (39)

Since $d\psi_p = 0$ in Eq. (32), Eq. (39) will be rewritten as

$$dy_p = \frac{1}{G} dq$$ (40)

in which

$$G = \frac{\partial f}{\partial y_p}$$ (41)

The coefficient $G$ is a function of $(\psi_p, y_p)$ and easily determined from the deviator stress-deviator strain curve obtained by the laboratory tests of soils. The link between the coefficient $G$ and the isotropic normal consolidation pressure $p'$ is replaced by the link between $G$ and $\psi_p$.

In order to take into account the behavior of soil under the general loading process which involves the plastic volume change in the soil skeleton, it may be necessary to introduce an additional yield surface which intersects with the yield surface in Eq. (37). Since the isotropic normal consolidation, which does not induce the deviator strain (i.e. $dy_p = 0$), has to be also explained by this yield surface, this imposes the constraint on the yield surface as

$$dy_p = 0$$ (42)

From this flow rule, the assumption of normality condition in Eq. (35) yields the following differential equation.

$$\frac{dp'}{dq} = 0$$ (43)

Thus, the additional yield surface of the soil skeleton is given by integrating Eq. (43) as

$$p' = \text{const.}$$ (44)

This can also be viewed as of a kinematic hardening type as discussed in Eqs. (38) through (41). The hardening rule can be specified from the volumetric stress strain curve at the isotropic normal consolidation of soils.

With the introduction of two intersecting yield surfaces, the present approach poses no "anomaly" or "uncertainty" associated with the isotropic normal consolidation which seems to have been a somewhat controversial issue (Roscoe et al., 1963; Schofield and Wroth, 1968).

The use of two intersecting yield surfaces has become one of popular practice in the constitutive modeling of soil and it is not the intention of the present study to claim that this is new. The remark is made only to point out the fact that the use of the two intersecting yield surface was implicitly involved in Cam-clay model and the present study simply makes it more explicit than it was.

It may also be pointed out that the hardening rule for specifying the deviator stress strain relation which is exactly the same as that of Cam-clay model can be adopted as one of reasonable candidates but this is not mandatory. Thus, some of the existing limit in the capability of Cam-clay model may be improved along this line of effort. It is again not the intention of the present study to claim that this line of effort is new. The remark is made only to point out the fact that this line of effort is not out of the scope of Cam-clay model but rather within it; the present study simply makes it more easy to be seen.

In the present chapter, yield surfaces were considered only for the soil skeleton. An interesting possibility may be given by considering the yield surfaces for the void skeletons as well (see Appendix).

**A NEW LOOK AT THE CRITICAL STATE**

When the soil can continue to be sheared without changes in its stress or volume, the soil is said to be at the
critical state (Roscoe et al., 1963; Schofield and Wroth, 1968). Hence, at the critical state

\[ dp' = 0 \]  \hspace{1cm} (45)
\[ dq = 0 \]  \hspace{1cm} (46)
\[ do = 0 \]  \hspace{1cm} (47)

At the critical state, there is no change in the effective mean stress as seen in Eq. (45). This demands that there will be no elastic volumetric strain change and hence, from Eq. (47), there will be no plastic volumetric strain change, such that

\[ dv_p = 0 \]  \hspace{1cm} (48)

Since the effective strain “properly” represents the current stress condition, there will either be no plastic effective volumetric strain change, such that

\[ dv'_p = 0 \]  \hspace{1cm} (49)

Thus, from Eqs. (15), (48) and (49), there will either be no volume change in the void, such that

\[ dv_v = 0 \]  \hspace{1cm} (50)

These conditions characterize the soil skeleton and the void at the critical state.

In the discussion of the critical state, a rather obvious hypothesis will be made in the present study as an additional hypothesis as follows.

**HYPOTHESIS 4**: the tangential stiffness of soil is continuous; i.e. no jump occurs in the increment of the plastic strain when the critical state is reached.

With this hypothesis, the slope of the critical state line in the \( p' - q \) space will be shown to coincide with \( M \) as follows, provided that the soil specimen deforms uniformly and the critical state is approached from the wet side.

The plastic energy transmitted across the boundaries to the unit volume of the soil element is given from Eqs. (12) and (48) as

\[ dE_p = qdy_p \]  \hspace{1cm} (51)

**HYPOTHESIS 4** demands that, if the critical state is approached from the wet side (i.e. \( dv'_p = 0 \)), Eq. (34) should also apply to the critical state. Comparison of the dissipated energy given in Eq. (34) to the transmitted energy given in Eq. (51) through the energy balance relation in Eq. (18) yields the condition for the critical state as

\[ q = Mp' \]  \hspace{1cm} (52)

Thus, if we demand that the soil deform uniformly and the critical state be approached from the wet side, Eq. (52) together with Eq. (46) imposes the constraint on the hardening rule for the deviator deformation of the soil skeleton; the tangential stiffness of the soil in the deviator deformation should be zero at the stress condition given by Eq. (52). On the other hand, the conditions for the critical state will impose no constraint on the hardening rule for the volumetric deformation of the soil skeleton; the conditions given by Eqs. (45) and (49) are simply complementary with each other.

**GENERALIZED CRITICAL STATE AND PHASE TRANSFORMATION LINE**

The foregoing observation on the critical state implies that, without the special requirement on the uniformity of the soil deformation or on the stress path, the critical state, in general, is governed by the hardening rule for the deviator deformation of the soil; it is not vice versa. Let us write the stress condition at which the tangential stiffness of soil in the deviator deformation continues to be zero as

\[ q/p' = M_f \]  \hspace{1cm} (53)

Under the general condition in which the uniformity of the soil deformation may no longer exist, Eqs. (12), (51) (or Eq. (1)) and (52) may no longer be applicable but the rest of the relations are still applicable if these relations are considered as a limiting case in which the representative soil element in concern shrinks to a point within the soil specimen. Under the general loading process, the state line at which the increment of the plastic volumetric strain of soil becomes zero (i.e. \( dv_p = 0 \)) is given from Eq. (28) as

\[ q/p' = M + dv'_p/dy_p \]  \hspace{1cm} (54)

The state line specified by Eq. (54) divides the stress space into dilative and contractive zones as

\[ q/p' < M + dv'_p/dy_p; \text{ contractive zone (i.e. } dv_p > 0) \]  \hspace{1cm} (55)
\[ q/p' > M + dv'_p/dy_p; \text{ dilative zone (i.e. } dv_p < 0) \]  \hspace{1cm} (56)

**HYPOTHESIS 4** demands that no jump occurs in \( dv_p \) when the critical state is reached; i.e. \( dv_p \) should gradually approach zero as imposed by Eq. (48). Consequently, the following relations should be satisfied at the critical state.

\[ M_f = M + dv'_p/dy_p \]  \hspace{1cm} (57)

This may be called the generalized critical state. This implies that \( M_f \) can be either greater than or equal to \( M \) depending on the types of the hardening rules of the soil.

It should be noted in Eqs. (55) and (56) that the loading process initiating from the isotropic stress condition at first goes through the contractive zone. Under the undrained condition, there will be a decrease in the effective mean stress. This demands \( dv'_p = 0 \) until the effective mean stress becomes greater than the pre-consolidation pressure. The condition \( dv'_p = 0 \) demands the existence of another state line at which the increment of the plastic volumetric strain of soil becomes zero (i.e. \( dv_p = 0 \)); this is given from Eqs. (54) through (56) by imposing \( dv'_p = 0 \) as

\[ q/p' = M \]  \hspace{1cm} (58)

and the state line specified by Eq. (58) divides the stress space into dilative and contractive zones as

\[ q/p' < M; \text{ contractive zone (i.e. } dv_p > 0) \]  \hspace{1cm} (59)
\[
q/p' > M: \text{ dilative zone (i.e. } dv_p < 0) \tag{60}
\]

Under the undrained condition, the stress point goes through the contractive zone given by Eq. (59), goes through the state line given by Eq. (58), goes into the dilative zone given by Eq. (60), then enters the stress zone involving the plastic volumetric change in the soil skeleton which gradually increases the slope of the critical state line by Eq. (54), and eventually is overtaken by the critical state line.

Laboratory data which support this implication are abundant for sands. The paper by Ishihara et al. (1975) may be the first to point out this significant fact based on the laboratory study; they called the line given by Eq. (58) as the phase transformation line. The examples are shown in Fig. 5 after Ishihara et al. (1991).

In the case of the clays on the dry side, the existence of the phase transformation line is less obvious. The reason for this is not clearly known to the present author.

DISCUSSIONS

The observation made in the previous chapter leads us to think carefully about the behavior of soils at the dilatant zone. It may be noted from Eq. (20) that, even when the current state is in the dilatant zone, the volume of the contractive void continues to decrease; the total volume increase in the soil is merely the results of the volume increase in the dilatant void in accordance with the increasing stress ratio. This observation is believed to become very important when the present line of thought is extended for understanding the soil behavior under the cyclic loading condition; liquefaction and cyclic mobility may be understood as the cumulative effect of the contractive void with the temporary recovery of effective mean stress due to the dilatant void.

It may also be noted that, whether the sand is on the wet side or on the dry side in the \( e - \log p' \) plane, the undrained loading process initiating from the isotropic

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**Fig. 5.** Effective stress paths at the undrained triaxial tests of Toyoura Sand (after Ishihara et al. (1991) with supplements by the present author); (a) at \( e=0.735, D_r=38\% \) and (b) at \( e=0.833, D_r=64\% \)
stress state always induces the stress point to go through
the contractive zone in the \( p' - q \) plane, then to go into
the dilative zone after passing through the phase transfor-
mation line and finally to reach the critical state. This
may be also seen in Fig. 5.

As seen in the present study, the concept of the effec-
tive strain can be a very useful tool in the constitutive
modeling of granular materials. Some of the implications
on the constitutive modeling were offered with respect to
the stability and the associated flow rule in the previous
study (Iai, 1993b). The results of the present study indi-
cate that the simple hypothesis made in Eq. (19), together
with the concept of the effective strain and other
hypotheses, may be a useful alternative to the convention-
al flow rule specified in terms of the strain of soil such as
given by Eq. (1), which has been used in the majority of
the constitutive modeling of granular materials.

CONCLUSIONS

An attempt was made to bring out the essential implica-
tions hidden behind the stress dilatancy relation in Cam-
clay model. In the present study, this stress dilatancy rela-
tion was used mainly because of its simplicity to explain
certain aspects of the soil behavior through the concept
of effective strain; it was not the intention of the present
study to claim that the stress dilatancy relation in Cam-
clay model provides a thorough explanation of the soil
behavior. Conclusions obtained from the present study
may be summarized as follows.

(1) The hypotheses made in deriving Cam-clay model
are reduced to the following four hypotheses:

**HYPOTHESIS 1:** the void consists of contractive and
dilative voids.

**HYPOTHESIS 2:** the increment of the volumetric
strain in the contractive void is proportional to the incre-
ment of the plastic deviator strain of soil through the con-
tant \( M \).

**HYPOTHESIS 3:** the energy dissipated by the void
skeleton of the dilative void is none.

**HYPOTHESIS 4:** the tangential stiffness of soil is con-
tinuous.

(2) In the general types of granular materials such as
sands, it is suggested that the phase transformation line,
which divides the stress space into dilative and contract-
tive zones, should be distinguished from the critical state
line.

(3) In particular, even in the dilative zone, the volume
of the contractive void continues to decrease; the total
volume increase in the soil is merely the results of the
volume increase in the dilative void in accordance with
the increasing stress ratio.

(4) In the constitutive modeling of granular materials,
**HYPOTHESIS 1**, together with the concept of effective
strain and other hypotheses, may be a promising alterna-
tive to the conventional stress dilatancy relation specified
in terms of the strain of soil.

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NOTATION

\( D \): a coefficient in the incremental relationship between the effec-
tive strain and the distance change between the particle centers
in the virtual granular material

\( E_p \): plastic energy transmitted to the soil

\( l \): length of a branch

\( k_c \): contact spring coefficient

\( n \): unit direction vector along a branch

\( N \): normal component of contact force

\( P_c \): contact force

\( p' \): effective mean stress \((= (\sigma_{1} + 2\sigma_{3})/3) \); compression positive

\( q \): deviator stress \((= \sigma_{1} - \sigma_{3}) \)

\( u_{c} \): displacement of a branch vector

\( u_{n} \): displacement of a contact point

\( V \): volume of representative element

\( v \): volumetric strain (compression positive)

\( v' \): effective volumetric strain (compression positive)

\( v_{c} \): volumetric strain in the void due to dilatancy (compression posi-
tive)

\( v_{c} \): volumetric strain in the void due to dilatancy (compression posi-
tive)

\( v_{c} \): contractive component of the volumetric strain in the void due
to dilatancy

\( v_{c} \): dilative component of the volumetric strain in the void due
to dilatancy

\( v_{c} \): plastic volumetric strain \((= (\sigma_{1} - 2\sigma_{3})/3) \); compression positive

\( v_{c} \): plastic effective volumetric strain (compression positive)

\( W_{c} \): plastic energy dissipated in the soil

\( W_{c} \): plastic energy dissipated in the soil skeleton

\( \gamma_{c} \): plastic deviator strain \((= (2/3)(\varepsilon_{1} - \varepsilon_{3})) \)

\( \varepsilon_{0} \): volumetric strain due to dilatancy

\( \varepsilon_{0} \): strain tensor

\( \varepsilon_{0} \): effective strain tensor \((= \varepsilon_{1} - \varepsilon_{3} \varepsilon_{3}) \)

\( \varepsilon_{0} \): plastic axial strain

\( \varepsilon_{0} \): plastic radial strain

\( \eta \): deviator stress ratio \((= q/p') \)

\( \sigma_{1}, \sigma_{3} \): stress tensor

\( \sigma_{1}, \sigma_{3} \): axial effective stress

\( \sigma_{1}, \sigma_{3} \): radial effective stress

\( M \): a constant of Cam-clay model

\( M \): a deviator stress ratio at which the tangential stiffness of soil
continues to be zero

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APPENDIX

YIELD SURFACES FOR VOID SKELETONS

In the present study, Eq. (26) specifies a flow rule for the void skeleton. For the completeness of the discussion in the present study, the yield surface for the void skeleton will be given as follows. With the assumption of normality condition with respect to the strain of the void skeleton, the direction of the incremental strain vector \((dv_0, dv_2)\) should be at a right angle with that of the incremental stress vector \((dp', dq)\) representing the yield surface. This demands

\[
\frac{dq}{dp'} = -(dv_0, dv_2)
\]  

(A.1)
The differential equation for specifying the yield surface is obtained from Eqs. (26) and (A.1) as

\[
\frac{dq}{dp'} = -M + q/p'
\]  

(A.2)
Integration of Eq. (A.2) by introducing the deviator stress ratio \(\eta = q/p'\) will yield the same yield surface as was given for Cam-clay model and it will not be reproduced here.

A more interesting possibility is to separately consider the yield surfaces for the contractive and the dilative void skeletons. The flow rules for these void skeletons are specified by Eqs. (20) and (21), respectively.

The differential equation for specifying the yield surface for the contractive void skeleton is obtained from Eqs. (20) and (A.1) as

\[
\frac{dq}{dp'} = -M
\]  

(A.3)
Thus, the yield surface for the contractive void skeleton is obtained by integrating Eq. (A.3) as

\[
Mp' + q = \text{const.}
\]  

(A.4)
This can be viewed as of a kinematic hardening type.

Similarly, the differential equation for specifying the dilative void skeleton is obtained from Eqs. (21) and (A.1) as

\[
\frac{dq}{dp'} = q/p'
\]  

(A.5)
This is rewritten after some manipulation by introducing the deviator stress ratio \(\eta = q/p'\) as

\[
\frac{d\eta}{dp'} = 0
\]  

(A.6)
Thus, the yield surface of the dilative void skeleton is given by integrating Eq. (A.6) as

\[
\eta = q/p' = \text{const.}
\]  

(A.7)
This can also be viewed as of a kinematic hardening type. It may be noted that the yield surface given by Eq. (A.7) is consistent with the failure surface of Mohr-Coulomb type in \(p' - q\) space.

Constants of the integration in Eqs. (A.4) and (A.7) are determined from the deviator stress strain relation so that the hardening rules for the contractive and the dilative void skeletons should correctly reproduce the same deviator strain as given by the hardening rule for the soil skeleton.

A drawback involved in the present approach is that the yield surfaces for the soil skeleton and two kinds of the void skeletons produces, as a summation, three times of the plastic deviator strain. This drawback may be overcome if we introduce a “complementary” yield surface for one of the voids so that it produces the same volumetric strain in the void but, at the same time, produces the negative value of the plastic deviator strain. This may be done, for example, by introducing a yield surface in place of Eq. (A.4) as

\[
Mp' - q = \text{const.}
\]  

(A.8)
The constant of the integration in Eq. (A.8) should be the same as that given in Eq. (A.4) so that the yield surface given by Eq. (A.8) continuously moves along positive direction of \(p'\) axis in the \(p' - q\) plane in accordance with the loading process associated with the “complementary” deviator stress of \((-q)\).

The results of the present appendix indicate that the volumetric strain in the void can be specified by two simple intersecting yield surfaces. It is also indicated that Cam-clay model can be reconstructed as a four intersecting yield surface model; the yield surfaces are given by Eqs. (37), (44), (A.1), and (A.8).