EXTREME LOAD OF SOILS REINFORCED BY COLUMNS: THE CASE OF AN ISOLATED COLUMN

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ABSTRACT

The improvement of bearing capacity for soft purely cohesive soils reinforced by columns was studied. Using the Yield Design theory, the extreme load of an isolated column was determined based on the plane strain assumption, and then in axisymmetry. Two loading cases were considered, and the influence of gravity was investigated. In order to ensure the improvement of the soil's bearing capacity, the minimum values of the mechanical characteristics of the column material are identified. Quasi-exact values and acceptable bounds for the extreme load were compared to results presented in the literature. The method developed here is considered relevant for better performance.

Key words: axisymmetric analysis, column, extreme load, limit analysis, plane strain, reinforcement, soft soils, Yield Design (IGC: E4/K3)

INTRODUCTION

Improving the load bearing capacity of soft soils is of extensive interest for foundation engineering problems. It can be accomplished in different ways (soil nailing, geotextiles, reinforced earth, etc.). Other advantages can also be obtained including reduction of settlement and of primary consolidation time. These advantages are provided by the column reinforcement technique: Hughes et al. (1974), Aboshi et al. (1979), Soyez (1985), etc. Common materials for these columns are mainly well graded clean sands or gravels. When the existing soil is stabilized by adding a low percentage of lime, the same performance can be attained, Broms (1982). Through previous papers it was shown particularly that, with this column technique, weak soils can be significantly reinforced: Madhav et al. (1978), Brauns (1978), etc.

In this present investigation, a discussion about the modelling of the loading process of a soil reinforced by columns is given, and then the improvement of bearing capacity is investigated for an isolated column. The latter aspect was studied by using the Yield Design theory for which important tools were given by Salençon (1983).

The minimum mechanical characteristics of columns ensuring the reinforcement of the soil were determined. Considering both the undrained and the drained behaviors of the reinforced soil, calculations were made assuming plane strain and with axisymmetry. The influence of gravity was studied. Based on the plane strain assumption,

an upper bound of the bearing capacity for an undrained composite ground is proposed if some geometrical conditions are fulfilled. Finally, the most important conclusions are summarized. The main notations used are presented at the end of the paper.

MODELLING OF THE LOADING PROCESS

Statement of the Problem

Consider a horizontal half space made up of a purely cohesive soft soil reinforced by cylindrical columns with an infinite height and a total cross section area denoted by A_c . The column material is cohesive-frictional. The reinforced soil is assumed to be completely saturated. The purpose of this investigation is to determine the maximum value of a vertical force F which can be maintained in equilibrium by the reinforced soil. This force F is applied (in a center position) by a rigid foundation (Fig. 1). The following assumptions were adopted.

—The columns are constituted with a perfectly drained material.

—When the soft soil is loaded, two different behaviors are observed:

*The undrained behavior where all stresses induced in the soft soil are transmitted to water in terms of pore water pressure excesses. Consequently, effective stresses are negligible.

*The drained behavior where pore water pressure excesses initially created are transmitted completely to

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Fig. 1. Non-uniform loading case

the solid mass of the soft soil.

Taking into account these considerations and denoting by A the loaded cross section of the reinforced soil, we distinguish two conditions in order to study the vertical effective stress distribution (at the surface of the reinforced soil) imposed by the loaded rigid foundation.

Condition 1 In the undrained behavior, the loaded foundation induces a vertical effective stress only on the section of the columns where no pore water pressure excess occurs. Since the column material is perfectly drained (Fig. 2(a)), for this condition we have $A=A_c$. □

Condition 2 In the drained behavior when all pore water pressure excesses are dissipated, the loaded foundation induces a vertical effective stress distribution defined by:

— σ'_s on the soft soil section in contact with the foundation;

— σ'_c on the column section.

Therefore, A is the complete of the foundation section. □

Starting with these two conditions described above, and considering the vertical effective stress distribution at the surface of the reinforced soil, the ratio defined by: $k_a=A_c/A$ becomes:

— $k_a=1$, for the undrained behavior;

— $0 < k_a < 1$, for the drained behavior.

In terms of effective stresses; for the condition $k_a=1$, since we neglect the contribution of pore water pressure to the equilibrium of the external load F , the maximum value of F is in practice greater than the value calculated here. This simplification leads to a larger safety factor.

In this paper, the case of an isolated column was first developed before studying the more practical case of a group of columns in future work.

Fig. 2. Effective stress distribution on a reinforced soil for one column

Geometrical and Mechanical Parameters of the Reinforced Soil

The calculation of the extreme load of an isolated column is achieved by using the following notation and hypotheses.

Assuming a smooth contact between the rigid foundation and the reinforced soil, the resulting load F due to the vertical stress distribution σ_{zz} is given by:

$$F = - \int_A \sigma_{zz} dA \quad (1)$$

Concerning the mechanical characteristics of the material; the soil is assumed to be homogeneous, and cohesive with a cohesion c , and governed by Tresca's strength criterion which is expressed by:

$$f(\underline{\sigma}(x)) = \sup \{ \sigma_i - \sigma_j - 2c, i, j = 1, 2, 3 \} \leq 0; \quad (2)$$

The column material can be either purely cohesive, or cohesive-frictional with a friction angle ϕ and a cohesion c_1 . In the latter case Coulomb's strength criterion is considered; it is expressed by:

$$f(\underline{\sigma}(x)) = \sup \{ \sigma_i(1 + \sin \phi) - \sigma_j(1 - \sin \phi) - 2c_1 \cos \phi, i, j = 1, 2, 3 \} \leq 0; \quad (3)$$

Calculations are made using axisymmetry or based on a plane strain assumption. In the latter case, the column is in fact a trench of breadth d_1 loaded on a breadth d . Considering one unit length, the sections A_c and A are given by: $A_c=d_1$, $A=d$.

Gravity is a loading parameter which has to be considered. For an isolated column (Fig. 3) γ and γ_1 denote respectively the specific weight of the soil and the column's material one.

The following dimensionless ratios are also introduced: $k_g=\gamma_1/\gamma$; $k_c=c_1/c$; $k_d=d_1/d$.

In this work, the Yield Design theory is used to determine bounds F^- and F^+ for the extreme load F^* for an isolated column. These bounds are calculated in terms of the resulting load defined by (1) and not in terms of the effective vertical stress distribution on the surface of the reinforced soil. The exact value of the extreme load is expressed by a dimensionless ratio denoted by (F^*/cd) in the plane strain analysis, and (F^*/cA) in axisymmetry.

Fig. 3. Loading parameters

CALCULATION IN THE PLANE STRAIN ANALYSIS WITHOUT GRAVITY

The reinforced soil, with a trench of width d_1 supports a strip footing loaded symmetrically with width d (Fig. 3). The analysis considers the (xz) plane to determine a lower (or an upper) bound for the extreme load F^* . The ultimate bearing capacity of footings on stabilized soils has been studied by Madhav et al. (1978) using plane strain analysis, where a generalized Prandtl's mechanism was considered. By equating the work of the external load to the dissipated energies, ultimate bearing capacity factors were derived. This analysis is similar to (but not equivalent to) the kinematic approach of the Yield Design theory.

The Static Approach

Using the static approach, a lower bound F^- of the extreme load F^* is obtained when a stress field, in equilibrium with F^- and satisfying the strength criterion at every point x , is constructed.

The stress fields considered here are constructed by subdividing the half space $z \leq 0$ into several zones with constant values of the stress tensor in each zone. The equilibrium equations are given by:

$$\sum_{j=1}^3 \partial \sigma_{ij} / \partial x_j = 0; i=1, 2, 3$$

(without gravity) and are automatically satisfied in each zone.

A separating surface between two given zones (I) and (II), where the stress tensors denoted by $\sigma^{(I)}$ and $\sigma^{(II)}$ are different, is qualified as a stress discontinuity surface. Denoting by $\underline{n}$ the unit normal vector to the stress discontinuity surface, continuity of the stress vector is defined by: $T_i = \sigma_{ij} n_j$ across all the stress discontinuity surfaces, and ensures equilibrium for the entire half space $z \leq 0$, Salençon (1990). A stress field satisfying equilibrium equations, continuity of the stress vector and Eq. (1) is called a statically admissible (s.a.) stress field.

A stress field with five zones

The half space defined by $z \leq 0$ is subdivided into five zones (Fig. 4). The stress field is constructed with constant values of σ for each zone and satisfy Eq. (1) and continuity of the stress vector at the surfaces (AA'), (BB'), (DD'), and (FF'). By applying Coulomb's criterion given by (3) in zone (3) the stress component $\sigma_{zz}^{(3)}$ is determined (Fig. 4), and the following lower bound of F^* is given:

$$4[1 + k_d(\sin \phi / (1 - \sin \phi))] + 2k_d[k_c(\cos \phi / (1 - \sin \phi)) - 1] \leq (F^* / cd) \quad (4)$$

From ineq. (4) a lower bound for the purely cohesive column material is obtained by setting $\phi=0$:

$$4 - 2k_d + 2k_d k_c \leq (F^* / cd) \quad (5)$$

In addition, for $k_d=1$, from ineq. (4) a lower bound of F^* for the undrained behavior is obtained. For this case

Fig. 4. Stress field with five zones

the five zones in Fig. 4 are reduced to three zones as shown in Fig. 5(a). A lower bound for a cohesive-frictional column material is easily deduced:

$$2 \tan^2(\pi/4 + \phi/2) + 2k_c \tan(\pi/4 + \phi/2) \leq (F^* / cd) \quad (6)$$

For purely cohesive material columns ($\phi=0$), when $k_d=1$, the lower estimate obtained from ineq. (6) is written:

$$2(1 + k_c) \leq (F^* / cd) \quad (6a)$$

By setting $k_c=0$ in ineq. (6), a lower bound for a purely frictional column material is obtained:

$$2 \tan^2(\pi/4 + \phi/2) \leq (F^* / cd) \quad (6b)$$

The extreme load of a homogeneous purely cohesive soil being equal to $(\pi+2)c$, Salençon (1983), from ineq. (6) one can deduce that reinforcement definitely occurs for the values of ϕ verifying:

$$\tan(\pi/4 + \phi/2) = (-k_c + \sqrt{k_c^2 + 4 + 2\pi})/2 \quad (6c)$$

For a purely frictional column material, reinforcement occurs from: $\phi=26^\circ$. This gives a justification of the value commonly recommended by several authors: Aboshi et al. (1979), Soyez (1985).

When comparing the lower bound given by ineq. (4) and the extreme load of a homogeneous purely cohesive soil: $(\pi+2)c$, an improvement of the extreme load for the drained behavior case definitely occurs for couples (ϕ, k_c) satisfying the following necessary condition:

Fig. 5(a). Stress field with three zones

$$k_c > (1 - 3 \sin \phi) / \cos \phi \quad (6d)$$

For a purely frictional column material, ineq. (6d) is fulfilled when: $\phi \geq 19.5^\circ$.

From ineq. (4), it can be deduced that the improvement of the extreme load occurs, in the drained behavior case, for values of k_d given by:

$$k_d \geq \left(\frac{\pi}{2} - 1 \right) (1 - \sin \phi) / (-1 + k_c \cos \phi + 3 \sin \phi) \quad (6e)$$

The minimum values of k_d verifying ineq. (6e) are shown in Fig. 5(b).

A stress field with eleven zones in the undrained behavior case ($k_d=1$)

This stress field (Fig. 6) has been used by Tristan-Lopez (1981). It leads to a better estimate of the lower bound. This latter is determined with the help of the Mohr stress representation. For a cohesive-frictional column material, (calculation is detailed in appendix A1) the following result is obtained:

$$h(\beta_1) + f(\beta_1, \phi) \left[\cos 2\beta - \frac{\sin 2\beta}{\sin 4\beta_2} + \frac{\sin 2(\beta + 2\beta_2)}{\sin 4\beta_2} \right] - \frac{\sin 4\beta_1}{\sin 2\beta_2} \leq (F^* / cd) \quad (7)$$

where:

$$h(\beta_1) = 1 - 2 \cos 2\beta_1 - \cos 4\beta_1 \quad (7a)$$

Fig. 5(b). Minimum values of k_d vs friction angle ϕ

Fig. 6. Stress field with eleven zones

The function $f(\beta_1, \phi)$ and the angle β are defined in appendix A1.

The lower bound estimate, for a given value of ϕ , is obtained when the left side of the ineq. (7) is maximum with respect to β_1 and β_2 ; this maximum being determined by a numerical procedure. The evolution of the lower bound estimate with friction angle ϕ is shown in Fig. 7(a). The stress field with eleven zones indicates that reinforcement occurs from the value $\phi = 17^\circ$; while for $\phi = 26^\circ$ the bearing capacity is significantly improved, by more than 38%.

For a purely cohesive column material ($\phi = 0$), by using the same eleven zones stress field, the expression of the lower bound estimate is determined (as detailed in appendix A2) by:

$$h(\beta_1) + k_c [1 + (2 / \sqrt{1 + Y}) + (Y - 1) / (Y + 1)] \leq (F^* / cd) \quad (8)$$

where:

$$Y = 2k_c \left[\frac{(k_c / \sin 4\beta_1) - \sqrt{(k_c^2 / \sin^2 4\beta_1) - 1}}{\sin 4\beta_1} \right] - 1$$

and $h(\beta_1)$ is given by Eq. (7a).

The maximum of the lower bound estimate with respect to β_1 is calculated numerically from ineq. (8). The evolution of the lower bound estimate with the cohesion ratio k_c is shown in Fig. 7(b).

The Kinematic Approach

The kinematic approach is based on the principle of virtual work and involves the construction of kinematically admissible (k.a.) velocity fields. A velocity field $\underline{v}$ is called

Fig. 7(a). Lower bounds estimate vs friction angle ϕ

Fig. 7(b). Lower bounds estimate vs cohesion ratio k_c ($k_d=1$)

(k.a.) if it is piecewise continuously differentiable and satisfies the given boundary conditions for velocity, Salençon (1990). The maximum resisting work is determined:

$$P(\underline{v}) = \int_{\Omega} \Pi(\underline{d}) d\Omega + \int_{\Sigma} \Pi(\underline{n}, [\underline{v}]) d\Sigma \quad (9)$$

where $[\underline{v}]$ is the velocity jump across the velocity discontinuity surfaces Σ in the domain Ω , and $\underline{d}$ denotes the strain rate tensor. Expressions of functions Π appearing in Eq. (9) are defined in appendix A3.

The external forces work is the scalar product of the load vector $\underline{F}$ with its associated kinematic vector $\underline{\dot{q}}(\underline{v})$. Using notation given in Fig. 3, this product is expressed by:

$$\underline{F} \cdot \underline{\dot{q}}(\underline{v}) = FU - \int_{\Omega} \gamma v_z d\Omega \quad (10)$$

where U is a given constant vertical displacement rate imposed along the width d on the plane $z=0$. The extreme load F^* is given by:

$$F^* = \inf \{ \underline{F}, \underline{F} \cdot \underline{\dot{q}}(\underline{v}) - P(\underline{v}) \leq 0, \forall \underline{v}(\text{k.a.}) \} \quad (11)$$

A load F^+ satisfying $\underline{F}^+ \cdot \underline{\dot{q}}(\underline{v}) - P(\underline{v}) \leq 0$ for all $\underline{v}$ in a particular family of (k.a.) velocity fields is an upper bound of F^* : $F^* \leq F^+$.

The undrained behavior case ($k_d=1$)

Consider the half space $z \leq 0$ subjected to the force F (Fig. 3). Several mechanisms can be imagined in this system where the block (OBB') is virtually constrained to a rigid motion with a constant velocity U (Figs. 8 and 9).

The mechanism of five blocks shown in Fig. 8 involves only rigid blocks motion along surfaces (OB), (ON), (BN), (NM) and their symmetrical parts where discontinuities of velocity occur. The Prandtl's mechanism (Fig. 9) induces deformations in blocks (2) and (2') and discontinuities of velocity along surfaces (OB), (NB), (MN), (OFN) and their symmetrical parts.

Discontinuities of velocity can be determined by constructing the hodograph. For the mechanism of five blocks two cases are considered on surfaces (BN) and (B'N') (Fig. 8): when the discontinuity of velocity occurs in the soil, being as a Tresca's material, it should be tangent to (BN) (case b1 in Fig. 8); while it should deviate by an angle β (case b2 in Fig. 8), such that: $\phi \leq \beta \leq (\pi - \phi)$, when this discontinuity occurs in the column being considered as a Coulomb's material.

The drained behavior case: ($0 < k_d < 1$)

Consider a mechanism with rotation (Fig. 10) where R , and angles β , θ_1 , θ_2 , and θ_3 are involved as parameters. The corresponding upper bound estimate, as detailed in appendix A4, is given by:

$$\begin{aligned} (F^*/cd) \leq & ((1+k_d)/\sin \theta_2)^2 [((\theta_1 - \theta_2)/2) \\ & + (k_c/4) \{ \exp(2(\theta_2 - \theta_3) \operatorname{tg} \beta) - 1 \} \cot \phi \\ & + ((\alpha + \theta_3)/2) \exp(2(\theta_2 - \theta_3) \operatorname{tg} \beta)] \end{aligned} \quad (12)$$

The minimum of the right handside of ineq. (12) occurs with respect to β for $\beta = \phi$, while it is determined numeri-

Fig. 8. Mechanism with five blocks

Fig. 9. Prandtl's mechanism

Fig. 10. Mechanism with rotation

cally with respect to θ_2 and θ_3 . Angles θ_1 and α are expressed in terms of θ_2 and θ_3 (see appendix A4).

The variation of the upper bound estimate, as a function of the angle ϕ , using the mechanism with rotation shows that reinforcement may occur, for all values of k_d , when $\phi > 15.5^\circ$ (Fig. 11). Table 1 summarizes the results obtained from the Yield Design theory and values of the ultimate load proposed by Madhav et al. (1978). The values of the ultimate load bounded by the upper and the

Fig. 11. Upper bound estimate for the mechanism with rotation

Table 1. Results for a purely frictional column material

k_d	Lower bound by the static app. (five zones)		Ultimate load by Madhav et al. (1978)		Upper bound by the kinematic app. (rotation)	
	$\phi=25^\circ$	$\phi=30^\circ$	$\phi=25^\circ$	$\phi=35^\circ$	$\phi=25^\circ$	$\phi=30^\circ$
0.2	4.40	4.68	5.85	6.25	6.33	6.75
0.4	4.80	5.35	6.70	7.30	7.22	8.14
0.6	5.20	6.03	7.50	8.60	8.32	9.78

lower bounds are presented here.

By considering in this case ($0 < k_d < 1$) a mechanism of five blocks similar to that shown in Fig. 8, and by applying (10), (11) and calculating the function $P(v)$ from Eq. (29b), the upper bound estimate is determined for a purely cohesive column material, as:

$$(F^*/cd) \leq (k_{cd}/\sin \alpha \cos \alpha) + (1/\sin \delta \cos \delta) + \tan \alpha + \tan \delta \quad (13a)$$

where:

$$k_{cd} = 1 + k_d(k_c - 1) \quad (13b)$$

The minimum of the right handside of ineq. (13a) is obtained analytically for α and δ given by: $\tan \alpha = (\sqrt{2}/2)$ and $\tan \delta = \sqrt{k_{cd}(1+k_{cd})}$. The upper bound estimate given by ineq. (13a) then becomes:

$$(F^*/cd) \leq 2.828 + 2\sqrt{k_{cd}(1+k_{cd})} \quad (13c)$$

The undrained behavior case ($k_d=1$)

Consider the mechanism of five blocks (Fig. 8) for a cohesive-frictional column material, and study the cases (b1) and (b2) for discontinuities of velocity on surfaces (BN) and (B'N'). By the same calculation method detailed in appendix A4 for the mechanism of rotation, the maximum resisting work is determined by using Eqs. (9), (29b) and (29d). Using Eq. (10) and (11), with $\gamma=0$, the following upper bounds estimates are obtained:

case b1:

$$(F^*/cd) \leq \frac{k_c \cot \phi \sin \beta}{\cos \alpha \sin (\alpha - \beta)} + \left[1 + \frac{\tan \delta}{\tan (\alpha - \beta)} + \frac{2}{\sin 2\delta \tan (\alpha - \beta)} \right] \tan \alpha \quad (14a)$$

case b2:

$$(F^*/cd) \leq \frac{k_c \cot \phi \sin \beta}{\cos \alpha \sin (\alpha - \beta)} \left[1 + \frac{\sin (\delta + \alpha - \beta) \sin \alpha}{\cos (\delta + \beta)} \right] + \frac{\cos (\alpha - 2\beta) \tan \alpha}{\sin (\alpha - \beta) \cos (\delta + \beta) \sin \delta} \quad (14b)$$

The minimization of the right handside of ineqs. (14a) and (14b) with respect to β gives $\beta = \phi$; in case b1 the minimum with respect to δ is reached when $\tan \delta = \sqrt{2}/2$. These values lead respectively to the following upper bounds:

case b1:

$$(F^*/cd) \leq \frac{k_c \cos \phi}{\cos \alpha \sin (\alpha - \phi)} + \left[1 + \frac{2.828}{\tan (\alpha - \phi)} \right] \tan \alpha \quad (14c)$$

case b2:

$$(F^*/cd) \leq \frac{k_c \cos \phi}{\cos \alpha \sin (\alpha - \phi)} \left[1 + \frac{\sin (\delta + \alpha - \phi) \sin \alpha}{\cos (\delta + \phi)} \right] + \frac{\cos (\alpha - 2\phi) \tan \alpha}{\sin (\alpha - \phi) \cos (\delta + \phi) \sin \delta} \quad (14d)$$

with conditions:

$$\alpha < \phi < \frac{\pi}{2}, \quad 0 < \delta < \frac{\pi}{2} - \phi.$$

The resulting minimum, for case b1 (resp. case b2), is determined numerically with respect to α (resp. α and δ). Results are given below.

For a purely cohesive column material, by setting: $k_d=1$ in Eq. (13b) and in ineq. (13c), the upper bound of the extreme load is written:

$$(F^*/cd) \leq 2.828 + 2\sqrt{k_c(1+k_c)} \quad (15)$$

For a cohesive-frictional column material, the Prandtl's mechanism related to the undrained behavior case (Fig. 9) yields the following upper bound estimate by applying Eq. (10), (11) and determining the maximum resisting work from Eqs. (29a), (29b), (29c) and (29d):

$$\frac{F^*}{cd} \leq k_c [(\cot \beta + \cot \alpha) \exp (2\alpha \tan \beta) - \cot \beta] + (\pi - 2\theta + \tan \theta) \left(1 + \frac{\tan \beta}{\tan \alpha} \right) \exp (2\alpha \tan \beta) \quad (16)$$

The minimum of the right handside of ineq. (16) is obtained when: $\beta = \phi$, and $\theta = \pi/4$. The minimum with respect to α is looked for numerically. The upper bound given by (16) becomes:

$$\frac{F^*}{cd} \leq k_c [(\cot \phi + \cot \alpha) \exp (2\alpha \tan \phi) - \cot \phi] + \left(1 + \frac{\pi}{2} \right) \left(1 + \frac{\tan \phi}{\tan \alpha} \right) \exp (2\alpha \tan \phi) \quad (16a)$$

For a purely cohesive column material, the upper bound estimate obtained with Prandtl's mechanism is given by:

$$(F^*/cd) \leq (\tan \alpha + \pi - 2\alpha)k_c + \tan \theta + \pi - 2\theta \quad (17)$$

The minimum of the right handside of ineq. (17) occurs when: $\alpha = \theta = \pi/4$; hence the upper bound is given by:

$$(F^*/cd) \leq (1 + (\pi/2))(1 + k_c) \quad (17a)$$

For the undrained behavior case ($k_d = 1$), the approach of Madhav et al. (1978) gives the same results as those obtained from the Prandtl's mechanism for a cohesive-frictional column material.

The Extreme Loads Bounds

The relative difference between the upper and lower bounds is calculated to quantify the error of the value of the extreme load.

The undrained behavior case ($k_d = 1$)

For a purely cohesive column material, by using ineq. (8) and ineq. (15) with a numerical maximization over β_1 , the extreme load is bounded with a relative difference ranging from 6.7% for $k_c = 2$ to 1.7% for $k_c = 10$ (Fig. 12).

For a purely frictional column material, after maximization (or minimization) of the bounds given by ineqs. (7), (14c) and (14d) with respect to the geometrical angles defining the involved fields, the relative difference between the upper and lower bounds is calculated. We can then check that this difference is less than 5.9% (percentage corresponding to $\phi = 42^\circ$, Fig. 13). In addition, the soil reinforcement begins with a value of ϕ satisfying: $16^\circ \leq \phi \leq 17^\circ$.

For a cohesive-frictional column material, the highest relative difference between the two bounds is equal to 4.8% for $\phi = 37^\circ$ (Fig. 14). These results are obtained from ineqs. (7), (14c) and (14d) for $k_c = 0.2$.

Fig. 12. Extreme load vs cohesion ratio k_c ($k_d = 0$)

Fig. 13. Extreme load vs friction angle ϕ ($k_c = 0$)

The three relative differences given in Figs. 12, 13 and 14 are sufficiently small, that the extreme loads in these cases are well approximated by the corresponding lower or upper bounds.

The drained behavior case ($0 < k_d < 1$)

For a purely cohesive column material, by using ineqs. (5) and (13c) the extreme load is bounded with a relative difference decreasing from 15% for $k_c = 2$ to 7% for $k_c = 10$ (Fig. 15).

For a cohesive-frictional column material, by using ineqs. (4) and (12) the extreme load is bounded with a relative difference ranging between 17% and 22% (Fig. 16).

The usefulness of the results presented here consists of the determination of the minimum values of the mechanical characteristics ensuring the soil reinforcement.

Fig. 14. Extreme load vs friction angle ϕ ($k_c \neq 0$)

Fig. 15. Extreme load vs cohesion ratio k_c ($k_d \neq 0$)

Fig. 16. Extreme load vs friction angle ϕ ($k_d \neq 0$)

CALCULATIONS IN THE AXISYMMETRIC ANALYSIS WITHOUT GRAVITY

The soil, supporting a circular foundation of diameter d (Fig. 3), is reinforced with a cylindrical column of diameter d_1 . Cylindrical coordinates (r, θ, z) are adopted to determine the bounds for the extreme load F^* .

The Static Approach

The stress field with five zones (Fig. 4) becomes a field with three zones in the axisymmetric analysis. The lower bounds expressed by (6), (6a), and (6b) remain valid in the axisymmetric case by setting: $\sigma_{rr} = \sigma_{xx}$ and $\sigma_{\theta\theta} = \sigma_{xx}$ (where σ_{xx} is given in Fig. 4) and substituting “ d ” by the section A, “ d_1 ” by the section A_c , and the ratio k_d by k_a . The lower bound estimate is then written from ineq. (4):

$$4[1 + k_a(\sin \phi / (1 - \sin \phi))] + 2k_a[k_c(\cos \phi / (1 - \sin \phi)) - 1] \leq (F^*/cA) \quad (18)$$

For the undrained behavior case ($k_a = 1$), for a purely frictional column material, ineq. (18) becomes:

$$\sigma_c/c = (F^*/cA) \geq 2 \operatorname{tg}^2((\pi/4) + (\phi/2)) \quad (18a)$$

The lower bound: $F^- = 2cA \operatorname{tg}^2(\pi/4 + \phi/2)$ deduced from ineq. (18a) gives the results based on pressometer observations proposed by Hughes et al. (1974), where F^- is expressed, in terms of the limit lateral pressure considered positively in compression: $\sigma_{pl} = \sigma_{rr} = \sigma_{\theta\theta} = 2c$, and the limit vertical stress $\sigma_v = (F^-/A)$, by:

$$F^- = A\sigma_v = A\sigma_{pl} \operatorname{tg}^2(\pi/4 + \phi/2) = 2cA \operatorname{tg}^2(\pi/4 + \phi/2)$$

Studying the same problem with a surface load q , Bell (1915) proposed an equivalent approach to the static one with two zones (Fig. 17), and gave the following value of the vertical stress on the column section:

$$\sigma_c = (q + 2c) \operatorname{tg}^2((\pi/4) + (\phi/2))$$

This value corresponds to the lower bound, given by ineq. (18a), for the case where $q = 0$.

The Kinematic Approach

The kinematic approach is developed by using the mechanism of five blocks (Fig. 8) for the undrained behavior case. Due to the cylindrical form of the column, this mechanism is called with three blocks in the axisym-

metric analysis.

For a purely cohesive column material, the velocity field considered (k.a.) is defined by:

—block (1):

$$\underline{v} = -U\underline{e}_z$$

—block (2):

$$\underline{v} = u\underline{e}_r - \left(u \frac{z}{r} + \frac{U}{2} \right) \underline{e}_z$$

—block (3):

$$\underline{v} = u \left[\underline{e}_r + \left(2 \operatorname{tg} \delta - \frac{d}{2r} (\operatorname{tg} \alpha + \operatorname{tg} \delta) - \frac{z}{r} \right) \underline{e}_z \right]$$

with:

$$u = (U/4 \operatorname{tg} \alpha)$$

By applying (10) and (11) and calculating the function $P(\underline{v})$ from Eqs. (29a) and (29b), the upper bound of the extreme load is written:

$$(F^*/cA) \leq k_c(1 + 1/\sin 2\alpha) + (\operatorname{tg} \alpha + \operatorname{tg} \delta)(2 + 1/\operatorname{tg}^2 \delta) + 2 \operatorname{tg} \alpha / \operatorname{tg} \delta \quad (19)$$

The minimization of the right handside of ineq. (19) with respect to $\alpha > 0$ and $\delta < (\pi/2)$, is done numerically. The bounding of the extreme load is obtained from ineqs. (18) and (19) (Fig. 18).

For a homogeneous soil ($k_c = 1$), Salençon et al. (1982) have determined the exact value of the bearing capacity of a circular shallow foundation, being equal to: $1.2(\pi + 2) = 6.16$. As shown in Fig. 18, the two estimates proposed here bound the exact value: $4 \leq (F^*/cA) = 6.16 \leq 7.31$. This proves that the mechanism of three blocks leads to acceptable bounds for the extreme load in the axisymmetric case.

For a cohesive-frictional column material, the velocity field considered (k.a.) is presented in appendix A5.

—When the discontinuity of velocity occurs in the column, for a purely frictional column material, by using Eq. (10) and (11) and calculating the function $P(\underline{v})$, the upper bound estimate is written:

$$(F^*/cA) \leq 4f_1(\alpha, \phi)g_1(\alpha, \delta, \phi)[(\operatorname{tg} \alpha + \operatorname{tg} \delta)/\sin^2 \delta] + 2 \operatorname{tg} \alpha / \operatorname{tg} \delta \quad (20a)$$

Fig. 17. Bell's approach (1915)

Fig. 18. Extreme load vs cohesion ratio k_c ($k_a = 1$)

The upper bound given by (20a) is valid when the friction angle of column material is verified as $\phi \geq 19.5^\circ$. This condition is not restrictive since reinforcement of soft soils only is considered.

—When the discontinuity of velocity occurs in the soil, for a cohesive-frictional column material, the upper bound estimate is written:

$$\left(\frac{F^*}{cA}\right) \leq (k_c \cot \phi) \left[\frac{4}{3} f_1(\alpha, \phi) \left\{ \frac{1+2K_A}{\cos \alpha} + 2(1-K_A) \right\} - \frac{1}{\cos \alpha} \right] + 4f_1(\alpha, \phi) \left[\operatorname{tg} \delta - 2 \operatorname{tg} \alpha + \frac{1+(2 \operatorname{tg} \alpha / \operatorname{tg} \delta) + (2 \operatorname{tg} \alpha / \operatorname{tg} \delta)^2}{3 \sin \delta \cos \delta} + 4 \frac{\operatorname{tg} \alpha}{\operatorname{tg} \delta} \left(1 + \frac{2 \operatorname{tg} \alpha}{3 \operatorname{tg} \delta} \right) \right] \quad (20b)$$

with:

$$K_A = \operatorname{tg}^2 ((\pi/4) - (\phi/2)); \quad \phi \leq \alpha < \frac{\pi}{2}$$

The expressions of functions $f_1(\alpha, \phi)$ and $g_1(\alpha, \delta, \phi)$ are given in appendix A5. The minima of the right hand sides of ineqs. (20a) and (20b) with respect to α and δ are calculated numerically.

Comparing the two estimates issued from ineqs. (20a) and (20b), for the case of a purely frictional column material ($k_c=0$), the best estimate of the extreme load is given by ineq. (20a) for all values of the friction angle ϕ .

For a purely frictional column material, Brauns (1978) proposed an approach based on the study of a limit equilibrium occurring in the column (passive stresses) and in the soil (Fig. 19). The ultimate stress applied on the column section is written:

$$(\sigma_c/c) = ((q/c) + 2/\sin 2\delta)(1 + (\operatorname{tg} \delta_0 / \operatorname{tg} \delta)) \operatorname{tg}^2 \delta_0 \quad (21)$$

The value of σ_c is determined by an analytical minimization of Eq. (21) with respect to δ ; particularly when $q=0$, the following condition should be fulfilled: $\operatorname{tg} \delta_0 = (\operatorname{tg}^2 \delta - 1)/2 \operatorname{tg} \delta$.

The Brauns's approach does not use a Yield Design reasoning but it combines simultaneously the static and the kinematic approaches.

For a purely frictional column material the mechanism of three blocks, with regard to the Brauns's approach,

Fig. 19. Brauns's approach (1978)

Fig. 20. Bounding of extreme load for purely frictional column material

provides a significant decrease in the upper bound for all values of the friction angle ϕ (Fig. 20). For $\phi=30^\circ$, the ultimate load value proposed by Brauns (1978) is overestimated by 53% with regard to the upper bound estimate obtained by the kinematic approach. For $\phi=45^\circ$, this overestimate reaches 210%.

The improvement of bearing capacity for the reinforced soil is certain for a friction angle satisfying: $\phi \geq 30.7^\circ$, (Fig. 20).

For a cohesionless column material, the cohesion is reduced to the minimum value of ϕ ensuring soil reinforcement as follows: for $k_c=0.2$: $\phi \geq 27.8^\circ$; and for $k_c=0.4$: $\phi \geq 24.9^\circ$.

THE INFLUENCE OF GRAVITY

Gravity is easily taken into account by applying the kinematic approach of the Yield Design theory and using Eq. (10). The mechanism of five blocks (Fig. 8), for the undrained behavior case without gravity, leads to upper bounds which are close to the extreme load value. This mechanism will be considered to study the influence of gravity on the upper bound estimate. Through calculations of the upper bound, it has been verified that ratios k_c and k_g do not affect the evolution of the extreme load. The influence of gravity is related only to the dimensionless ratio $(\gamma d/c)$. For a cohesive-frictional column material, the contribution of gravity consists of adding the following terms:

—in the plane strain analysis: when the discontinuity of velocity occurs in the soil, add to the right hand side of ineq. (14c) the term:

$$(\gamma d/c)[(\operatorname{tg}^2 \alpha)/2 \operatorname{tg}(\alpha - \phi) - (k_g \operatorname{tg} \alpha)/2] \quad \text{for: } \alpha > \phi$$

—in the axisymmetric analysis: when the discontinuity of velocity occurs in the column, add to the right hand side of ineq. (20b) the term:

$$\frac{\gamma d}{c} \left[f_1(\alpha, \phi) g_1(\alpha, \delta, \phi) \left\{ \operatorname{tg} \alpha - 2 \frac{\operatorname{tg}^2 \alpha}{\operatorname{tg} \delta} \right\} - k_g \frac{\operatorname{tg} \alpha}{6} \right]$$

with:

$$0 < \delta < \frac{\pi}{2} - \phi$$

In the plane strain analysis the influence of the ratio $(\gamma d/c)$

Fig. 21. Influence of gravity on upper bound

c is negligible when $\phi < 30^\circ$ (Fig. 21). For material columns with high value of the friction angle (gravel, ballast: $\phi \geq 45^\circ$), however, the extreme load value is increased with regard to a weightless material by more than 30%. In the axisymmetric case, the influence of gravity is negligible for all values of the friction angle.

For a purely cohesive column material, by considering the plane strain analysis or the axisymmetric one, calculation of the extreme load for the drained and the undrained behavior cases show that the influence of gravity is negligible.

REMARK ON THE BEARING CAPACITY OF A COMPOSITE GROUND

Using the results obtained from the kinematic approach, the bearing capacity F_{gr} of a reinforced soil by a group of trenches, as concerned in practical applications, can be bounded based on the above under conditions detailed in the following;

Consider first the example of the mechanism of five blocks (Fig. 8) in the undrained behavior case ($k_d=1$). Under the plane strain assumption, the upper bounds obtained from this mechanism correspond to optimal values of angles α and δ denoted respectively by α_{op} and δ_{op} . For a group of trenches, let d_m be the distance between the trench axes (Fig. 22 for two trenches). It is clear that when the condition:

$$d_m \geq d \left[1 + 2 \frac{\tan \alpha_{op}}{\tan \delta_{op}} \right]$$

Fig. 22. Determination of bearing capacity for a soil reinforced by a group of trenches

is satisfied, the mechanism of five blocks for an isolated trench can be duplicated around each trench of the group without interference between any two adjacent trenches. It follows that an upper bound of the bearing capacity of soil reinforced by a group of " n " trenches is given by:

$$F_{gr}^+ = nF^+$$

where F^+ is the right handside of ineq. (14c) or ineq. (14d) in the case of a cohesive-frictional column material, and the right handside of ineq. (15) in the case of a purely cohesive column material.

CONCLUSION

Making use of the Yield Design theory, the determination of the extreme load for soils reinforced by columns has been studied for the case of an isolated column. Following the stress distributions on the soil and on the columns, two loading cases have been presented. In the plane strain analysis, quasi-exact solutions have been established for the undrained behavior case. Acceptable bounds of the extreme load's value have been established for the drained behavior case. Under the axisymmetric analysis, bounds of the extreme load's value have also been established for the undrained behavior case. The minimum mechanical characteristics of the column material ensuring soil reinforcement have been specified.

For the case of purely cohesive soils and column materials, when considering a surface load surrounding the loaded foundation, all lower bounds (determined by the static approaches) are added by the surface load value.

The influence of gravity has also been studied. This parameter increases considerably the extreme load value in the plane strain analysis when the columns material has a high friction angle. For the approaches developed here, several results announced by Bell (1915), Brauns (1978), Aboshi et al. (1979), have been improved by using the Yield Design theory.

The results presented here have also shown that better performance can be accomplished in terms of the improvement of bearing capacity. This can be validated by experimental investigations. Finally, by using the values of the upper bounds determined based on plane strain assumption, an estimate of the bearing capacity for a reinforced soil by a group of trenches has been proposed.

NOTATIONS

(e_r, e_θ, e_z) are vectors of the local basis for axisymmetric coordinates;
 v is a virtual velocity field;
 $\bar{n}$ is the normal vector to a given discontinuity surface;
 $\bar{\sigma}$ denotes a stress tensor, and $\bar{\sigma}(x)$ its value at point x ;
 $\bar{\sigma}_i$, $1 \leq i \leq 3$, denote the principal stresses;
 $\bar{d}$ is the strain rate tensor; its components are defined as: $d_{ij} = [(\partial v_i / \partial x_j) + (\partial v_j / \partial x_i)] / 2$, $i, j = 1 \text{ à } 3$. D_i , $1 \leq i \leq 3$, denote the principal deformation rates;
 $f(\bar{\sigma}(x))$ is a scalar function defining the strength criterion as follows:

$$\bar{\sigma}(x) \in G(x) \Leftrightarrow f(\bar{\sigma}(x)) \leq 0$$

Ω is a given three-dimensional domain;

Σ is a velocity discontinuity surface.

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APPENDIX A1

In the case of cohesive-frictional column material, the eleven zones stress field corresponds to the Mohr stress representation shown in Fig. 23. Using notations indicated in this figure, the continuity of the stress vector on surfaces (b), (c) and (d) is ensured by the following conditions respectively:

$$\sin 2\delta_1 = \sin 2\beta_1: \text{ then: } \beta_1 = \frac{\pi}{2} - \delta_1 \quad (22)$$

Circle (p) corresponds to zone number p

Fig. 23. Mohr stress representation for the case of cohesive-frictional column material (stress field with eleven zones)

$$r_3 \sin 2(\delta_2 - \beta_2) = c \sin 4\beta_1, \text{ with } \beta_2 = \frac{\pi}{2} - \alpha_2 \quad (23a)$$

$$r_4 \sin 2\beta_2 = r_3 \sin 2\delta_2 \quad (23b)$$

where r_3 and r_4 denotes respectively the radii of circles (3) and (4) which should verify:

$$0 < r_3 < r_4; r_3 = |O'O_3| \sin \phi; r_4 = |O'O_4| \sin \phi \quad (24)$$

We require that states of stress (σ , τ) on surfaces (c) and (d), situated on circles (3) and (4), belong respectively to arcs (I_3T_3) and (I_4T_4). Referring to Fig. 23, this condition leads to:

$$\psi_3 \leq \frac{\pi}{2} - \phi \text{ and } \psi_4 \leq \frac{\pi}{2} - \phi$$

Taking into account the following conditions (Fig. 23):

$$\pi = \psi_3 + 2\alpha_2 + 2\delta_2; \pi = \psi_4 + 2\beta_2; \text{ and } \phi < \frac{\pi}{2}$$

the relationships

$$\delta_2 \geq \phi \quad (25a)$$

$$\beta_2 \geq \phi \quad (25b)$$

are obtained.

By using the Mohr stress representation of Fig. 23, it was confirmed that conditions (22) to (25b) ensure continuity of the stress vector on surfaces (e), (f), (g), (h) and (i).

The expression of the vertical stress in zone (4) is determined from Fig. 23 and taking into account relationships (22), (23a) and (23b), $(\sigma_{zz})_4 = -|OO''|$, is obtained and therefore:

$$(\sigma_{zz})_4 = c \left\{ h(\beta_1) + \frac{\sin 4\beta_1}{\sin 2(\delta_2 - \beta_2)} \left[\cos 2(\delta_2 - \beta_2) + \frac{\sin 2\delta_2}{\sin 2\beta_2} - \frac{\sin 4\beta_1}{\sin 2\beta_2} \right] \right\} \quad (26)$$

where:

$$h(\beta_1) = 1 - 2 \cos 2\beta_1 - \cos 4\beta_1$$

Using Eqs. (22), (23b) and (24) and relationship (26), the lower bound of the extreme load is written:

$$h(\beta_1) + f(\beta_1, \phi) \left[\cos 2\beta - \frac{\sin 2\beta}{\sin 4\beta_2} + \frac{\sin 2(\beta + 2\beta_2)}{\sin 4\beta_2} \right] - \frac{\sin 4\beta_1}{\sin 2\beta_2} \leq \frac{F^*}{cd}$$

where:

$$f(\beta_1, \phi) = \frac{[k_c \cos \phi + h(\beta_1) \sin \phi]}{1 - \cos 2\beta \sin \phi}$$

$$\beta = \text{Arctg} [(1 + \sqrt{\Delta})g(\beta_1, k_c, \phi) / ((1 + \sin \phi) \sin 4\beta_1)]$$

$$\Delta = 1 + [\sin^2 4\beta_1 / g^2(\beta_1, k_c, \phi)] \cos^2 \phi$$

$$g(\beta_1, k_c, \phi) = h(\beta_1) + k_c / \tan \phi$$

APPENDIX A2

For the case of a purely cohesive column material, the eleven zones stress field corresponds to the Mohr stress representation shown in Fig. 24. Continuity of the stress vector required on surfaces (b), (d) and (c) is ensured respectively by Eq. (22) and the following:

$$\sin 2\delta_2 = \sin 2\beta_2 \quad \text{then} \quad \beta_2 = \frac{\pi}{2} - \delta_2 \quad (27a)$$

$$\sin 2(\delta_1 + \alpha_1) = k_c \sin 2(\delta_2 + \alpha_2), \quad (27b)$$

Based on the principle of Mohr circle representation, and using notations of Fig. 6 and Fig. 24, the equation following is obtained: $\alpha_1 = (\pi/2) - \beta_1$ and $\alpha_2 = (\pi/2) - \beta_2$. Then, by Eqs. (22) and (27a), relationship (27b) leads to:

$$\sin 4\beta_1 = k_c \sin 4\beta_2 \quad (27c)$$

By using the Mohr stress representation in Fig. 24, it was confirmed that conditions (22), (27a) and (27b) ensure continuity of the stress vector on surfaces (e), (f), (g), (h) and (i).

The expression of the vertical stress in zone (4) is determined from Fig. 24 and taking into account relationships (22), (27a) and (27b), $(\sigma_{zz})_4 = -|OO''|$, is obtained and therefore:

$$(\sigma_{zz})_4 = c(1 - 2 \cos 2\beta_1 - \cos 4\beta_1) + c_1(1 - 2 \cos 2\beta_2 - \cos 4\beta_2) \quad (28)$$

Eq. (27c) gives the expression $(\tan \beta_2)$ as a function of k_c and the angle β_1 , which is used to express the lower bound of the extreme load in terms of k_c and angle β_1 as follows:

$$h(\beta_1) + k_c [1 + (2/\sqrt{1+Y}) + (Y-1)/(Y+1)] \leq (F^*/cd)$$

where:

$$h(\beta_1) = 1 - 2 \cos 2\beta_1 - \cos 4\beta_1$$

APPENDIX A3

According to Salençon (1983, 1990), the Π functions appearing in Eq. (9) are given as follows:

*For Tresca's material with cohesion c :

Circle (p) corresponds to zone number p

Fig. 24. Mohr stress representation for the case of purely cohesive column material (stress field with eleven zones)

$$\Pi(\underline{d}) = \begin{cases} c(|D_1| + |D_2| + |D_3|); & \text{if: } D_1 + D_2 + D_3 = 0 \\ +\infty; & \text{otherwise} \end{cases} \quad (29a)$$

$$\Pi(\underline{n}; [\underline{v}]) = \begin{cases} c|[\underline{v}]| & \text{if: } [\underline{v}] \cdot \underline{n} = 0 \\ +\infty; & \text{otherwise} \end{cases} \quad (29b)$$

*For Coulomb's material with cohesion c and a friction angle ϕ :

$$\Pi(\underline{d}) = \begin{cases} (D_1 + D_2 + D_3)c \cotg \phi; & \text{if: } (D_1 + D_2 + D_3) \geq (|D_1| + |D_2| + |D_3|) \sin \phi \\ +\infty; & \text{otherwise} \end{cases} \quad (29c)$$

$$\Pi(\underline{n}; [\underline{v}]) = \begin{cases} (c \cotg \phi)[\underline{v}] \cdot \underline{n} & \text{if: } \underline{v} \cdot \underline{n} \geq |[\underline{v}]| \sin \phi \\ +\infty; & \text{otherwise} \end{cases} \quad (29d)$$

In Eqs (29b) and (29d) $[\underline{v}]$ denotes the discontinuity of velocity.

The property that $\Pi(\underline{n}; [\underline{v}])$ is finite implies that the discontinuity of velocity $[\underline{v}]$ is tangent to the discontinuity surface in the case of a Tresca's material; and it is deviated by an angle $\beta \in [\phi; \pi - \phi]$ for a Coulomb's material.

APPENDIX A4

Considering the mechanism with rotation shown in Fig. 10 and using notations of this figure, the external forces work is given by:

$$\underline{F} \cdot \underline{\dot{q}}(v) = F v_z(M) = F \omega \left(R \sin \theta_1 - \frac{d}{2} \right) \quad (30)$$

where $v_z(M)$ is the vertical component of velocity at point M .

To ensure a positive external work in Eq. (30), the condition: $R \sin \theta_1 > (d/2)$, is required. This mechanism, with a rigid bloc motion, induces discontinuities of velocity along the circular arcs (AA_1) and (A_2A_3) in the soil, and along the logarithmic spiral (A_1A_2) . This spiral is characterized by a constant angle $((\pi/2) - \beta)$ between the tangent vector at any point S and the vector OS (Fig. 10).

By using Eqs. (9), (29b) and (29d), the maximum resisting work is written:

$$P(v) = c \omega R^2 \left[(\theta_1 - \theta_2) + \frac{k_c}{2} \{ \exp(2(\theta_2 - \theta_3) \tg \beta) - 1 \} \cotg \phi + (\alpha + \theta_3) \{ \exp 2(\theta_2 - \theta_3) \tg \beta \} \right] \quad (31)$$

Using (11) and Eq. (30) an upper bound of F^* is given by:

$$F^+ = \text{Min} \left[P(v) / \omega \left(R \sin \theta_1 - \frac{d}{2} \right) \right] \quad (32)$$

where the minimum is taken over parameters defining the mechanism: R , β , θ_1 , θ_2 and θ_3 .

Minimization with respect R leads to:

$$R \sin \theta_1 = d \quad (33a)$$

Hence, the following geometrical relationships are satisfied:

$$R \sin \theta_2 = (d + d_1)/2; \quad R_1 \sin \theta_3 = (d - d_1)/2 \quad (33b)$$

$$\cos^2 \alpha = (R/R_1)^2 - ((d + d_1)/R_1) \sin \theta_3 - ((d^2 + d_1^2)/2R_1^2) \quad (33c)$$

using (33a), (33b), (33c), (31) and (32) we get the following upper bound of the extreme load:

$$\begin{aligned} (F^*/cd) \leq & ((1 + k_d)/\sin \theta_2)^2 [((\theta_1 - \theta_2)/2) \\ & + (k_c/4) \{ \exp(2(\theta_2 - \theta_3) \tg \beta) - 1 \} \cotg \phi \\ & + ((\alpha + \theta_3)/2) \exp(2(\theta_2 - \theta_3) \tg \beta)] \end{aligned}$$

APPENDIX A5

In the axisymmetric case, by using the mechanism of three blocks (Fig. 8) when the discontinuity of velocity occurs in the soil, the following (k.a.) velocity field has been considered (calculation will be made using the origin coordinates at point O in Fig. 8):

—in block (1):

$$\underline{v} = -U \underline{e}_z \quad (34a)$$

—in block (2):

$$\underline{v} = \lambda r \underline{e}_r + \lambda' z \underline{e}_z \quad (34b)$$

where λ and λ' are two constant parameters.

To ensure a positive maximum resisting work, using condition (29c), the following condition is required:

$$(2\lambda + \lambda') \geq (2|\lambda| + |\lambda'|) \sin \phi \quad (35a)$$

By considering: $\lambda' < 0$, the condition (35a) is verified for:

$$\lambda' = -2\lambda K_A \quad (35b)$$

with: $K_A = \tg^2((\pi/4) - (\phi/2))$.

To ensure a positive maximum resisting work, the condi-

tion (29d) along surface (BOB') is fulfilled by verifying the inequation:

$$E(\lambda r) \geq 0 \quad (36)$$

where:

$$\begin{aligned} E(\lambda r) &= A(\lambda r)^2 - 2B(\lambda r) + C \\ A &= [1 + 4K_A(1 + K_A \cos^2 \phi)] \operatorname{tg}^2 \alpha - \sin^2 \phi \\ B &= U(1 + 2K_A \cos^2 \phi) \operatorname{tg} \alpha \\ C &= U^2 \cos^2 \phi \\ 0 &\leq r \leq \frac{d}{2} \end{aligned}$$

This calculation is made using (34a) and (34b) to determine discontinuities of velocity $[v]$ and by using the relationship (35b). To the equation $E(\lambda r) = 0$, correspond the roots:

$$(\lambda r)_1 = \frac{B - \sqrt{\Delta}}{A} \quad \text{and} \quad (\lambda r)_2 = \frac{B + \sqrt{\Delta}}{A}$$

with:

$$\Delta = U^2(\operatorname{tg}^2 \alpha + \cos^2 \phi) \sin^2 \phi > 0$$

For $\alpha > \phi$, we have $A > 0$, then $(C/A) > 0$, and these two roots satisfy; $0 < (\lambda r)_1 \leq (\lambda r)_2$. Ineq. (36) is verified when: $0 < (\lambda r) \leq (\lambda r)_1$ or $(\lambda r)_2 \leq (\lambda r)$.

The constant λ is determined by setting: $(\lambda r) = (\lambda d/2)$, then:

$$\lambda = (2U/d \operatorname{tg} \alpha) f_1(\alpha, \phi)$$

where:

$$f_1(\alpha, \phi) = \frac{1 + 2K_A \cos^2 \phi - \sin \phi \sqrt{1 + (\cos^2 \phi / \operatorname{tg}^2 \alpha)}}{1 + 4K_A(1 + K_A \cos^2 \phi) - (\sin^2 \phi / \operatorname{tg}^2 \alpha)}$$

—in block (3):

1) When the discontinuity of velocity occurs in the column, the appropriate velocity field is given by:

$$\underline{v} = b_1 \left[\underline{e}_r + \left(2 \operatorname{tg} \delta - \frac{d}{2r} (\operatorname{tg} \alpha + \operatorname{tg} \delta) - \frac{z}{r} \right) \underline{e}_z \right] \quad (37)$$

where:

$$b_1 = \frac{\lambda d}{2} g_1(\alpha, \delta, \phi) \quad (37a)$$

and the following two conditions deduced from Fig. 8 (case b2) have finite discontinuities of velocity:

$$\alpha > \beta \geq \phi \quad \text{and} \quad 0 < \delta < \frac{\pi}{2} - \phi$$

The function $g_1(\alpha, \delta, \phi)$ in Eq. (37a) is determined from condition (29d) to ensure a positive maximum resisting work on the cylindrical surface (BNB'N'), this condition becomes:

$$\left(b_1 - \frac{\lambda d}{2} \right) \geq \sqrt{\left(b_1 - \frac{\lambda d}{2} \right)^2 + \left\{ b_1 (\operatorname{tg} \delta - \operatorname{tg} \alpha) - (b_1 - \lambda d K_A) \frac{2z}{d} \right\}^2 \sin \phi} \quad (38)$$

with:

$$-\frac{d \operatorname{tg} \alpha}{2} \leq z \leq \frac{d \operatorname{tg} \alpha}{2} \quad (38a)$$

Through Eq. (37a) and ineq. (38), the function $g_1(\alpha, \delta, \phi)$ must confirm that:

$$g_1(\alpha, \delta, \phi) \geq 1 \quad (38b)$$

By writing the condition (38) in the form:

$$C^2 \geq \left\{ A - \left(\frac{2z}{d} \right) B \right\}^2 \quad (39)$$

where:

$$A = (\operatorname{tg} \delta - \operatorname{tg} \alpha) g_1(\alpha, \delta, \phi) \quad (39a)$$

$$B = g_1(\alpha, \delta, \phi) - 2K_A \quad (39b)$$

$$C = [g_1(\alpha, \delta, \phi) - 1] \cotg \phi; C > 0 \quad (39c)$$

When $B > 0$ is considered, then ineq. (39) is satisfied for all z given by (38a), and if we have simultaneously:

$$\frac{A - C}{B} \leq -\operatorname{tg} \alpha \leq \operatorname{tg} \alpha \leq \frac{A + C}{B} \quad (40)$$

Taking account of Eqs. (39a), (39b) and (39c), and choosing the angles α and δ such that:

$$2K_A \operatorname{tg} \alpha \leq \operatorname{tg} \delta < 1/\operatorname{tg} \phi \quad (41a)$$

and

$$\delta > \alpha > \phi \quad (41b)$$

the choice:

$$g_1(\alpha, \delta, \phi) = \frac{1 - 2K_A \operatorname{tg} \alpha \operatorname{tg} \phi}{1 - \operatorname{tg} \delta \operatorname{tg} \phi} \quad (42)$$

ensures condition (38b) and satisfies (40) and consequently (39) for all z given by (38a). From (38b), it is easy to verify that for:

$$0 < K_A < 0.5 \quad (\text{or } \phi > 19.5^\circ), \text{ we have } B > 0 \quad (43)$$

This choice is made in order to calculate the upper bound of the extreme load when discontinuity of velocity occurs in the column material. $\square$

2) When the discontinuity of velocity occurs in the soil, the considered velocity field is written:

$$\underline{v} = \lambda [r \underline{e}_r + (3 \operatorname{tg} \delta - d(\operatorname{tg} \alpha + \operatorname{tg} \delta) - 2z) \underline{e}_z]$$

by verifying:

$$\operatorname{tg} \delta \geq 2 \operatorname{tg} \alpha \quad \square$$

Applying (11), by using Eq. (10) and calculating the maximum resisting work from Eqs. (29a) to (29d), the following upper bounds are obtained:

—When the discontinuity of velocity occurs in the column, for a cohesive-frictional column material, the upper bound estimate is written:

$$\left(\frac{F^*}{cA}\right) \leq (k_c \cotg \phi) \left\{ \left[\frac{4}{3} f_1(\alpha, \phi) \left(\frac{1+2K_A}{\cos \alpha} + 2(1-K_A) \right) - \frac{1}{\cos \alpha} \right] + 4f_1(\alpha, \phi)(g_1(\alpha, \delta, \phi) - 1) \right\} \\ + 4f_1(\alpha, \phi)g_1(\alpha, \delta, \phi)[((\tg \alpha + \tg \delta)/\sin^2 \delta) + 2 \tg \alpha / \tg \delta]$$

—When the discontinuity of velocity occurs in the soil, for a cohesive-frictional column material, the upper bound estimate is written:

$$\left(\frac{F^*}{cA}\right) \leq (k_c \cotg \phi) \left[\frac{4}{3} f_1(\alpha, \phi) \left\{ \frac{1+2K_A}{\cos \alpha} + 2(1-K_A) \right\} - \frac{1}{\cos \alpha} \right] \\ + 4f_1(\alpha, \phi) \left[\tg \delta - 2 \tg \alpha + \frac{1 + (2 \tg \alpha / \tg \delta) + (2 \tg \alpha / \tg \delta)^2}{3 \sin \delta \cos \delta} + 4 \frac{\tg \alpha}{\tg \delta} \left(1 + \frac{2 \tg \alpha}{3 \tg \delta} \right) \right] \quad \square$$