IMPERFECTION-SENSITIVE BIFURCATION OF CAM-CLAY UNDER
PLANE STRAIN COMPRESSION WITH UNDRained BOUNDARIES

AKIRA ASAOKA and TOSHIHiro NODA

ABSTRACT

Imperfection-sensitive bifurcation of a rectangular (12 cm × 36 cm) Cam-Clay specimen emerging from a state of homogeneous in-plane compression with undrained boundaries is simulated by soil-water coupled finite deformation analysis, in which the vertical displacement at a constant rate is imposed on the top of the specimen. The analysis is carried out using a non-linear finite element method through an up-dated Lagrangean scheme, in which the bifurcation points corresponding to the first few bifurcation modes are found to be well approximated by the limit points of external load by introducing cosine-curve geometrical imperfection on both sides of the specimen. This is particularly true when a fast rate of testing is considered.

Even with undrained boundaries, when the specimen includes initial imperfection, the imperfection causes the migration of pore water from the beginning. The effect of migration on the shear deformation, therefore, should be highly dependent on the rate of vertical displacement. When a slow rate of displacement (9.12×10^{-3}%/min) is applied, it is found, in this study, that the mode-switching occurs due to migration and a “peak” in the load-displacement curve occurs which naturally yields a relatively low undrained shear strength. One of the possible reasons for this rate dependency of undrained shear strength of saturated clay is ascribed to the effect of pore water migration on the imperfection-sensitive bifurcation behaviour of saturated clay.

Key words: bifurcation, Cam-Clay, finite element method, numerical simulation, plane strain, undrained condition (IGC: D6)

INTRODUCTION

Figure 1 shows the results of a series of triaxial compression tests under constant cell pressure on the remoulded and normally consolidated Kawasaki clay (Asaoka et al., 1994), in which \( q_f \) denotes the deviator stress at an axial strain of 17% and \( \dot{e}_a \), the strain rate of loading. Two different drainage conditions were considered in this testing: the upper curve shows the test results under a partially drained condition in which the drainage valve was kept open during shear, while the lower s-shaped curve in the figure shows the results for “undrained” condition with the valve closed. The two curves giving the variation of shear strength \( q_f \) in the same range of \( \dot{e}_a \) suggest that the migration of pore water should equally be taken into consideration although it is usual to attribute the rate dependent characteristics of a saturated clay mainly to the inherent nature of the clay like visco-plasticity (e.g., Adachi and Okano, 1974; Sekiguchi, 1977, 1984; Adachi et al., 1987).

It is already known, however, that the migration of pore water alone may be insufficient to illustrate the difference in \( q_f \) between Test A and Test B as shown in the lower curve in Fig. 1. A series of numerical simulations for the undrained triaxial compression tests was also carried out by Asaoka et al. (1994) using the inviscid original Cam-

![Fig. 1. Strength variation with axial strain rate](image)

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Footnotes:

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Clay model (Schofield and Wroth, 1968) with necessary modification for finite strain. Their results are summarized in Figs. 2 and 3. Figure 2 shows the perfectly plastic behaviour observed only in the testing with a very large strain rate in which no migration is expected (Test A). The typical behaviour of a soil element during shear with an extremely slow strain rate (Test B) is shown in Figs. 3(a) and (b). Since almost the full migration of pore water occurs, the hardening with drainage (Fig. 3(a)) and the softening with swelling (Fig. 3(b)) are both observed in a single soil specimen. The notations of the symbols for these figures are given in Table 1, while the boundary conditions for these tests, is shown in Fig. 4. Although Fig. 2 shows a remarkable contrast to Fig. 3, it should be noted that no significant difference was observed in the computed load-displacement curves between Test A and Test B (see Fig. 5), in which the load and the displacement are expressed in terms of apparent deviator stress and apparent axial strain respectively. One of the possible reasons for this unexpected agreement may be ascribed to the fact that the shape of the soil specimen has been asked in computation to follow axial, rotational and furthermore reflectional symmetry from the start to the end of the shear deformation, (see Fig. 4).

The main objective of this present study is to observe the effect of migration of pore water on the undrained shear deformation of saturated clay by means of the numerical simulation using the original Cam-Clay model, but in which every restriction for keeping the aforementioned symmetry is released. Soil-water coupled finite deformation analysis was applied to simulate both the bifurcation and the post-bifurcation behaviour of a rectangular Cam-Clay specimen emerging from a state of homogeneous in-plane compression under undrained boundary conditions. For the sake of simplicity, the plane strain condition was considered.

The analysis was made by means of the non-linear finite element method through an up-dated Lagrangean scheme, in which the bifurcation point was approximated by the so-called “limit (maximum or minimum) point” of external load (Ikeda and Muromata, 1990 (a) and (b); Ikeda and Goto, 1993). A small geometrical imperfection was introduced, therefore, from the beginning of the loading to yield the formation of non-symmetrical shear deformation, the procedure for which is almost the same as that employed in the sample problem presented in

Table 1. Definitions of the technical terms used in the figures

<table>
<thead>
<tr>
<th>Term</th>
<th>Definition</th>
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<tbody>
<tr>
<td>Mean effective stress</td>
<td>$p' = -\text{tr}\ T' / 3$ normalized by $p_0$.</td>
</tr>
<tr>
<td>Deviator stress</td>
<td>$q = 3S'S / 2$ normalized by $p_0$.</td>
</tr>
<tr>
<td>$S = T' + p' I$.</td>
<td></td>
</tr>
<tr>
<td>Excess pore pressure</td>
<td>$u / p_0$, $u$ normalized by $p_0$.</td>
</tr>
<tr>
<td>Specific volume</td>
<td>$v = \nu_0 \det F$</td>
</tr>
<tr>
<td>$\nu_0 = 1 + e_0$, initial specific volume,</td>
<td></td>
</tr>
<tr>
<td>$e_0$, initial void ratio,</td>
<td></td>
</tr>
<tr>
<td>$F$, deformation gradient</td>
<td></td>
</tr>
<tr>
<td>Shear strain</td>
<td>$\varepsilon_s = 2\pi^{1/2} e_0^{-3/2}$</td>
</tr>
<tr>
<td>$e' = e - (\text{tr} e / 3)I$, $e = (I - (FF^T)^{-1}) / 2$.</td>
<td></td>
</tr>
</tbody>
</table>
Fig. 3. Behaviour of soil specimen with $\dot{\varepsilon}_s = 9.1 \times 10^{-3}$/s
(a) Hardening due to drainage, (b) Softening due to swelling

Simo and Meschke (1992). Although the imperfection was installed from the beginning, it will be noted later that the formation of asymmetry itself should be triggered off not from the beginning but at the loading stage near/at the bifurcation point. After this point, the behaviour of the soil-water system became highly dependent on the migration of pore water, the detailed observation of which was an objective of this study.
GOVERNING EQUATIONS AND BOUNDARY CONDITIONS

Governing equations for a soil-water system based on Terzaghi's effective stress concept and Darcy's law are summarized as follows:

**Equation of Equilibrium of Forces**

The incremental constitutive model is considered to be an equation of equilibrium of forces using a rate type, where the body force due to gravity is neglected:

\[
\text{div} \, \dot{\mathbf{S}}_f = 0, \tag{1}
\]

in which the material time derivative is expressed by an upper dot and

\[
\dot{\mathbf{S}}_f = \dot{T} + (\text{tr} \, D)T - TL^T \tag{2}
\]

is the nominal stress rate (Yatomi et al., 1989). In Eq. (2), \( T, L \) denote the Cauchy (total) stress and the velocity gradient of soil skeleton respectively while \( D = (L + L^T)/2 \) is the stretching.

**Effective Stress and Pore Water Pressure**

The total stress \( T \) is divided into two terms:

\[
T = T' - uI \tag{3}
\]

where \( T' \) denotes the effective stress, \( I \), the unit tensor and \( u \), the pore water pressure.

**Linear Rate Type Constitutive Equation for Soil Skeleton**

The stress rate first proposed by Green and Nagdhi (1965)

\[
\dot{T'} = \dot{T'} + T' \Omega - \Omega T', \quad \Omega = \dot{R}R^T \tag{4}
\]

was employed in this study, the characteristics of which can be found elsewhere (e.g., Dienes (1979), Nishimura and Achenbach (1986)). The \( R \) in the above equation denotes the rotation obtained from the polar decomposition of the deformation gradient \( F \). The linear rate-type constitutive relation between \( \dot{T'} \) and \( D \) is expressed as,

\[
\dot{T'} = \mathcal{D}[D]. \tag{5}
\]

**Compatibility Condition**

Constraint conditions imposed upon six components of \( D \) can be derived from

\[
L = \text{grad} \, v \left( \frac{\partial v}{\partial x} \right), \tag{6}
\]

in which \( v \) and \( x \) are the velocity vector and current position vector of the material point \( X \) of the soil skeleton respectively.

**Continuity Condition of the Soil-Water System**

The rate of volume compression of the soil skeleton is set equal to the rate of the water discharged from the soil skeleton:

\[
\left( \frac{1}{V} \int_v dV \right) = - \int_v \text{tr} \, Ddv = - \int_v v' \cdot n \, da, \tag{7}
\]

in which \( v' \) is the discharge velocity of pore water, while \( n \), the unit outward normal vector at the boundary surface of the soil skeleton.

**Darcy's Law**

The discharge velocity \( v' \) following Darcy's field equation is,

\[
v' = -k \frac{\partial h}{\partial x} = -k \frac{\partial}{\partial x} \left( z_0 + \frac{u}{\gamma_w} \right), \tag{8}
\]

in which \( k \) is the coefficient of permeability and assumed here a scalar valued constant, while \( h \), the sum of elevation head \( z_0 \) and pressure head \( u/\gamma_w \).

The boundary conditions considered in this study are of two types. One is the velocity boundary \( \Gamma_v \) and the other "constant cell pressure" boundary \( \Gamma_c \), on which the traction force is constant, i.e.,

\[
t = cn \text{ on } \Gamma_c, \quad c: \text{ const.} \tag{9}
\]
The nominal traction rate is now expressed simply as,
\[ \dot{s} = c' da \cdot c (nda) = c (tr D)T - L d \] on \( \Gamma_v \).  
(10)

Concerning the discharge velocity \( v' \),
\[ v' = -k \frac{\partial h}{\partial x} = 0 \text{ on } \Gamma_v, \]
(11)
in which \( \Gamma_v \) is the discharge velocity boundary, is introduced at all the boundary surfaces in order to simulate undrained testing condition.

THE CAM-CLAY MODEL

Dividing the stretching \( D \) into elastic and plastic components as \( D = D^e + D^p \), the total volume change of soil skeleton becomes,
\[ \int_0^v J tr D^e d\tau = \int_0^v J tr D^e d\tau + \int_0^v J tr D^p d\tau, \]
(12)
where
\[ J = \det F = \frac{1 + e}{1 + e_0}, \]
(13)
in which \( 1 + e \) and \( 1 + e_0 \) are the specific volume at current and reference states respectively. Introducing \( p' \) and \( p_0 \) to be \( -tr T'/3 \) at current (time \( t \)) and reference (time \( t_0 \)) state, respectively and a material parameter \( \tilde{k} \), the first term on the right side of Eq. (12) can be expressed as,
\[ \int_0^v J tr D^e d\tau = -\frac{\tilde{k}}{1 + e_0} \ln \frac{p'}{p_0}, \]
(14)
Then the elastic response
\[ D^e = E^{-1} \dot{T}', \]
(15)
can be found to have the following inverse relation:
\[ \dot{T}' = \left( \tilde{k} - \frac{2}{3} \tilde{G} \right) (tr D^e) I + 2 \tilde{G} D^e, \]
(16)
where
\[ \tilde{k} = \frac{J(1 + e_0)}{\tilde{k}} p' \quad \text{and} \quad \tilde{G} = \frac{3(1 - 2v)}{2(1 + v)} \tilde{k} \]
(17)
are the non-linear bulk modulus and shear modulus respectively.

Following Asaoka et al. (1994), the yield function of the original Cam-Clay is extended to the finite deformation regime as,
\[ F = f(q, p', p_0) + \int_0^v J tr D^p d\tau \]
\[ = 2MD \ln \frac{p'}{p_0} + D \frac{q}{p'} + \int_0^v J tr D^p d\tau = 0, \]
(18)
in which \( M \) and \( D \) are material constants with the relation
\[ D = \frac{\lambda - \tilde{k}}{M(1 + e_0)}, \]
(19)
where \( \lambda \) and \( \tilde{k} \) denote compression index and swelling in-
dex respectively, and
\[ q = \sqrt{3S_S S/S}, \quad S = T' + p'I \]
(20)
is the generalized deviator stress. The associated flow rule gives the plastic response of Cam-Clay:
\[ D^p = \frac{\partial f}{\partial T'}, \]
(21)
where
\[ \lambda = \frac{\partial f}{\partial T'} \cdot \dot{T}', \quad \frac{\partial f}{\partial T'} \cdot \dot{T}' = \frac{\partial f}{\partial T'}, \]
(22)
is the plastic multiplier.

The essentials of the Cam-Clay should come from the fact that the loading condition, \( \lambda > 0 \), is subdivided into two parts:
\[ \frac{\partial f}{\partial T'} \cdot \dot{T'} > 0 \cdots \text{hardening when } q < Mp', \]
\[ \frac{\partial f}{\partial T'} \cdot \dot{T'} < 0 \cdots \text{softening when } q > Mp' \]
(23)

Now, considering the inverse relation of Eq. (15), the \( \lambda \)

is also known to have another representation in terms of stretching:
\[ \Lambda(= \lambda) = \frac{\frac{\partial f}{\partial T'} \cdot ED}{J tr \frac{\partial f}{\partial T'} + \frac{\partial f}{\partial T'} \cdot E \frac{\partial f}{\partial T'}}, \]
(24)
which yields the linear rate-type constitutive relation of Cam-Clay:
\[ \dot{T'} = ED - \Lambda E \frac{\partial f}{\partial T'}, \]
(25)

It should be noted that material parameters of the Cam-Clay are required to make the denominator of Eq. (24) positive. Only in this case, the following intuitive interpretation of the loading criterion (Hashiguchi, 1989, 1993; Zienkiewicz and Taylor, 1991) is applicable to the original Cam-Clay:
\[ \frac{\partial f}{\partial T'} \cdot ED > 0 \cdots \text{loading } (D^p \neq 0), \]
\[ \frac{\partial f}{\partial T'} \cdot ED < 0 \cdots \text{unloading } (D^p = 0) \]
(26)

APPLICATION OF THE FINITE ELEMENT METHOD

Using Eqs. (3), (4) and (6) above, the weak form of Eq. (1) is expressed as
\[ \int_a^b \left( \dot{T} \cdot \delta D + (tr D)T \cdot \delta L - TL \cdot \delta L \right) dv = \int_a^b \vec{u} \cdot (tr D) dv \]
\[ = \int_a^b \dot{s} \cdot \delta v dv - \int_a^b (Omega T' - T' \cdot \omega) \cdot \delta D dv \]
(27)
in which $\delta \rho$ is the virtual velocity satisfying the necessary boundary conditions, while $\delta L$ and $\delta D$ are the consequent virtual velocity gradient and virtual stretching respectively. The first integrand part on the left hand side of Eq. (27) yields the global stiffness matrix $K$ of the soil skeleton in finite element discretization. The coupling equations, Eqs. (7) and (8) are discretized based on the physical model proposed by Christian (1968) and Akai and Tamura (1978), details, see Asaoka et al. (1994). The finite element formulation of these governing equations are then summarized in the following set of simultaneous equations:

$$
\begin{bmatrix}
K & -LT \\
-L & \partial H \\
\end{bmatrix}
\begin{bmatrix}
\Delta u \\
\{ u \}_r + sf \\
\end{bmatrix}
=
\begin{bmatrix}
\Delta f \\
0 \\
\end{bmatrix}
+ \begin{bmatrix}
-LT\{ u \}_r \\
-(1 - \partial)H\{ u \}_r \\
\end{bmatrix}
$$

(28)

where the necessary notations are given in Appendix A.

**IMPERFECTION-SENSITIVE BIFURCATION OF A RECTANGULAR CAM-CLAY SPECIMEN**

The isotropically, normally consolidated plane strain rectangular soil specimen ($B=12$ cm, $H=36$ cm), the material constants of which are tabulated in Table 2 is now considered together with the initial stress conditions. The soil specimen is discretized in finite element computation by $12 \times 36 (= 432)$ iso-parametric plane strain elements and boundary conditions of the problem are given in Fig. 6. As indicated, monotonically increasing vertical displacement $u_v$ is imposed on top of the specimen with the rate that of Test B in Fig. 1, i.e., $v_2/H (=u_v/H)=9.12 \times 10^{-3}$/min.

In Fig. 7 the main path of the perfect system with no imperfection is shown, in which the deviator “stress” denotes the ratio of axial load to initial width $B$ minus cell pressure and the axial “strain” $\varepsilon_a$ merely means the ratio of axial displacement at the top of specimen to its initial height $H$. The stress-strain curve in Fig. 7 therefore, means essentially no more than the load-displacement curve. The element-wise soil behaviour, as illustrated for one element in Fig. 8, is found that all elements follow an undrained path towards the critical state with no migration of pore water even under this very slow “strain” rate. The shape of the specimen also remains rectangular.

Of particular interest, is the deviation from the main path of the perfect system with introduction of an extremely small geometrical imperfection. With a scratch of $1/100$ cm deep, that was introduced at point A in Fig. 6 as an initial imperfection at the beginning, the computation showed the load-displacement curve as in Fig. 9. Initially the curve follows the main path almost exactly and then a deviation occurs at $\varepsilon_a=1.9\%$ and the curve begins to follow a different path. The soil element configuration at $\varepsilon_a=7.6\%$ is shown in Fig. 10(a), while the cosine curve is shown in Fig. 10(b) for comparison,

$$u_i = a \cdot \cos \left( n \pi x_i / h \right),$$

$n = 1$, and $h$: the current height of specimen

(29)

with the same amplitude and period as that of Fig. 10(a). It is clear that the cosine curve simulates the shape of specimen after deformation well and no difference can be seen between Fig. 10(a) and Fig. 10(b) even long after the bifurcation point at $\varepsilon_a=1.9\%$.

The same cosine curve can also be obtained from the

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**Table 2. Cam-Clay parameters**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
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<tbody>
<tr>
<td>$\lambda$</td>
<td>0.108</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>0.025</td>
</tr>
<tr>
<td>$\nu$</td>
<td>0.3</td>
</tr>
<tr>
<td>$M$</td>
<td>1.55</td>
</tr>
<tr>
<td>$\varepsilon_0$</td>
<td>1.83</td>
</tr>
<tr>
<td>$p_c$ (kPa)</td>
<td>294</td>
</tr>
<tr>
<td>$k$ (cm/sec)</td>
<td>$3.7 \times 10^{-8}$</td>
</tr>
</tbody>
</table>
perfect system with no imperfection using the stiffness matrix $K$ at strain level of $\varepsilon_s = 1.9\%$. Since the matrix $K$ is found to have an almost zero-eigenvalue around this strain level, that may be due to the terms underlined in Eq. (27) which account for the changes in geometry, the characteristic equation corresponding to the zero-eigenvalue

$$
\begin{bmatrix}
\Delta u_1 \\
\Delta u_2 \\
\vdots \\
\Delta u_{m-1} \\
1
\end{bmatrix}
\begin{bmatrix}
\Delta u_1 \\
\Delta u_2 \\
\vdots \\
\Delta u_{m-1} \\
1
\end{bmatrix} = 0
$$

yields the characteristic mode in terms of increment of the displacement $\Delta u$, the solution for which is found to be exactly identical to the cosine curve mentioned above. The illustration of the mode with unit amplitude is given in Fig. 11. It should be noted here that the matrix $K'$ in Eq. (30) is the matrix obtained from the original $K$ by eliminating the given displacement boundary. A similar procedure has been tried by Yatomi et al. (1993) to analyze bifurcation points and modes. Although they have employed the “non-coaxial” Cam-Clay constitutive model (Yatomi et al., 1989), there is, of course, no sig-

Fig. 8. Perfectly plastic behaviour of an element in the perfect system

Fig. 9. Behaviour of specimen with a small scratch

Fig. 10. Comparison of the shape of specimen ($\varepsilon_s = 7.6\%$) with cosine curve, (a) specimen, (b) cosine curve
significant difference in the shape of the bifurcation modes.

The intuitive interpretation of both the bifurcation and limit point is given next using the finite element formulation. The first row of Eq. (28) can be rewritten as

\[ K \Delta u = \Delta f' \]  \hspace{1cm} (31)

in which \( \Delta f' \) denotes the virtual load increment that is effective to the mobilized displacement increment and is naturally introduced when a soil-water coupled system is considered (Asaoka and Ohtsuka, 1986), (see Appendix A). The stiffness matrix K is assumed here to be diagonalizable for simplicity. In this case the spectral resolution of Eq. (31) yields the following set of independent linear equations:

\[ \lambda_i \Delta u_i = \Delta f_i' \]

in which \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are the distinct eigenvalues of K. The \( \Delta u_1, \Delta u_2, \ldots, \Delta u_1 \) and \( \Delta f_1, \Delta f_2, \ldots, \Delta f_k \) are the corresponding resolution of \( \Delta u \) and \( \Delta f' \) respectively. When a simple loading condition is considered in a symmetrical system as in this study it naturally follows that some of the components of \( \Delta f' \), say \( \Delta f'_1, \Delta f'_j, \ldots, \Delta f'_k \) be equal to zero from the beginning. Since all the eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_n \) should not be zero (usually positive) at the beginning of loading, some of the equations of Eqs. (32) should have the solution \( \Delta u_1, \Delta u_2, \ldots, \Delta u_k = 0 \) at the beginning. Now, among these \( i, j, \ldots, k, \) assume that the \( i \)-th stiffness \( \lambda_i \) becomes zero as loading proceeds to a certain stage. Then, at this critical point where \( \lambda_i = 0 \), the equation concerned becomes

\[ 0 \Delta u_i = 0 \]  \hspace{1cm} (33)

which may have the solution either \( \Delta u_i = 0 \) or \( \Delta u_i = \beta \Delta I_1 \) (bifurcation of the solution), where \( \Delta I_1 \) is the \( i \)-th unit eigenvector of K, although the magnitude \( \beta \) is still indeterminate.

If some geometrical imperfection exists in this symmetrical system from the beginning, \( \Delta f'_i \) may not be absolutely zero and may in fact, have some small value. Therefore, \( \Delta u_i \) should also have, in this case, some value. Since \( \lambda_i \) approaches zero as loading proceeds, however, the equilibrium equation requires \( \Delta f'_i \) also to become zero at this critical point (stationary point of external load), which is sometimes referred to as the limit point. At this point, the system again holds the condition of Eq. (33), after which the system follows not the main path but the bifurcation path because \( \Delta u_i \neq 0 \), the magnitude of which is naturally dependent on the magnitude of the initial imperfection. When the limit point approximates to the bifurcation point, then the problem is called in this study, the problem of the imperfection-sensitive bifurcation.

With regard to the cosine curve, in the succeeding sections, based on the aforementioned findings on the cosine curve, and also following Hill and Hutchinson (1975), the initial imperfection of the \( n \)-th cosine curve

\[ (u_i)_{t=0} = A \cdot \cos (n\pi X_c/H), \]

\[ 0 < X_c < H \text{ and } n = 1, 2, \ldots \]  \hspace{1cm} (34)

will be introduced on the both sides of the rectangular specimen, which is referred to in this study as the \( n \)-th mode of initial geometrical imperfection. Two times the amplitude \( A \) in Eq. (34) represents the magnitude of the imperfection in this study. In order to observe several bifurcation points distinctively, quite a long and narrow Cam-Clay column is compressed as shown in Appendix B.

\[ \text{RATE-DEPENDENT POST-PeAK BEHAVIOUR OF CAM-CLAY} \]

In some cases the stiffness \( \lambda_i \) changes its sign from positive to negative at the critical point (declining bifurcation path). The post-bifurcation behaviour, then, may be called the post-peak behaviour, which is rather normal in traditional soil mechanics.

Two different "strain" rate tests are examined here for the specimens with several types of initial imperfection. One is the fast test with a rate of displacement, \( \varepsilon_c / \)

![Fig. 12. Strain stress curves of specimen with geometrical imperfection under \( \varepsilon_c = 4.9\% / \text{min} \) (fast test)](image-url)
$H=4.9\%/\text{min}$ (the rate of Test A in Fig. 1) and the other, the slow test with the rate, $9.12 \times 10^{-3}\%/\text{min}$ (Test B). The results are shown in Fig. 12 (fast test) and Fig. 13 (slow test). When the 2nd mode of initial imperfection is introduced to the tests, the difference in the shape of the soil specimen between the fast and the slow rates is as given in Fig. 14. When almost no migration is expected (Fig. 12), only the geometrical imperfection with the 1st mode gives a peak and the other higher order imperfections give an increasing trend in the load-displacement curves after critical points (rising bifurcation path) although they all lie below the main path. Unexpected results come from the slow test. As shown in Figs. 13 and 14(b) all types of initial geometrical imperfection yield an asymmetric shape almost similar to the shape resulted from the 1st mode of imperfection, although the higher mode gives the slightly higher peak than that of the 1st mode (Fig. 13).

When a slow rate of displacement is considered in the testing, since initial geometrical imperfection develops inhomogeneity from the beginning, that should be of course due to the migration of pore water, even the higher mode of imperfection yields the 1st-mode bifurcation at the critical point. This may be firstly because the Fourier cosine series expansion of the inhomogeneity mentioned above can include the first mode, the situation of which is very much similar to the case when a small scratch was considered in the slow test in the former section, which also yielded the 1st-mode bifurcation. The mode-switching of this kind in a slow test is also found in Appendix B, although the switching occurs at a higher mode than three when the specimen is quite tall and narrow. The mode-switching in a slow test is thus found to be one of the possible reasons for the apparent rate dependency of the "undrained shear strength" of saturated clay.

The typical post-peak behaviour of soil specimens at $\varepsilon_s=7.6\%$ in a series of the fast tests is shown in Figs. 15 to Figs. 18. Figures 15(a), (b) and (c) show the spatial distributions of shear strain, specific volume and excess pore pressure respectively of the soil specimen with the 1st mode of initial imperfection. The typical soil element behaviour is shown in Figs. 16(a) and (b). Figure 16(a) shows a typical undrained path, while it is seen in Fig. 16(b) that a small amount of swelling occurs after the stage of unloading with no volume change. The behav-

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**Fig. 13.** Strain-stress curves of specimen with geometrical imperfection under $\varepsilon_r=9.12 \times 10^{-3}\%/\text{min}$ (slow test)

**Fig. 14.** Shape-comparison of specimen with the 2nd mode of imperfection at $\varepsilon_r=7.6\%$, (a) $\varepsilon_r=4.9\%/\text{min}$ (fast test), (b) $\varepsilon_r=9.12\%/\text{min}$ (slow test)

**Fig. 15.** Distribution of (a) shear strain, (b) specific volume and (c) excess pore pressure ($\times 98\text{kPa}$) in the fast with the 1st mode
Fig. 16. Behaviour of soil elements, (a) perfect plasticity, (b) undrained unloading

Fig. 17. Distribution of (a) shear strain, (b) specific volume, (c) excess pore pressure (×98 kPa) in the fast with the 2nd mode
Fig. 18. Behaviour of soil elements, (a) perfect plasticity, (b) undrained unloading

Fig. 19. Distribution of (a) shear strain, (b) specific volume, (c) excess pore pressure ($\times 98$ kPa) in the slow test
Fig. 20. Behaviour of soil elements, (a) hardening due to drainage, (b) softening due to swelling

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Fig. 21. The effect of the magnitude of the initial geometrical imperfection on the stress-strain curve in the fast testing

Fig. 22. The effect of the magnitude of the initial geometrical imperfection on the stress-strain curve in the slow testing

Fig. 23. Imperfection sensitivity on peak load in terms of Koiter's law

Remarks on Mesh-Size Dependency and Mesh-Type Dependency of Load-Instability Calculations

Two types of mesh-size were examined for both fast and slow tests using the specimens with both the 1st and 2nd mode of imperfection, (see Fig. 24). As shown in Figs. 25(a) and (b), there is no significant mesh size dependency on the instability and/or softening calculation. This should be mainly because the volume change of the soil skeleton is governed by Darcy's field equation and then the constitutive relation of the soil skeleton has the so-called non-local characteristics when it is applied to the soil-water coupling problem.

Figure 26, however, shows another 'mesh-size dependency' which appears when describing the cosine curve with a limited number of finite element nodes. This must be stated here because the coarse mesh gives a mode-switching from the 2nd to the 1st mode even in the fast test, (compare Fig. 26(a) with Fig. 12). When the slow test is considered, there is no significant mesh-size dependency (Fig. 26(b)).

When incompressible material is considered, the 4CST (four constant strain triangles) finite element mesh is sometimes used, for which the characteristics can be
found elsewhere (Desai and Abel, 1972; Negategaal et al., 1974; Kikuchi, 1983). In order to see the shear band formation more clearly, it is suggested that the inner node should be transferred from step to step to the intersection point of two diagonals (Negategaal et al., 1974). Although this procedure was omitted in this study to avoid ambiguity in the evaluation of deformation gradient and stress history, the 4CST elements were still attempted. The typical results are shown in Figs. 27, 28, and 29. When the fast test of the specimen with 1st mode of imperfection is attempted (Fig. 27), there are no significant changes when compared with Fig. 15. Even in the fast test, however, when the specimen with 2nd mode of imperfection is examined, the strain localization is more exaggerated in 4CST than in iso-parametric computation (see Figs. 28(c) and (d) and compare them with Fig. 17).

When the slow test with almost full migration of pore water is examined, the strain localization is also more exaggerated in 4CST than in iso-parametric computation even in the specimen with the 1st mode of initial imperfection (compare Figs. 29(c) and (d) with Figs. 19(a) and (b)). It must be pointed out, however, that when a high amount of pore water migration is expected with a significant volume change of the soil skeleton, the use of 4CST mesh does not always give reasonable behaviour in the soil elements, although the strain localization is apparently well recognized.

Fig. 24. Mesh array  
(a) 6 × 18 elements, (b) 12 × 36 elements

Fig. 25. Mesh-size dependency of stress–strain curve of specimen with the 1st mode imperfection, (a) \( \dot{e}_s = 4.9\% / \text{min} \), (b) \( \dot{e}_s = 9.12 \times 10^{-3}\% / \text{min} \)

Fig. 26. Mesh-size dependency, with the 2nd mode imperfection,  
(a) \( \dot{e}_s = 4.9\% / \text{min} \), (b) \( \dot{e}_s = 9.12 \times 10^{-3}\% / \text{min} \)
Fig. 27. Behaviour of specimen with the 1st mode of imperfection under fast test ($\varepsilon_e = 4.9\% / \text{min}$) computed by 4CST element, (a) comparison of stress – strain curve with iso-parametric element, (b) deformed mesh array, distribution of (c) shear strain, (d) specific volume and (e) excess pore pressure, $\varepsilon_e = 7.6\%$

Fig. 28. Behaviour of specimen with the 2nd mode imperfection under fast test ($\varepsilon_e = 4.9\% / \text{min}$) computed by 4CST element, (a) comparison of stress – strain curve with iso-parametric element, (b) deformed mesh array, distribution of (c) shear strain and (d) specific volume, $\varepsilon_e = 7.6\%$

Fig. 29. Behaviour of specimen with the 1st mode imperfection under slow test ($\varepsilon_e = 9.12 \times 10^{-3}\% / \text{min}$) computed by 4CST element, (a) comparison of stress – strain curve with iso-parametric element, (b) deformed mesh array, distribution of (c) shear strain and (d) specific volume, $\varepsilon_e = 7.6\%$
SUMMARY AND CONCLUSIONS

Soil-water coupled finite deformation analysis was applied to simulate imperfection-sensitive bifurcation behaviour of a rectangular (12 cm × 36 cm) Cam-Clay specimen emerging from a homogeneous state of in-plane strain compression under undrained boundary conditions, in which monotonically increasing vertical displacement was imposed on the top of the soil specimen. The analysis was carried out by means of the non-linear finite element method through an up-dated Lagrangean scheme, in which the bifurcation points corresponding to the first few bifurcation modes were found to be well approximated by the limit points of external load by the introduction of cosine-curve geometrical imperfection on both sides of the specimen. This is particularly true when a fast rate of testing is considered.

Even under undrained boundary conditions, however, when the specimen includes initial imperfection, the imperfection initiates the migration of pore water from the beginning. Therefore, the effect of the migration on the shear deformation should be highly dependent on the rate of vertical displacement. When a slow rate of displacement (9.12 × 10^{-3} %/min) was applied to the specimen with the higher mode of imperfection, it was found in this study that the mode-switching occurred due to migration and a "peak" in the load-displacement curve resulted which naturally yielded a relatively low undrained shear strength compared with the strength under a fast rate of displacement.

Although the quantitative findings given in this study can not be extended to a different size specimen and for a different rate of loading, one of the possible reasons for the rate dependency of the undrained shear strength of saturated clay may be ascribed to the effect of pore water migration on the imperfection-sensitive bifurcation behaviour of saturated clay.

REFERENCES

APPENDIX A

In Eqs. (28) and (31), following notations are employed:

$K$: the tangent stiffness matrix of soil skeleton

$L$: the rectangular matrix that transforms the increment of nodal displacements to the volume change of each element

$L^T$: the transpose of $L$ that transforms pore pressures to the seepage force of the element

$H$: the fluid "stiffness" matrix

$\Delta u$: the vector that represents the increment of nodal displacements

$\Delta f$: the vector that represents the increment of applied force which includes the spin terms in the stress rate and the effect of changes in geometry

$\{u\}_t$: the pore pressure vector at time $t$

$\theta$: $0 < \theta < 1$ that gives the implicit finite difference scheme, and

$$\Delta f' = \Delta f + L^T \{ (u)_{t+\Delta t} - (u)_t \}$$  \hspace{1cm} (A-1)

APPENDIX B

The bifurcation behaviour of the rectangular soil specimen of 4 cm wide and 72 cm tall under a slow rate of vertical displacement is examined. Other boundary and loading conditions are the same as those in the main text. Fig. B-1 shows the load-displacement curves in terms of the apparent deviator stress and the apparent axial strain, while Fig. B-2 and B-3 are the typical stress paths of the soil elements. As shown in those figures, the elastic response due to unloading is observed after bifurcation. Furthermore, the mode switching was found to occur at the mode higher than three when the specimen is quite tall and narrow.

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Fig. B-1. Distinct bifurcation points on the main path

Fig. B-2. Sudden unloading due to bifurcation in the early stage of loading and soil specimen at $\varepsilon = 3.8\%$
Fig. B.3. Elastic response after bifurcation