CALIBRATION OF A COMPREHENSIVE HYPOPLASTIC MODEL FOR GRANULAR MATERIALS

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ABSTRACT

Based on the concept of hypoplasticity, a comprehensive constitutive equation describing the three-dimensional nonlinear stress-strain behaviour of granular materials was presented by Gudehus (1996). This companion paper focuses more on experimental and numerical investigations for the calibration of this constitutive model. The representations of the functions chosen are discussed in detail. The consideration of special state paths permits an easy assignment of the constitutive parameters to the experimental data from standard and element tests. Numerical simulations of various element tests indicate that the model is able to reproduce the salient features of granular materials in a wide range of pressures, deformations and densities.

Key words: constitutive equation of soil, critical state, deformation, granular material (IGC: D0/D3/D5/D6/D9/E0)

1 INTRODUCTION

The consideration of granular bodies as a continuum admits the use of the notions density, stress and stretch (or strain, or deformation) to describe the mechanical aspects of changes of states. Herein the concept of so-called simple materials (e.g. Truesdell and Noll, 1965) implies that the material properties are fully revealed under homogeneous deformations. Consequently, it is sufficient to base the experimental and theoretical investigations on stress and stretching paths of homogeneously deformed bodies, i.e. on the so-called element tests of soil mechanics, especially oedometer, triaxial, biaxial and simple shear tests. Based on a careful evaluation of such experiments several general mechanical features of granular materials have already been detected by different authors, e.g. the asymptotical behaviour of proportional deformation paths or stress paths, the reaching of limit states, and the dependence of stiffness and shear strength on pressure level and density. These characteristics of granular materials are of fundamental importance for the development of the present hypoplastic constitutive equation and its calibration.

The basic idea of hypoplasticity was developed by Kolymbas (1985). He formulated the behaviour of an inelastic material by using a single nonlinear tensorial function of the rate-type. Based on the general concept of hypoplasticity various aspects of the mechanical behaviour of granular materials have been investigated in the

past few years, e.g. shear banding (Wu and Sikora, 1991), cyclic loading (Bauer and Wu, 1993), rate-dependence (Wu et al., 1993) and cohesion (Bauer and Wu, 1995). Critical states have been incorporated in order to better describe the influence of pressure level and density on the material behaviour (Wu and Bauer, 1993). It has been demonstrated in these papers that the hypoplastic model can describe the dependence of the material behaviour on the pressure level (called barotropy) and on the density (called pyknotropy) with a single set of material constants.

A constitutive model is of practical use only if it can be adapted to experimental results for a special material. This procedure is called calibration and generally consists of two steps, namely the formulation of approximation functions (called calibration equations) and the determination of their constants. Of course, the calibration equations must be consistent with the constitutive equation for the chosen boundary conditions and the direction of deformation. From a mathematical point of view the adaption of the experimental data to a function is an inverse problem. The existence and uniqueness of a solution are of fundamental importance (Hadamard, 1923). The selection of suitable states and paths required for the calibration is not only of theoretical interest, but the simplicity of handling them is also of great practical importance. In this paper it is shown that a certain factorial representation of the hypoplastic constitutive equation permits the determination of the constitutive constants

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from separate calibration equations. This separability also facilitates the physical interpretation of the constitutive constants, which was difficult in earlier versions. A further advantage of the factorized form is the easy exchange of suitable approximation functions in the general hypoplastic constitutive equation, in particular the functions of the limit surface, of the compression behaviour and of the relations describing the interaction between barotropy and pyknotropy.

For the present hypoplastic constitutive equation, some modifications of earlier versions were made. The constitutive assumptions are summarized and discussed in Section 2 of this paper. The Critical State concept (Schofield and Wroth, 1968) is included in the rate-type description and extended to a wider range of pressures. In contrast to the earlier versions, we can now describe the reaching of critical states for any unlimited and monotonic shearing with the basic hypoplastic equation proposed by the Author (Sec. 3). The surface of critical stress states is now determined by only one constitutive parameter, namely the critical friction angle defined for a triaxial compression test. In contrast to the Mohr-Coulomb surface the proposed hypoplastic critical stress surface is smooth. With a unified description of pyknotropy, barotropy and their interaction the model appears to be capable of reproducing many of the mechanical features of granular materials observed in a wide range of pressures and densities. The calibration outlined here is orientated toward granular materials for which the rate-dependence can be neglected. The proposed calibration procedure (Sec. 4) is based on data from asymptotical states, viz. the minimum and maximum void ratios, the critical void ratio, the shear resistance and the compression behaviour. A numerical simulation of several element tests is outlined in Section 5.

2 QUALITATIVE ASPECTS OF SEPARABILITY

In accordance with the notations used by Gudehus (1996) we start with the following rate-type equation:

$$\dot{T}_s = F(e, T_s, D_s). \quad (1)$$

Herein $\dot{T}_s$ represents the objective stress rate tensor as a function of the current void ratio e , the Cauchy granulate stress tensor T_s and the stretching tensor of the granular skeleton D_s . For a detailed formulation of Eq. (1) it is possible to use the general representation theorem for isotropic tensor-valued functions given by Wang (1970). From this general representation suitable terms can be selected which relate to the essential mechanical features of granular materials (Bauer, 1992). Based on the hypoplasticity concept introduced by Wu and Kolymbas (1990) we consider a subclass of a hypoplastic constitutive equation where the tensor valued function F of Eq. (1) is decomposed into two parts:

$$\dot{T}_s = A(e, T_s, D_s) + B(e, T_s) \|D_s\|. \quad (2)$$

The first part $A(e, T_s, D_s)$ of Eq. (2) is linear in D_s , while

the second part $B(e, T_s) \|D_s\|$ is nonlinear in D_s . $\|D_s\|$ stands for the Euclidian norm $\sqrt{\text{tr } D_s^2}$. Both functions are positively homogeneous of the first degree in D_s . Thus Eq. (2) is rate-independent and incrementally nonlinear as required in hypoplasticity.

In order to make particular representations of Eq. (2) more transparent and its calibration easier, it is helpful to factorize the functions $A(e, T_s, D_s)$ and $B(e, T_s)$ in a suitable way. To visualize the qualitative aspects of a factorization we can use the concept of the so-called response envelopes (Gudehus, 1979). The response envelope for a known stress state T_s and a certain void ratio e is defined as the surface covered by all stress rates $\dot{T}_s$ which correspond to different stretching rates D_s of unit magnitude, i.e. with $\|D_s\| = 1$. As an example we consider the response envelope for an axisymmetric stress state (vector t : $T_{11}, T_{22}=T_{33}$) and coaxial stretchings ($D_{11}, D_{22}=D_{33}$) in the space of the principal stress rates (Fig. 1). The stress rates $\dot{a} = A(e, T_s, D_s)$ derived from the linear part of Eq. (2) form an ellipsoid with the centre at the origin of the space of stress rates. The cross-section of this ellipsoid with the $\dot{T}_{11} - \sqrt{2} \dot{T}_{22}$ -plane is an ellipse (dashed response envelope $\dot{a}$ in Fig. 1). For two particular stretchings $\bar{D}_s$ and $-\bar{D}_s = \bar{\bar{D}}_s$, we obtain two stress rates with the same length but in opposite directions: $\dot{a}(\bar{D}_s) = -\dot{a}(\bar{\bar{D}}_s)$. The stress rates $\dot{b} = B(e, T_s) \|D_s\|$ from the nonlinear part of Eq. (2) have the same values irrespective of the direction of stretching. The sum of the linear and the nonlinear parts, $\dot{t} = \dot{a} + \dot{b}$ (full line in Fig. 1), yields a transla-

Fig. 1. Stress rate response for axially symmetric unit stretchings

Fig. 2. Response envelope for the limit state

tion of the dashed response envelope $\dot{a}$ by the constant value $\dot{b}$. The resulting response envelope is also elliptic and therefore smooth and convex. Particular stretchings $\bar{D}_s$ and $\bar{\bar{D}}_s$ lead to stress rates $\dot{\bar{t}}(\bar{D}_s)$ and $\dot{\bar{\bar{t}}}(\bar{\bar{D}}_s)$ with different lengths and directions: $\dot{\bar{t}}(\bar{D}_s) \neq -\dot{\bar{\bar{t}}}(\bar{\bar{D}}_s)$. Therefore incremental non-linearity can be accounted for with a single equation, so that there is no need to distinguish between loading and unloading explicitly.

Limit states characterized by a vanishing incremental stiffness, i.e. $\dot{T}_s = 0$, are also included in the hypoplastic constitutive equation. For such states the origin of the stress rates is coincident with one point of the response envelope. Thus the responses $\dot{a}$ and $\dot{b}$ are equal in magnitude but opposite in direction, i.e. $\dot{a} = -\dot{b}$ (Fig. 2).

2.1 Pyknotropy

In order to separate more clearly the dependence of the mechanical behaviour of granular materials on their density, the following factorizing of the functions A and B in Eq. (2) was found to be appropriate (Wu and Bauer, 1993):

$$A(e, T_s, D_s) = f_e L(T_s, D_s)$$

and

$$B(e, T_s) = f_e f_d N(T_s). \quad (3)$$

Herein the density factors f_e and f_d were assumed to be dimensionless scalar functions of the relative void ratios (Sec. 3.2). The factor f_e causes a proportional expansion of the response-envelope for $f_e > 1$, which is equivalent to an increase of the stiffness (Fig. 3). The effect of the second factor $f_d \geq 0$ can be visualized by a translation of the response-envelope along the direction of $N(T_s)$ (Fig. 4). Therefore f_d controls the shear strength depending on density and pressure.

For the special case $f_d = 0$ the second part of Eq. (2) vanishes, and the centre of the response-envelope derived from the first part of Eq. (2) is coincident with the origin of the stress rates. Consequently the tensor function $L(T_s, D_s)$ can be called hypoplastic according to the

 Fig. 3. Expansion and translation of the response-envelope by f_e

 Fig. 4. Translation of the response-envelope by f_d

definition by Truesdell (1955).

2.2 Asymptotic Behaviour

Following the experimental observations of sand made by Goldscheider and Gudehus (1973) with a truly triaxial apparatus, we assume the existence of a class of proportional strain (stress) paths where the resulting stress (strain) paths are also proportional. More precisely, this means that for a fixed stretching rate the obtained stress rate ratio is equal to the stress ratio:

$$\frac{\dot{T}_s}{\|\dot{T}_s\|} = \frac{T_s}{\|T_s\|}. \quad (4)$$

This property is restricted to successions of states (e, T_s) which correspond to a particular stretching rate and for which the value of the density factor f_d is constant. The factor f_e is insignificant for the direction of a proportional stress path as follows by inserting Eq. (3) into Eq. (4). To describe such a behaviour for a particular and constant stretching rate it is sufficient that the constitutive equation is homogeneous in T_s :

$$\begin{aligned} L(\lambda T_s, D_s) + f_d N(\lambda T_s) \|D_s\| \\ = \lambda^m [L(T_s, D_s) + f_d N(T_s) \|D_s\|]. \end{aligned} \quad (5)$$

Herein λ is an arbitrary positive scalar, and m denotes the degree of homogeneity. Equation (5) admits a further factorizing of the functions A and B :

$$A(e, T_s, D_s) = (\text{tr } T_s)^m f_e L(\hat{T}_s, D_s)$$

and

$$B(e, T_s) = (\text{tr } T_s)^m f_e f_d N(\hat{T}_s). \quad (6)$$

Herein the dimensionless granular stress ratio tensor is defined as $\hat{T}_s = T_s / \text{tr } T_s$ (Gudehus, 1996). If the functions $L(\hat{T}_s, D_s)$ and $N(\hat{T}_s)$ were homogeneous of the first degree in $\hat{T}_s$, the expansion of the response-envelope would be proportional to the pressure level $\text{tr } T_s$.

As a consequence of the homogeneity in T_s for particular paths, the constitutive equation can also describe the asymptotical behaviour of proportional deformations or stress paths independently of the initial state. This asymptotical behaviour can be interpreted as a gradual sweeping out of the memory of the initial state. Due to the fundamental importance of asymptotical states for developing constitutive models, the so called SOM-states (Swept Out of Memory) were discussed by Gudehus et al. (1977) and experimentally investigated by several other authors, e.g. for sand by Goldscheider (1982) and Bauer (1992), for clay by Topolnicki (1987). It is worth noting that for proportional stretching paths with $f_d \neq \text{const}$ the corresponding stress paths are nonlinear (Sec. 4.1).

2.3 Limit States

As discussed before, the concept of hypoplasticity is also able to describe states which are characterized by a vanishing stress rate (Fig. 2). We will now specify the differences between states of maximum shear resistance (called peak states), and steady flow states (called critical states). From Eqs. (2) and (6) we obtain for vanishing stress rates:

$$L(\hat{T}_s, D_s) + f_d N(\hat{T}_s) \|D_s\| = 0. \quad (7)$$

One consequence of Eq. (7) is the dependence of the maximum shear resistance on the density factor f_d . In accordance with experimental observations of peak states, the relation for f_d should be a function of both the instantaneous density and the mean pressure.

A further restriction of the function f_d is necessary to describe critical states (Schofield and Wroth, 1968), i.e. states which are characterized by a vanishing stress rate and a constant void ratio under continued shear deformation:

$$\dot{T}_{sc} = 0 \quad \text{and} \quad \dot{e}_c = 0. \quad (8)$$

Experimental observations of sand specimens under undrained conditions (Casagrande, 1936; Castro, 1969) and drained conditions (e.g., Lee and Seed, 1967; Bouvard and Stutz, 1986) indicate that the critical void ratio e_c decreases with the pressure level. This means that e_c is not a material constant. It can be assumed that the void ratio in a critical state is directly related to the mean granulate pressure and independent of the deformation history (Sec. 3.3). Consequently, the factor f_d must be a function

of the critical void ratio. If we postulate that for states with $e/e_c = 1$ the value of the density factor is $f_d = 1$, then the constitutive equation for critical states can be written as:

$$L(\hat{T}_{sc}, D_{sc}) + N(\hat{T}_{sc}) \|D_{sc}\| = 0. \quad (9)$$

The set of all possible states which satisfy Eq. (9) uniquely defines a surface in the stress space, which is called critical stress surface.

The assumption that critical states can be reached for any unlimited, monotonic shearing implies a consistency condition for particular representations of the functions $L(\hat{T}_s, D_s)$ and $N(\hat{T}_s)$ as shown in the following. First we rewrite Eq. (9) as:

$$\mathbf{L} : \bar{D}_{sc} + N(\hat{T}_{sc}) = 0. \quad (10)$$

Herein $\mathbf{L}$ is a fourth order tensor which can be obtained using the Euler theorem for homogeneous functions, i.e. $\mathbf{L} = \partial L(\hat{T}_{sc}, D_{sc}) / \partial D_{sc}$. The colon $:$ in Eq. (10) denotes an inner product of two tensors. $\bar{D}_{sc}$ in Eq. (10) denotes the stretching rate ratio corresponding to the vanishing stress rate:

$$\bar{D}_{sc} = \frac{D_{sc}}{\|D_{sc}\|} = -\mathbf{L}^{-1} : \mathbf{N}. \quad (11)$$

We can obviously write:

$$\bar{D}_{sc} : \bar{D}_{sc} = \|\bar{D}_{sc}\|^2 = 1. \quad (12)$$

Inserting Eq. (11) into (12) gives a scalar equation depending only on the stress ratio tensor:

$$(\mathbf{L}^{-1} : \mathbf{N}) : (\mathbf{L}^{-1} : \mathbf{N}) - 1 = 0. \quad (13)$$

From this derivation it follows that Eq. (13) initially fulfills only the general condition for limit states, i.e. $\dot{T}_{sc} = 0$. Consequently Eq. (13) represents the critical stress surface if the second condition of (8), namely $\text{tr } \bar{D}_{sc} = 0$ (i.e. $\dot{e} = 0$), is also satisfied:

$$\mathbf{I} : (\mathbf{L}^{-1} : \mathbf{N}) = 0. \quad (14)$$

Herein $\mathbf{I}$ is the unit tensor, and Eq. (14) is again a scalar function of the stress ratio tensor only. Thus Eqs. (13) and (14) must be satisfied for particular functions $L(\hat{T}_s, D_s)$ and $N(\hat{T}_s)$ if the constitutive equation is also able to describe critical states for any deviatoric direction.

3 REPRESENTATION OF FUNCTIONS

With the barotropy factor $f_b = f_b(\text{tr } T_s)$ and the abbreviation for the stiffness factor $f_s = f_b f_e$ the proposed hypoplastic constitutive equation has the following general form:

$$\dot{T}_s = f_s [L(\hat{T}_s, D_s) + f_d N(\hat{T}_s) \|D_s\|]. \quad (15)$$

Based on the requirements defined in the previous section and on numerical tests special representations have been developed for the functions $L(\hat{T}_s, D_s)$, $N(\hat{T}_s)$, f_s and f_d . It can be expected that a refined reproduction of the observed material behaviour usually requires a higher number of material constants and also a higher number of ex-

perimental data for calibration. In order to make the calibration easier and the constitutive functions more transparent some simplifications are considered below. It is worth noting that our hypoplastic constitutive equation is not confined to specific representations of the included functions. Thus, alternative versions of the functions are conceivable within the general representation of Eq. (15). The factorized form of the constitutive equation enables the exchange of functions so that each function in Eq. (15) can be understood as a module subjected to several requirements.

3.1 Proposed Functions for L and N

For the tensor-valued functions $L(\hat{T}_s, D_s)$ and $N(\hat{T}_s)$ the following representation are used, namely:

$$L(\hat{T}_s, D_s) := a_1^2 D_s + \hat{T}_s \text{tr}(\hat{T}_s D_s), \quad (16)$$

$$N(\hat{T}_s) := a_1 (\hat{T}_s + \hat{T}_s^*). \quad (17)$$

Herein a_1 is a dimensionless scalar factor, and the deviatoric part $\hat{T}_s^*$ of $\hat{T}_s$ is given by:

$$\hat{T}_s^* = \hat{T}_s - \frac{1}{3} \mathbf{1}.$$

Functions (16) and (17) represent modifications of the functions:

$$L = C_1 \text{tr} T_s D_s + C_2 T_s \text{tr}(T_s D_s) / \text{tr} T_s$$

and

$$N = C_3 T_s^2 / \text{tr} T_s + C_4 T_s^{*2} / \text{tr} T_s$$

with

$$T_s^2 = T_s T_s \quad \text{and} \quad T_s^* = T_s - \frac{1}{3} \text{tr} T_s \mathbf{1},$$

which were given by Wu (1992). The stress tensor T_s was initially rewritten using the stress ratio tensor $\hat{T}_s$, so that the functions (16) and (17) are independent of the unit of stress and the magnitude of the pressure level. The pressure level $\text{tr} T_s$ and the common part of constants C_1 , C_2 , C_3 and C_4 are now included in the barotropy factor f_b . The second considerable modification refers to function N . In order to describe critical states the constants in the function N have to be $C_3 = -C_4$. Thus the modification: $C_3 T_s^2 / \text{tr} T_s + C_4 T_s^{*2} / \text{tr} T_s = \text{tr} T_s C_3 (\hat{T}_s^2 - \hat{T}_s^{*2}) = \text{tr} T_s (C_3 / 3) (\hat{T}_s + \hat{T}_s^*) \mathbf{1}$, yields a similar function as proposed in (17).

It can be shown that the requirements for critical states, i.e. Eqs. (13) and (14), are satisfied for the functions (16) and (17). For the critical stress surface the following equation is obtained:

$$\sqrt{\text{tr}(\hat{T}_s^2) - 1/3} - a_1 = 0. \quad (18)$$

It follows from Eq. (18) that the critical stress surface is a cone with its apex at the origin in the principal stress space. In order to explain the factor a_1 in Eq. (18), we consider the cross-section of the conical limit surface with the deviator plane $\text{tr} T_s = 1$ as shown in Fig. 5. Due to the isotropy of Eq. (18) it is sufficient to consider a 60°-section in a deviator plane, since the limit curve has three symmetry axes. Herein a_1 is the radius of the critical

Fig. 5. Critical stress surface in the unit deviator plane

stress points, i.e. it is equal to the magnitude of the deviatoric stress ratio tensor:

$$a_1 = \|\hat{T}_s^*\|. \quad (19)$$

For a constant a_1 the curve in the deviator plane is a circle as with the Drucker-Prager limit condition. For granular materials it is known that the friction angles for triaxial extension and biaxial deformation implied by this condition are too high. To be more realistic a_1 must be a function of the deviatoric stress direction. It is advantageous to introduce the Lode-angle θ for a_1 , which is defined as

$$\cos(3\theta) := -\sqrt{6} \frac{\text{tr}(\hat{T}_s^{*3})}{[\text{tr}(\hat{T}_s^{*2})]^{3/2}}.$$

The dependence of a_1 on the direction of the deviatoric stress can adequately be described by the function:

$$a_1^{-1} = c_1 + c_2 \|\hat{T}_s^*\| [1 + \cos(3\theta)], \quad (20)$$

wherein c_1 and c_2 are dimensionless constants. Their determination is possible by assuming the same critical friction angle for triaxial compression and extension (Sec. 4.2). For special states with $\theta = 60^\circ$ and for isotropic stress states relation (20) is reduced to $a_1 = 1/c_1$. In addition it should be noted that the reproduction of various conical limit surfaces is possible by a suitable adaptation of the function for a_1 .

3.2 Density Factors f_d and f_e

With the density factor f_d the concept of the critical state is included in the present hypoplastic model with $f_d = 1$ for $e = e_c$. The increase of the peak friction angle with density can now be accounted for in the constitutive equation with a density factor $f_d < 1$. A linear decrease of f_d with the relative void ratio $r_e := (e - e_d) / (e_c - e_d)$ was earlier proposed by Wu and Bauer (1993), viz. $f_d = (1 - f_o)r_e + f_o$ as shown by the dashed line in Fig. 6. For $e = e_d$ this relation yields a value $f_d = f_o > 0$. On the other hand, observations of small shear amplitudes in resonant column tests suggest that the constitutive equation for the maximum densification, e_d , should become nearly elastic. As discussed in the previous section, the hypoplastic constitutive equation includes hypoplastic

Fig. 6. Dependence of the density factor f_d on relative void ratio r_e

material behaviour as a special case for $f_d=0$. Therefore it is reasonable to combine the requirement $f_d=1$ for $e=e_c$ and the additional requirement $f_d=0$ for $e=e_d$ in one function. These conditions are satisfied by the function

$$f_d := \left(\frac{e - e_d}{e_c - e_d} \right)^\alpha. \quad (21)$$

Herein α is a dimensionless positive constant: $0 < \alpha < 1$. Relation (21) for the density factor is nonlinear as shown by the solid line in Fig. 6.

It is worth noting that extensions starting from $e=e_d$ can lead to void ratios $e < e_d$ which are not included in the allowed range of function (21). For the sake of simplicity, we set $f_d=0$ for $e < e_d$. Thus f_d is defined for the whole range of void ratios.

For the density factor f_e a function which relates the current void ratio to the critical void ratio is chosen, namely

$$f_e := \left(\frac{e_c}{e} \right)^\beta. \quad (22)$$

Herein the constant β slightly exceeds the value of 1. A decrease of the void ratio means an increase in f_e , and consequently an increase in the incremental stiffness for the same stress state (Fig. 3). Because $e \geq 0$ is satisfied, Eq. (22) can yield $0 \leq f_e \leq \infty$.

3.3 Pressure Dependence of the Void Ratios

There is ample experimental evidence that the maximum void ratio, called e_i , the minimum void ratio, called e_d , and the critical void ratio e_c of a granular material decrease with the mean pressure $p_s = -\text{tr } T_s/3$. Firstly, we will consider the pressure dependence of the maximum void ratio which can be related to a so-called perfectly isotropic compression starting from a suspension. Perfectly means test conditions without gravity and any non-mechanical effects so that the grain skeleton remains isotropic. The void ratio e_i is then considered as an upper bound of e for a given p_s . The evaluation of numerous tests has shown that the relation:

$$e_i = e_{i0} \exp [-(3 p_s / h_s)^n] \quad (23)$$

can approximate the isotropic compression behaviour of

various granular materials for a wide range of pressures. Herein e_{i0} denotes the value of e_i for $p_s=0$. The granular hardness h_s has the dimension of stress, and n is constant. For high pressures the void ratio in Eq. (23) tends toward zero, which can be explained by grain plastification and crushing.

As postulated by Gudehus (1996) the pressure dependence of the critical void ratio e_c and the minimum void ratio e_d is related to the one for e_i :

$$\frac{e_c}{e_i} = \frac{e_{c0}}{e_{i0}}, \quad \frac{e_d}{e_i} = \frac{e_{d0}}{e_{i0}}. \quad (24)$$

The relations (23) and (24) are consistent within a very wide pressure range (with the exception of states $p_s \equiv 0$ and $p_s \rightarrow \infty$, which are characterized by phase transitions of the material). The material constants e_{i0} , e_{c0} and e_{d0} are therefore not related to the real void ratios for zero granular stress states as outlined in Sec. 4.3.

In the Cam-Clay model the critical state curve and the normal compression curve are assumed to be parallel straight lines (Fig. 7(b)). This is in contrast to the proposed relations (23) and (24) as shown in Fig. 7(a). The disadvantage of the conventional description becomes obvious for wider pressure ranges. In addition, it is not objective as the reference pressure, e.g. the atmospheric pressure p_a , has nothing to do with the granular skeleton. As can be seen from experimental results (Fig. 11), in particular, the compression curve of sand can be well approximated for a wide range of pressures by Eq. (23).

Fig. 7. Isotropical and critical void ratios versus pressure: (a) proposed relation; (b) Cam Clay model

3.4 Barotropy Factor f_b

For consistency Eq. (15) must coincide with Eq. (23) for a perfectly isotropic compression. This condition permits the direct determination of the barotropy factor f_b as proposed by Gudehus (1996). Because of $f_e(e_{i0}/e_{c0})^\beta = (e_i/e)^\beta$ the stiffness factor is obtained as:

$$f_s = f_b f_e = \frac{h_s}{n h_i} \left(\frac{e_i}{e} \right)^\beta \frac{1 + e_i \left(\frac{3 p_s}{h_s} \right)^{1-n}}{e_i}, \quad (25)$$

wherein the constant h_i is given by:

$$h_i = \frac{1}{c_1^2} + \frac{1}{3} - \left(\frac{e_{i0} - e_{d0}}{e_{c0} - e_{d0}} \right)^\alpha \frac{1}{c_1 \sqrt{3}}. \quad (26)$$

4 DETERMINATION OF THE MATERIAL CONSTANTS

4.1 Cylindrical Compression and Extension

Some of the parameters, viz. the void ratios e_{i0} , e_{d0} , e_{c0} , the granular hardness h_s and the critical friction angle φ_c can be determined approximately by simple classification tests as outlined by Gudehus (1996). For practical purposes it is also useful to correlate the other constants n , α , β and $\lambda = e_i/e_{c0}$ with the grain size distribution and grain shape. In the following, however, we will discuss the calibration based on element tests and guided by the concept of separability.

States along oedometer and triaxial paths are used to formulate suitable calibration equations. These test paths are characterized by axisymmetric deformation without rotation of the principal axes. Thus the tensors $\hat{T}_s$, T_s and D_s can be represented by their components in a Cartesian coordinate system, i.e. $\hat{T}_{ii}$, T_{ii} and D_{ii} for $i=1, 2, 3$. Due to axial symmetry, we can use the ratios $\mu := D_{22}/D_{11} = D_{33}/D_{11}$ and $K := T_{22}/T_{11} = T_{33}/T_{11}$. Herein μ and K are not material constants. In particular, μ must not be confused with Poisson's ratio. For these conditions two differential equations are obtained from Eq. (15):

$$\dot{T}_{11} = f_s \left\{ a_1^2 D_{11} + \frac{1+2\mu K}{(1+2K)^2} D_{11} + f_d a_1 \left[\frac{2}{(1+2K)} - \frac{1}{3} \right] \sqrt{D_{11}^2 (1+2\mu^2)} \right\} \quad (27)$$

$$\dot{T}_{22} = f_s \left\{ a_1^2 \mu D_{11} + \frac{K(1+2\mu K)}{(1+2K)^2} D_{11} + f_d a_1 \left[\frac{2K}{(1+2K)} - \frac{1}{3} \right] \sqrt{D_{11}^2 (1+2\mu^2)} \right\}. \quad (28)$$

D_{11} can be set equal to -1 for compression and $+1$ for extension because of rate-independence. For compression $K \leq 1$ the factor $a_1 = 1/c_1$.

We will now consider asymptotical states of proportional stress and stretching paths which are useful to obtain calibration equations. For a fixed $D_s = D_{sp}$ (p stands for proportional paths) the corresponding asymptotic stress path is linear if the void ratio e_p fulfills the condition $e_p/e_i = e_{p0}/e_{i0}$ in each state. The value of the density

Fig. 8. Asymptotical behaviour for isotropic densification

factor, f_{dp} is therefore independent of pressure p_s :

$$f_{dp} = \left(\frac{e_{p0} - e_{d0}}{e_{c0} - e_{d0}} \right)^\alpha = \text{const.} \quad (29)$$

For compression $D_{11} < 0$ we can derive from Eqs. (27) and (28):

$$f_{dp} = \frac{3(K_p - \mu_p)}{c_1(1 - K_p) \sqrt{1 + 2\mu_p^2}}. \quad (30)$$

The comparison of Eqs. (29) and (30) yields a relation between e_{p0} , μ_p and K_p for proportional paths valid for $K_p < 1$ and $-0.5 < \mu < 1$:

$$\left(\frac{e_{p0} - e_{d0}}{e_{c0} - e_{d0}} \right)^\alpha = \frac{3(K_p - \mu_p)}{c_1(1 - K_p) \sqrt{1 + 2\mu_p^2}}. \quad (31)$$

For an isotropic densification starting from an isotropic state the stress path is isotropic independent of the initial density. If the initial void ratio is lower, i.e. $e_0 < e_{i0}$, the curve will tend toward the perfect one (Fig. 8). For other fixed stretching rates $\bar{D}_{sp}$ similar relations for proportional paths can be derived from the constitutive Eq. (15) in the same way.

4.2 Friction Constants c_1 and c_2

In order to determine the constants c_1 and c_2 separately it is advantageous to consider critical states. For two particular states the constants can then be determined from Eq. (19). Choosing critical limit states for triaxial compression (superscript ∇) and triaxial extension (superscript Δ) we obtain:

$$\|\hat{T}_s^*\|^\nabla = \frac{1}{c_1} \quad (32)$$

and

$$\|\hat{T}_s^*\|^\Delta [c_1 + 2c_2 \|\hat{T}_s^*\|^\Delta] = 1. \quad (33)$$

Inserting the corresponding stress ratios K^∇ and K^Δ into Eqs. (32) and (33) yields:

$$c_1 = \sqrt{\frac{3}{2}} \left(\frac{1 + 2K^\nabla}{1 - K^\nabla} \right)$$

and

$$c_2 = \frac{3(1 + 2K^\Delta)(2 + K^\nabla + K^\Delta - 4K^\Delta K^\nabla)}{4(1 - K^\nabla)(K^\Delta - 1)^2}.$$

Fig. 9. Section of the critical stress surface for $\varphi_c = 30^\circ$ with a deviator plane

As usual the stress ratios can be expressed by the friction angles, i.e.:

$$K^\nabla = (T_{22}/T_{11})^\nabla = \frac{1 - \sin \varphi^\nabla}{1 + \sin \varphi^\nabla}$$

$$K^\Delta = (T_{22}/T_{11})^\Delta = \frac{1 - \sin \varphi^\Delta}{1 + \sin \varphi^\Delta}.$$

Therefore different friction angles for triaxial compression and extension can be taken into account in the proposed function for a_1 . If the same magnitude for the friction angle is assumed, i.e. $\varphi^\nabla = -\varphi^\Delta = \varphi_c$, then the relations for the constants are:

$$c_1 = \sqrt{\frac{3}{8}} \frac{(3 - \sin \varphi_c)}{\sin \varphi_c}, \quad c_2 = \frac{3(3 + \sin \varphi_c)}{8 \sin \varphi_c}. \quad (34)$$

For $\varphi_c = 30^\circ$ the values

$$c_1 = 3.1 \quad \text{and} \quad c_2 = 2.6$$

are obtained. Figure 9 shows the critical limit curve in the deviator plane derived from Eqs. (19) and (34). The determination of the shape of this limit curve is usually rather subjective since the homogeneity of specimens is limited for large deformations by necking of the specimen in extension tests and due to shear band localizations (Wu and Kolymbas, 1991). The assumptions of stationarity and identical friction angles for triaxial compression and extension are therefore rather pragmatic in order to have an easy calibration, numerical robustness of the constitutive model and a limitation of dilatancy for large shearing.

4.3 Compression Constants h_s and n

As shown in Section 4.1, SOM states satisfy the condition $e_i/e_p = e_{i0}/e_{p0}$. Thus the densification behaviour for such paths is generally characterized by:

$$e_p = e_{p0} \exp [-(3p_s/h_s)^n]. \quad (35)$$

The value of e_{p0} defined in Eq. (35) for $p_s = 0$ cannot be measured with a material exposed to gravity. In addition, a specimen in a nearly stress-free state is influenced by various non-mechanical effects: honeycombs and other structures appear, which cannot be described by T_s and e

Fig. 10. Relation between h_s and n for proportional compression

alone. It is therefore advantageous to consider a state (e_r , $p_r > 0$) as a reference and to represent the experimental data in a plot $x := p_s - p_r$ versus $y := -\ln(e/e_r)$. Then Equation (35) can be written as:

$$y = \left(\frac{3x}{h_s}\right)^n. \quad (36)$$

With the corresponding values $\tan \alpha^* := dy/dx$ for $x = x^*$ and $y = y^*$ in Fig. 10 we obtain:

$$n = \frac{x^*}{y^*} \tan \alpha^* \quad \text{and} \quad h_s = \frac{3x^*}{y^{*(1/n)}}.$$

The theoretical value e_{p0} for zero stress can be calculated by extrapolation:

$$e_{p0} = e_r \exp \{(3p_r/h_s)^n\}. \quad (37)$$

A more refined adaption of h_s and n can be achieved by an optimization including experimental data from a higher pressure range. In this case an oedometer is more practical than an isotropic compression in the triaxial device, which is usually limited to low pressures. The relevant initial void ratio e_{00} for a proportional oedometric compression lies in the range $e_{c0} < e_{00} < e_{i0}$ and can be determined by Eq. (31) with $e_{p0} = e_{00}$ and $K = K_0$, i.e.:

$$e_{00} = e_{d0} + (e_{c0} - e_{d0}) \left[\frac{3K_0}{c_1(1 - K_0)} \right]^{1/\alpha}. \quad (38)$$

The values for e_{d0} , e_{c0} , α and K_0 must initially be estimated for a first approximation. It should be noted that the stress ratio in oedometric compression strongly depends on the initial density and that the formula given by Jaky: $K_0 = 1 - \sin \varphi_c$ yields a slight underestimation of K_0 for an initially very loose granular material (Wu and Bauer, 1993). A more precise value of K_0 for Eq. (38) can be determined experimentally by the so-called soft-oedometer device (Kolymbas and Bauer, 1993).

As an example we consider the calibration of h_s and n with results of oedometric compression tests. The material tested is medium sand, which mainly consists of subrounded quartz grains. Figure 11 shows the compression behaviour for three different initial void ratios in the pressure range from $p_s = -5$ kPa to -50 MPa. It can be observed that the curves of denser specimens turn asymptotically toward the one with the highest void ratio. In a dry state the conventional minimum void ratio is

Fig. 11. Oedometric compression behaviour of medium quartz sand. Experimental results for different initial densities

$e \approx 0.51$, and the conventional maximum void ratio is close to $e \approx 0.8$ whereas the maximum value with a moisture content of $\approx 3\%$ comes up to ≈ 1.2 . An estimation using Eq. (38) yields $e_{00} \approx 1.1$ for the perfect oedometric compression test. The adaption of h_s and n to this test data by Eq. (37) yields:

$$h_s = 190 \text{ MPa} \quad \text{and} \quad n = 0.4.$$

Note that an adaption to the curves with lower initial void ratios would be misleading.

4.4 Limiting Void Ratios

With known values h_s and n , and based on the proposed relations for e_i , e_o , e_c and e_d , it is sufficient to determine experimentally only one state (e, p_s) for each curve. An extrapolation of the theoretical value to zero pressure, i.e. e_{i0} , e_{d0} and e_{c0} is then also possible as shown for e_{p0} in Eq. (37). For medium quartz sand we obtain e.g. $e_{i0} \approx 1.2$, so that $\lambda = e_{i0}/e_{c0} \approx 1.46$.

4.5 Peak States

The calibration of α is based on a relation between a peak state in triaxial compression and the corresponding value of the density factor, f_{dP} . The subscript P denotes peak states. The necessary experimental values are the stress ratio $K_P = (T_{11}/T_{22})_P$ and the corresponding void ratio e_P (Fig. 12). Inserting $D_{11} = -1$ for compression and $\dot{T}_{11} = \dot{T}_{22} = 0$ for the limit state into Eqs. (27) and (28), the following calibration equation for α can be derived:

$$\alpha = \ln(f_{dP}) / \ln\left(\frac{e_P - e_d}{e_c - e_d}\right) \quad (39)$$

with

$$f_{dP} = 3 \frac{1 + c_1^2 + K_P(4 + 4K_P + 2c_1^2\mu_P)}{c_1(5 + 8K_P - 4K_P^2)\sqrt{1 + 2\mu_P^2}}$$

Fig. 12. Triaxial compression test: (a) stress ratio vs. axial strain; (b) void ratio vs. axial strain. Representative values $K_P = T_{11P}/T_{22P}$ and e_P at peak state

and

$$\mu_P = \frac{K_P(2 + c_1^2 + 8K_P) - 1 - c_1^2}{K_P(8 + 2c_1^2 - 4K_P - 2c_1^2K_P) + 5}.$$

A refined identification of the parameter α can be achieved by optimizing the peak states for various pressure levels and densities. Numerical investigations show that the parameter α lies in a narrow range between $0.1 < \alpha < 0.3$, e.g. for medium sand $\alpha = 0.15$. A comparison of the experimental data from triaxial compression tests with numerical simulations shows that the behaviour for loose and dense samples can be well approximated (Fig. 13). The lower limit of φ_P refers to critical states and initial void ratios $e_0 > e_c$. For $e_0 \sim e_d$ the maximum value of the peak friction angle will be reached, which increases

Fig. 13. Hypoplastic prediction and experimental data of peak friction angles for medium quartz sand

Fig. 14. Multi-step triaxial test with medium quartz sand: (a) stress path; (b) stress strain curve

with a decrease in the consolidation pressure.

For an efficient experimental investigation of peak states for various stress levels the multi-step triaxial test can be used. The application of this technique for non-cemented loose sand samples was developed in Karlsruhe (Bauer, 1993). Based on a special freezing technique for the preparation of homogeneous sand specimens, repeated increase in the cell pressure after reaching a peak state was possible up to three times (Fig. 14).

4.6 Density Factor f_e

The increase in the incremental stiffness with a decrease in the void ratio is accounted for by the density factor f_e . In order to estimate the constant β in Eq. (25) we consider an isotropic compression with $e < e_i$ (Fig. 15). For the stress state $T_{11}^* = T_{22}^*$ the corresponding void ratio is e^* and the stiffness is $d(T_{11}^*/p_s)/de^* = -\tan \phi^*$. Inserting these values and $D_{11} = D_{22} = -1$ for compression in Eq. (15), the following relation for β can be derived:

$$\beta = \frac{\ln h_\phi}{\ln(e_i/e_1)} \quad (40)$$

with

$$h_\phi = -h_\phi^* e^* \tan \phi^* \left(\frac{3nh_i}{h_s} \right) \left(-\frac{3T_{11}}{h_s} \right)^{n-1}$$

Fig. 15. Compression tests with different initial densities. Representative values T_{11}^* , e^* and ϕ^*

and

$$h_\phi^* = \frac{1}{c_1^2} + \frac{1}{3} - f_d \frac{\sqrt{3}}{c_1}$$

An alternative relation for β can be derived in a similar way by comparing oedometric compression curves for different initial void ratios.

5 NUMERICAL EXAMPLES

We will now demonstrate the capability of the present hypoplastic model with numerically simulating several element tests under drained and undrained conditions. The constitutive constants are:

$$\begin{aligned} c_1 &= 3.06186 & c_2 &= 2.625 \\ h_s &= 190 \text{ MPa} & n &= 0.4 \\ \alpha &= 0.15 & \beta &= 1.05 \\ e_{c0} &= 0.82 & e_{d0} &= 0.51 \\ \lambda &= 1.46 \end{aligned}$$

First we consider oedometer tests, which are characterized by zero radial deformation, i.e. $D_{22} = D_{33} = 0$. The loading and unloading responses of the present model for specimens with two different densities are represented together with the experimental results (Figs. 16 and 17). The stress paths and stress-strain curves for loose and dense sand are well reproduced. In accordance with the

Fig. 16. Oedometric compression and extension on dense sand ($e_0 = 0.55$): (a) radial stress vs. axial stress; (b) void ratio vs. axial stress. Experimental data by Bauer (1992)

Fig. 17. Oedometric compression and extension on loose sand ($e_0=0.77$): (a) radial stress vs. axial stress; (b) void ratio vs. axial stress. Experimental data by Bauer (1992)

Fig. 18. Triaxial compression under drained conditions for various initial densities and lateral pressures: (a) stress ratio vs. axial strain; (b) void ratio vs. axial strain

experiments the stress ratio $K_0 = T_{22}/T_{11}$ for the first loading is higher ($K_0=0.5$) for the initially loose specimen than for the denser one ($K_0=0.45$). During unloading the axial stress decreases more rapidly than the radial one.

Numerical triaxial compression experiments were next carried out for various lateral pressures and initial densities. All simulations began with hydrostatic states. It can be seen in Fig. 18 that for an initially dense sand the stress ratio at the peak is higher and the volume change is initially contractant and subsequently dilatant. With advanced deformation the void ratio approaches the critical value. For critical states the void ratio is determined by the mean pressure but it does not refer to the initial density. The influence of barotropy on the peak friction angle is more significant for lower pressures.

The dependence of the peak states on the pressure level is demonstrated by simulating triaxial compression and extension tests (Fig. 19). As discussed above the peak state is strongly influenced by the density. Consequently there exists a set of curves of the peak states depending on both the pressure level and the density. Numerical investigations for high pressures indicate that the peak states approach the critical stress surface. A detailed discussion of such results is given by Gudehus (1996).

Numerical triaxial compression tests under undrained conditions are shown in Fig. 20 for different confining pressures with an initial void ratio of $e_0=0.7$. In accord-

Fig. 19. Peak stress states (PSS) and critical stress states (CSS) in the Rendulic-plane

ance with the principle of effective stress, the total stress tensor is assumed to be the sum of the intergranular stress tensor and an isotropic tensor representing the fluid pressure in the voids. With full saturation and incompressibility of the grains and fluid, i.e. $\dot{e}=0$, the change of the intergranular stress is determined by the constitutive equation. The change in the fluid pressure

Fig. 20. Triaxial compression without drainage for various lateral pressures: (a) stress ratio vs. axial strain; (b) pore pressure vs. axial strain

Fig. 21. Simple shearing with full drainage ($e_0=0.6$, $T_{22}=-0.1$ MPa) and a large shear amplitude ($\tan \gamma = \pm 0.1$): (a) shear stress vs. shear angle; (b) void ratio vs. shear angle

can be obtained from the statical boundary condition. The behaviour of the material in the numerical tests is essentially determined by the density factor f_d , i.e. the intergranular pressure level tends to the corresponding critical value, which depends on the given void ratio. In Fig. 20(a), it is indicated that the stress ratio approaches a stationary value which is independent of the initial pressure level.

Cyclic numerical tests for simple shearing with a constant normal stress of $T_{22} = -0.1$ MPa were also performed (Figs. 21 and 22). In all the tests an initial void ratio of $e_0 = 0.6$ was assumed. With a shear angle of $\tan \gamma = \pm 0.1$ the volumetric strain is contractant immediately after shearing reversal and becomes dilatant with advanced deformation. With an increasing number of cycles the total volumetric strain is dilatant and becomes nearly stationary. The maximum magnitude of the stress ratio decreases in each cycle and gradually approaches a limit value. For a small shearing of $\tan \gamma = \pm 0.01$ the material becomes denser, and after several cycles a shake-down is reached. A similar behaviour of sand in simple shear experiments was observed by Wood and Budhu (1980). Figure 23 shows the numerical results of a simple shear test without drainage. The test starts from an isotropic state $T_{11} = T_{22} = T_{33} = -0.15$ MPa and a void ratio of $e = 0.65$. Cyclic shearing leads to the so-called cyclic mobility, which was experimentally observed by several authors, e.g. Tatsuoka (1988).

Fig. 22. Simple shearing with full drainage ($e_0=0.6$, $T_{22}=-0.1$ MPa) and a small shear amplitude ($\tan \gamma = \pm 0.01$): (a) shear stress vs. shear angle; (b) void ratio vs. shear angle

Fig. 23. Simple shearing without drainage ($e=0.65$ and $T_{11}=T_{22}=T_{33}=-0.15$ MPa): (a) shear stress vs. shear angle; (b) effective stress path

CONCLUSIONS

A hypoplastic model has been presented to describe the incrementally non-linear, pressure and density dependent behaviour of granular materials. In order to make the hypoplastic concept more comprehensive a factorized representation of the general form of the constitutive equation has been developed. This form opens a new avenue for the selection of suitable approximation functions with regard to a certain material group. In particular, it allows the exchange of functions describing the form of the critical stress surface, of the compression behaviour and of the relations describing the interaction between barotropy and byknотropy. With a unified description of barotropy and byknотropy the model can be applied for a wider range of pressures and densities using only one set of constitutive constants. The numerical simulation of element tests shows that the model appears to be capable of reproducing the salient features of granular material under both drained and undrained conditions.

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