A RIGID-PLASTIC ANALYSIS FOR GRANULAR MATERIALS

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ABSTRACT

A rigid-plastic analysis of an assembly of regularly arranged circular elements with an equal diameter was made to consider the relationship between the micro- and macro-mechanisms of cohesive granular materials. No elastic constant for elements was employed whereas rolling and plastic sliding at the contact points was considered. The equations of equilibrium of elements at the critical state were solved by using the linear programming method to minimize the upper bound of the applied force. Some of the deformation modes were compared with the results of an experiment where a set of coins are used. It was found that the applied force at critical state can be obtained in a quasi-closed form by considering the condition of equilibrium of the whole assembly. Finally the global cohesion is related with the local one in a simple way when the number of elements are increasing.

Key words: cohesion, granular material, (limit analysis), plasticity (IGC: D6/E1)

INTRODUCTION

It is well-known that there are two typical approaches to model the behavior of granular materials such as sands and powders. One way is to regard an assembly of grains as a continuous body where the theories of elasticity, plasticity and other continuum mechanics are applied directly to establish the governing differential equations. The other way is to observe the motion of each grain subject to the external forces in order to derive the equations of the motion or equilibrium. The former approach, where the appropriate material constants for the continuous body are assumed, does not consider the mechanical properties of each grain. The interpretation of the results is therefore limited within the macroscopic view point. The microscopic behavior of grains is taken into consideration in the latter approach. Both of the above two have been studied for a long period however the development of computer technology has facilitated the use of the microscopic approach extensively these days. For instance, the Distinct Element Method (DEM)(Cundall and Strack, 1979; Hakuno and Meguro, 1992), the Granular Element Method (GEM)(Kishino, 1989) and their revised methods are being used quite often in geotechnical engineering. A drawback of the microscopic approach is the difficulty in determining the material constants necessary for the numerical analysis since the elastic or plastic constants of grains cannot be easily measured through experiments. In DEM and GEM, a kind of spring constant of the inter-particles is required to calcu-

late the repulsive forces as a function of the overlapping distance of two particles. The real sand particle seems to be hard enough to prevent measurable deformation. Even if deformable, the spring coefficient must be deformation-dependent (Nakase et al., 1992).

It is more natural to assume that the deformation of an assembly of sands is mainly attributed to rolling and plastic sliding at each contact point than to the elastic behavior of the grains since the sand particles themselves are considered to be sufficiently hard. In this paper, a rigid-plastic deformation analysis was done from a microscopic view point without any elastic constant. The ultimate value of the applied force and the rates of the displacement and the rotation of all grains at the critical state were actually analyzed by means of the upper bound theorem using the linear programming method. Results are compared with data obtained by a simple experiment. The main purpose of this study was to determine the relationships between the micro- and macro-mechanisms of the deformation of granular materials. It will be possible, as a second step, to extend the present method to the analysis of the nonlinear deformation of the granular materials by considering the successive change of geometrical configuration.

ANALYTICAL PROCEDURES

Rigid-plastic analysis has been used thus far since it is much less complex than elastic or elasto-plastic analysis. This technique was developed originally for frame analy-

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Manuscript was received for review on June 22, 1995.

Written discussions on this paper should be submitted before April 1, 1997 to the Japanese Geotechnical Society, Sugayama Bldg. 4 F, Kanda Awaji-cho 2-23, Chiyoda-ku, Tokyo 101, Japan. Upon request the closing date may be extended one month.

sis in the structural engineering. Recently the rigid-plastic finite element method for the geotechnical structures is used more extensively to calculate the bearing capacity, the shear band and so on (e.g., Tamura et al., 1987). One of the advantages of the rigid-plastic analysis is that only a small number of material constants are required to obtain several quantities in the critical state. More precisely, the material constants required to express the yield condition are just sufficient for rigid-plastic analysis.

Assumptions

The numerical process of the rigid-plastic analysis for granular materials will be explained here in detail since no such an attempt can be found in the literature. For simplicity, the two-dimensional problem is considered. Referring to Fig. 1, the following assumptions are made initially.

- 1) No effect of the gravitational force is considered.
- 2) There are three-degree of freedoms in the motion of each particle; two components of the displacement and the rotation.
- 3) Each particle is rigid and circular with the equal diameter.
- 4) A force P applies in the negative x -direction through a rigid but frictional (cohesive) plate which is aligned in the y -direction.
- 5) A shear force works at each contact point the maximum value of which (shear resistance) is C_p for the inter-particle contact and C_w for the particle-wall contact, respectively. C_p and C_w , independent of the normal forces, are material constants. No sliding occurs if the shear force at the contact point is smaller than the shear resistance.

Frictional forces instead of cohesive forces should be considered for the granular materials. C_p or C_w is, however, used in this present paper as the normal-force independent frictional force to make the calculation easier since the normal forces at contact points, necessary to evaluate the active frictional forces, are in general unknown at the beginning of the analysis. This simplification will be eliminated later in an example.

Rigid-plastic analysis focuses on the critical state of the

whole structure when the applied force P is increased. The rates of displacement and rotation are obtained by solving the equations of equilibrium of particles. A practical and simple way is to assume a deformation mode for all particles. Once a set of motion of all particles is established, the contact points of slide can be identified. At sliding points, the shear forces are determined in magnitude and direction by using the plastic constants. If all equilibrium equations are satisfied with the above shear forces, such a set of motion provides the exact solution. If not, another mode of motion should be assumed until all the equations of equilibrium are satisfied. This method, although straightforward, is time-consuming for a large number of particles because the shear forces change the magnitude and the direction depending upon the deformation mode. To treat these difficulties, the upper bound theorem in plasticity is adopted for the exact motion.

A Simple Example of the Upper Bound Theorem

The upper bound of the applied force to the rigid plastic structure is obtained by equating the external work rate with the internal plastic dissipation. It was clear (Tamura et al., 1984) that the minimization of the upper bound is nothing but solving the equations of equilibrium. To formulate the upper bound, the relative velocity at each sliding contact point has to be measured. It is understood from Fig. 1 that

$$\dot{S}_1 = \frac{y_2 - y_1}{D}(\dot{x}_2 - \dot{x}_1) - \frac{x_2 - x_1}{D}(\dot{y}_2 - \dot{y}_1) + \frac{D}{2}(\dot{\theta}_2 + \dot{\theta}_1) \quad (1)$$

is equal to the relative sliding velocity at the contact point 1 in the clockwise direction where x_i, y_i are the x -, y -coordinates and the anti-clockwise rotation of the i -th particle, respectively and “ $\dot{}$ ” denotes the time derivative or the time rate.

$$\left(\frac{y_2 - y_1}{D}, -\frac{x_2 - x_1}{D} \right) \quad (2)$$

in Eq. (1) means the unit vector in the tangential line oriented clockwise at the contact point. Other sliding velocities at the contact points 2, 3 and 4 are easily determined by

$$\begin{cases} \dot{S}_2 = -\dot{y}_1 + \frac{D}{2}\dot{\theta}_1 \\ \dot{S}_3 = \dot{y}_2 + \frac{D}{2}\dot{\theta}_2 \\ \dot{S}_4 = \dot{x}_1 + \frac{D}{2}\dot{\theta}_1. \end{cases} \quad (3)$$

If two particles in Fig. 1 maintain contact with each other,

$$(x_2 - x_1)(\dot{x}_2 - \dot{x}_1) + (y_2 - y_1)(\dot{y}_2 - \dot{y}_1) = 0 \quad (4)$$

must be satisfied since this means that the relative velocity vector is perpendicular to the line which connects both centers. From Eq. (4) and the boundary conditions:

Fig. 1. Two circular elements

$$\begin{cases} \dot{x}_1 = \dot{y}_1 = 0 \\ \dot{x}_2 = -\dot{\delta} \end{cases}$$

the following relation is obtained.

$$\dot{y}_2 = -\frac{x_2 - x_1}{y_2 - y_1} \dot{x}_2 = \sqrt{3} \dot{\delta} \quad (6)$$

Introducing these equations into Eqs. (1) and (3), the upper bound of the applied force P is obtained as follows:

$$\begin{aligned} P\dot{\delta} &= C_p \dot{S}_1 + C_w (\dot{S}_2 + \dot{S}_3 + \dot{S}_4) \\ &= C_p \left| -2\dot{\delta} + \frac{D}{2}(\dot{\theta}_1 + \dot{\theta}_2) \right| + C_w |D\dot{\theta}_1| \\ &\quad + C_w \left| \sqrt{3}\dot{\delta} + \frac{D}{2}\dot{\theta}_2 \right|. \end{aligned} \quad (7)$$

It is reasonable to assume that

$$\dot{\theta}_1 = \Theta_1 \dot{\delta}, \quad \dot{\theta}_2 = \Theta_2 \dot{\delta} \quad (8)$$

where Θ_1 and Θ_2 are unknown constant. Since $\dot{\delta} > 0$, P is written as

$$P = C_p \left| -2 + \frac{D}{2}(\Theta_1 + \Theta_2) \right| + C_w |D\Theta_1| + C_w \left| \sqrt{3} + \frac{D}{2}\Theta_2 \right|. \quad (9)$$

When P , a function of Θ_1 and Θ_2 in this example, is minimized, two of the three absolute values on the right hand side can be regarded to be 0 otherwise P might get smaller by changing Θ_1 and Θ_2 . Hence there are three possibilities as shown in Fig. 2, at one among which P attains minimum. Each motion in Fig. 2 makes two terms of Eq. (9) vanish by turns. If $C_p < C_w$, the motion of Fig. 2(a) gives minimum of P by erasing the second and the third terms of Eq. (9). Actually the total moment of forces applied to the particle 2 of Fig. 2(b) and the particle 1 of Fig. 2(c) are out of equilibrium while the motion of Fig. 2(a) is never inconsistent with equilibrium and

$$P = (2 + \sqrt{3})C_p \quad (10)$$

is obtained by solving the equation of equilibrium of the particle 2. On the contrary if $C_p > C_w$, the motion of Fig. 2(b) gives the exact solution and the minimum value of P :

$$P = (2 + \sqrt{3})C_w \quad (11)$$

The motion of Fig. 2(c) cannot be the solution even if $C_p > C_w$ since P in this motion indicates a doubled value of that in Fig. 2(b). Finally if $C_p = C_w$, any positive combination of motions of Fig. 2(a) and Fig. 2(b) can be a exact solution. It is understood from this example that the minimization of the upper bound P gives an exact solution for the problem.

Analysis of Many Particles

An example shown in Fig. 3 will be considered where a number of circular elements of an equal size are regularly and most densely packed. The number of arrangements are M (=an even number) in the x -direction and N in the y -direction, respectively. The problem is to determine the value of the applied force P and the mode of the motion of all elements at the critical state under the same bound-

Fig. 2. Deformation modes: (a) sliding at 1, (b) sliding at 3, (c) sliding at 2 and 4

Fig. 3. (M, N) -elements with $\theta = \pi/6$

ary condition as in Fig. 2. The following assumptions are further considered.

6) The top and bottom contact points of each element (except elements on the bottom line of the region) are never subject to any forces when the external force P is applied in the negative x -direction. This means that such contact points will disappear after the deformation begins.

7) On the contrary, all of the other contact points will continue to maintain contact with each other.

These assumptions will be justified by the experiment explained later on. The total freedoms of displacement (without rotation) are $2MN$. Due to the boundary conditions and the contact conditions written as in Eq. (4), $(2N+M/2)$ and $(2N-1)(M-1)$ of freedoms are extracted. Therefore $(M/2-1)$ of freedoms of displacement still remain in addition to the freedom of rotation MN . The applied force P , a function of $(MN+M/2-1)$ variables, has to be minimized in this problem. This will be done effectively by the linear programming method.

EXPERIMENT

A simple experiment was performed using a set of regularly arranged coins to observe the mode of the velocity field of the granular materials. The procedure of the experiment is as follows.

1) As shown in Fig. 4, a set of (Japanese 10 yen) coins were regularly deposited on the vinyl-chloride sheet which is spread in a horizontal plane. Neither effect of the gravitational force nor frictional resistance of the sheet is considered.

2) The displacements of coins along the left and bottom boundaries were fixed in the horizontal and vertical directions, respectively.

3) The top side of the region was free to move.

4) Along the left, the right and the bottom boundaries, adhesive tape was attached to increase the friction effect against the coins. In other words, C_w is taken to be much higher than C_p .

5) The right hand side boundary was slightly displaced to the left by pushing the triangle carefully.

Fig. 4. Experimental setup

6) The mode of motion of the assembly of coins was photographed after pushing the triangle.

No measurement of the applied force P was made since the present experiment aimed to just observe the global motion of coins. Experiments were repeated several times and no widely scattered results could be found if they were based on the same conditions.

RESULTS AND CONSIDERATION

Before obtaining the numerical solutions for a large number of elements, the example of Fig. 2 ($M=2$, $N=1$) was first attempted to check the computational procedure which includes the linear programming method. As a result, the numerical solution was verified to completely coincide with the theoretical one. The normal forces and the shear forces for the non-sliding contact points were calculated after obtaining the solution of rates of displacement and rotation. It is actually not so cumbersome to solve the equations of equilibrium in this model if the mode of the motion is known.

It should be noted that there are some number of forces which cannot be determined by solving the equilibrium equations. For example, the normal forces at contact points 2 and 4 in Fig. 1 are indeterminate.

Comparison of Motions

Figure 5 ($M=10$, $N=9$) illustrates the numerical solution of the motion with $C_w=2C_p$ while Photo. 1 shows the experimental result for the same arrangement as Fig. 5. Figure 5 was made by multiplying the velocity fields of displacement and rotation by a short time period. Pairs of small segments stand for occurrence of sliding. Assumptions 6) and 7) mentioned above are not far from these results. A close similarity in displacement and rotation can be observed between the two results. It is particularly interesting to note in both results the clear slip lines

Fig. 5. Numerical result ($M=10$, $N=9$)

 Photo. 1. Experimental result ($M=10, N=9$)

or shear bands which run diagonally in the domain. Similar slip lines appear in Fig. 6 ($M=6, N=15$) which is obtained by the present numerical analysis under the same condition as before. Slip lines seem to turn 120° regularly whenever they attain the walls. But the experiment for this example is slightly difficult in keeping the alignment of the triangle. Contrasting against the similarity between Fig. 5 and Photo. 1, poor agreement was found be-

 Fig. 6. Numerical result ($M=6, N=15$)

 Fig. 7. Numerical result ($M=10, N=4$)

 Photo. 2. Experimental result ($M=10, N=4$)

tween Fig. 7 ($M=10, N=4$) and Photo. 2. All elements are going to move and rotate in Fig. 7 while a slip line, similar to that in the Coulomb's passive earth pressure theory, is formed as shown in Photo. 2. The reason why there is either good or poor agreement between the analysis and the experiment remains still unknown except for the following supposition. Coins are, as was mentioned before, frictional rather than cohesive in general and therefore some discrepancy between the two results might be observed if the normal forces at the contact points distribute quite unevenly through the domain as is the case in Fig. 7 or Photo. 2.

A Trial Analysis with Frictional Angles

An analysis was made assuming that the critical shear force T at the contact point depends on the normal force N :

$$\begin{cases} T = C_p + N \tan \phi_p & (\text{particle-particle}) \\ T = C_w + N \tan \phi_w & (\text{particle-wall}) \end{cases} \quad (12)$$

where ϕ_p and ϕ_w are, respectively, the inter-particle and the particle-wall frictional angles. A series of iterations were required to obtain the solution which satisfies Eq. (12). In the first iteration, all of the normal forces N were adequately assumed to be fixed. Getting the solution for the first step, each of N is modified for the second solution by using the velocity field. The solution suitable for Eq. (12) will be attained if all of N converge after a number of iterations. Figure 8(a) and (b) ($M=4, N=2$) shows the results of the cohesive case and the frictional case, respectively, where small arrows denote the shear

forces acting at sliding points. In the latter, C_p and C_w are assumed to be almost 0 with $\phi_p = \phi_w = 30^\circ$. A slight difference is found between them. Photograph 3 shows the corresponding results for the experiment. A careful observation of the rotation of coins could conclude the advantage of the friction model for coins. If a number of parti-

cles are large, however, the problem concerning the convergence of the linear programming procedure still remains.

Equilibrium and Critical Load

The following discussion will be again limited for the case without frictional angles.

Table 1 presents the total results of the critical load P with $C_w = 2C_p$ obtained by numerical analysis. It should be noted that there are clear cycles in the increment in each column of Table 1. (Four kinds of increments are remarked in the bottom of Table 1.) For the cases where $M=4$ for instance, the same increment value returns every third row. Figure 8(a) shows the mode of the movement for the case ($M=4, N=2$). The shear forces at sliding points are mobilized up to the cohesive strength C_p . If the equation of the moment of forces of each element is considered, some of the shear forces at non-sliding points are found to be C_p as well. For example, the shear force at the left bottom of element 1 has to be C_p otherwise the element will be accelerated to rotate. From this fact, the normal (axial) forces applied to element 1 are calculated as shown in Fig. 9. A similar consideration on element 2 leads to the normal forces in Fig. 10. It will therefore be understood that there are two kinds of normal forces which are running diagonally as shown in Fig. 11. One (thin line) passes through element 1 and the other

Fig. 8. Numerical result ($M=4, N=2$): (a) cohesive case, (a) frictional case

Photo. 3. Experimental result ($M=4, N=2$)

11			{ 49.294	{ 47.196		
10		{ 48.758	{ 43.196	{ 41.830	{ 41.830	
9		{ 42.660	{ 39.196	{ 37.830	{ 35.732	
8		{ 38.660	{ 35.196	{ 33.830	{ 31.732	
7		{ 33.294	{ 29.830	{ 27.732	{ 27.732	
6	((41.052	{ 27.196	{ 25.830	{ 23.732	{ 23.732	
5	((33.588	{ 23.196	{ 19.732	{ 19.732	{ 19.732	
4	((26.124	{ 17.830	{ 15.732	{ 15.732	{ 14.366	
3	((18.660	{ 11.732	{ 11.732	{ 10.366	{ 10.366	
2	((11.196	{ 7.732	{ 6.366	{ 6.366	{ 6.366	
1	((3.732	{ 2.366	{ 2.366	{ 2.366	{ 2.366	
N	M	2	4	6	8	10

((7.464 [5.366 (4.0 { 6.098 ...increments

Fig. 9. Equilibrium of element 1 in Fig. 8(a)

Fig. 10. Equilibrium of element 2 in Fig. 8(a)

(M=4 N=2)

Fig. 11. Lines of normal force (M=4, N=2)

(thick line) starts from element 2. Actually the system of normal and shear forces shown in Fig. 11 satisfies all of the equations of equilibrium and are consistent with the movement shown in Fig. 8. A similar situation is found in Fig. 12 ($M=4, N=5$) where all elements are in equilibrium with the shear forces C_p and normal forces. It is apparent to see the cyclic appearance of behaviors along y -direction in Fig. 12. For example, the contact points a, b , and c are subject to the same forces as those of contact points a', b' and c' . This explains the reason why the clear cycles appear in the increment in each column of Table 1. The period of cycle in Fig. 12 is 3 since the thick line returns back to the right hand side in every third row. In general, the period of the cycle for the case of M columns can be easily understood to be $(M-1)$

(M=4 N=5)

Fig. 12. Lines of normal force (M=4, N=5)

if Fig. 12 is referred to.

General Solutions

Although the critical values of P in Table 1 were calculated numerically, the cyclic property mentioned above might extend Table 1 to Table 2 which shows $P(M, N)$ for the case (M, N) with the general inclination angle θ (see Fig. 13) and $C_w = 2C_p$ where the following constants are used.

$$\begin{cases} \alpha = \frac{2}{\sin \theta} C_p \\ \beta = \frac{1 + \cos \theta}{\sin \theta} C_p \\ \gamma = \frac{2 + 2 \sin \theta + 3 \cos \theta + \cos^2 \theta + 2 \cos \theta \sin \theta}{\sin \theta (1 + \cos \theta + \sin \theta)} C_p \end{cases} \quad (13)$$

Table 2 is equivalently expressed by the equations:

$$P(2, N) = (2N-1)\beta \quad (14)$$

$$P(4, N) = \begin{cases} 2K(\alpha + \beta) - \beta + \gamma & (L=0) \\ 2K(\alpha + \beta) + \alpha + \beta & (L=1) \\ 2K(\alpha + \beta) + 2\alpha + \beta & (L=2) \end{cases} \quad (15)$$

$$P(M \geq 6, N) = \begin{cases} K\{(M-2)\alpha + 2\beta\} - \beta + \gamma & (L=0) \\ K\{(M-2)\alpha + 2\beta\} + L\alpha - \beta + \gamma & \left(1 \leq L < \frac{M}{2} - 2\right) \\ K\{(M-2)\alpha + 2\beta\} + L\alpha + \beta & \left(\frac{M}{2} - 2 \leq L \leq M-2\right) \end{cases} \quad (16)$$

where K and L are integers which denote, respectively, the quotient and the residue of division $(N-1)$ by $(M-1)$. Not all the values shown in Table 2 have yet been verified by solving the equilibrium equations simply

because such a manipulation is a little bit too complex for cases with many elements. In other words, Table 2 is considered to be a kind of prediction which totally supports the values in Table 1. Actually if $N=1$ or if

Fig. 13. Initial configuration with θ Fig. 14. $c_G(M)$ v.s. M Table 2. Values of critical loads (for general θ)

N	12				8	7		9	3	1	10	3		10	1	1
	11				7	7		8	3	1	9	3		9	1	1
	10				6	5	1	8	3		8	1	1	8	1	1
	9		17		6	5		7	3		7	1	1	8	1	
	8		15		5	5		6	3		6	1	1	7	1	
	7		13		4	3	1	5	1	1	6	1		6	1	
	6		11		4	3		4	1	1	5	1		5	1	
	5		9		3	3		4	1		4	1		4	1	
	4		7		2	1	1	3	1		3	1		3	-1	1
	3		5		2	1		2	1		2	-1	1	2	-1	1
	2		3		1	1		1	-1	1	1	-1	1	1	-1	1
	1		1			-1	1		-1	1		-1	1		-1	1
		α	β	γ	α	β	γ	α	β	γ	α	β	γ	α	β	γ
		M=2			M=4			M=6			M=8			M=10		

$$N = \frac{M}{2} + (M-1) \times I \quad (I: \text{positive integer}), \quad (17)$$

then it is easy to form the equilibrium state as shown in Figs. 11 and 12 which is consistent with the mode of plastic flow. Values for those cases have therefore proved to be quite accurate.

Local and Global Cohesions

It is very interesting to know the relationship between the local cohesion C_p and the global cohesion c_G , which is one of the keys to interpretation of the mechanism of deformation of the granular materials. Table 2 indicates that P values will be approximated by simple formulas

when N is large. The global cohesion $c_G(M)$, a function of M , can be defined as

$$c_G(M) = \lim_{N \rightarrow \infty} \frac{P}{2l} \quad (18)$$

where l is the length of the domain in y -direction. If N is large, l is approximated by

$$l = 2ND \sin \theta. \quad (19)$$

Referring to Table 2 or Eqs. (14)–(16), $c_G(M)$ is obtained as follows:

$$c_G(M) = \begin{cases} \frac{1 + \cos \theta}{2 \sin^2 \theta} \times \frac{C_p}{D} & (M=2) \\ \frac{3 + \cos \theta}{6 \sin^2 \theta} \times \frac{C_p}{D} & (M=4) \\ \frac{M + \cos \theta - 1}{2(M-1) \sin^2 \theta} \times \frac{C_p}{D} & (M \geq 6) \end{cases} \quad (20)$$

Figure 14 illustrates how $c_G(M)$ changes with M for several values of the inclination angle θ . It is understood that $c_G(M)$ converges quickly to the limit value:

$$c_G = \frac{1}{2 \sin^2 \theta} \times \frac{C_p}{D} \quad \left(\frac{\pi}{6} \leq \theta < \frac{\pi}{3} \right) \quad (21)$$

if M becomes large. Equation (21) relates the microscopic cohesion C_p with the macroscopic one c_G .

CONCLUSIONS

The following conclusions were made from this study.

1. The rigid-plastic analysis of regularly arranged circular elements can be made using the local cohesion.
2. Some results of velocity field determined by numerical analysis were confirmed to be similar to those of sim-

ple experiments for both displacement and rotation of elements.

3. Cyclic increments of the critical load were observed when the number of rows was increased.

4. Quasi-analytical values of the critical load were obtained.

5. A simple relationship between the local and the global constants for cohesion was derived.

The real microscopic structures of geotechnical materials such as sands are more complex than those presented in this paper. A detailed investigation is required to consider the effects of static and rolling friction (Sakaguchi and Ozaki, 1993), the random configuration at the initial state, the mixing of elements with different diameters and shapes (Sawada and Pradhan, 1994), in addition to the establishment of the stable and fast numerical calculation scheme combined with the linear programming method.

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