A SEARCH FOR THE CRITICAL SLIP SURFACE IN THREE-DIMENSIONAL SLOPE STABILITY ANALYSIS

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ABSTRACT

The studies dealing with slope stability analyses within the framework of limit equilibrium can be classified into two major categories: the first is formulation of factors of safety i.e. the problem of how to calculate the factor of safety for a given slip surface, and the second refers to minimization of factors of safety i.e. the problem of how to determine the minimum factor of safety and the corresponding critical slip surface. From the latter aspect of the studies, an efficient and practical minimization approach for the 3-D slope stability analysis, based on Dynamic Programming (Baker, 1980) and Random Number Generation, is proposed in this paper. This approach can simultaneously provide the minimum factor of safety, and the corresponding critical slip surface for a general 3-D slope. The 3-D simplified Janbu method, which was developed by Ugai (1988) and Ugai and Hosobori (1988), is incorporated with the proposed approach. The slope stability analysis presented in this paper has two noticeable characteristics: 1) the potential slip surface is not restricted to any particular shape; 2) the approach is available for either inhomogeneous problems or cases in which no symmetrical plane appears.

The developed analysis method was implemented into a computer program called 3D-DYRANUT. Using this program, several example problems, including a multilayered convex slope as well as an actual slope failure caused by an earthquake, were analyzed and the results are reported.

Key words: (dynamic programming), factor of safety, failure, (random number generation), slip surface, slope stability, three-dimensional stability analysis (IGC: E6/E13)

INTRODUCTION

All actual slope failures occur in three-dimensions. In the past twenty years, a number of researchers have contributed to the development and application of 3-D slope stability analysis. Several methods of analysis, which include limit equilibrium analysis, variational calculus and finite element techniques, have been proposed in the literature. Within the framework of limit equilibrium techniques, however, the research achievements concerning 3-D slope stability analysis have almost been concentrated on the formulation of factors of safety, while little attention has been paid to the problem of how to determine the 3-D critical slip surface. This paper presents an efficient and practical minimization approach for 3-D slope stability analysis, based on Dynamic Programming (DP for short in the following) and the method of Random Number Generation. Using this approach, the minimum factor of safety and the corresponding critical slip surface may be determined simultaneously for a general 3-D slope.

Searching for the 3-D critical slip surface has a close relationship with factor of safety equations. These can be classified into two major groups. The first assumes a slip surface to have a particular shape, for example, an extended circular arc (Baligh and Azzouz, 1975), a log spiral surface (Giger and Krizek, 1975), a cylindrical surface (Ugai, 1985), a surface of revolution (Gens et al., 1988), or an ellipsoidal surface (Xing, 1988), and the second is valid for an arbitrary slip surface. When the analysis of slope stability is carried out by the first group of methods, slip surfaces can be expressed analytically and the critical slip surface can be found easily through simple numerical computations. Based on the methods that are valid for slip surfaces of arbitrary shape, however, it is quite difficult to search for the critical slip surface because possible slip surfaces exist infinitely. As far as the authors know, only two papers (Thomaz and Lovell, 1988; Yamagami et al., 1991) considered this.

Thomaz and Lovell (1988) recommended a procedure for 3-D slope stability analysis using the method of random generation of surfaces, and analyzed the effects of

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soil parameters on the potential slip surfaces for several simple slopes. A similar analysis method using random generation of surfaces was also presented by one of the authors (Yamagami et al., 1991). It has been indicated, however, that the critical slip surface determined using such a method is probably not the most critical one since the convergence criterion for solutions with required precision does not exist (Yamagami, 1995).

Leshchinsky et al. (1985, 1986) presented a mathematical approach based on limit equilibrium and variational calculus for 3-D slope stability analysis. Solving the variational limit equilibrium equations makes it possible to obtain the minimum factor of safety and the associated critical slip surface at the same time. Their studies by the variational analysis were however limited to homogeneous problems, and assumptions were made that resulted in the analytical 3-D slip surfaces i.e. extended log-spiral surfaces (Leshchinsky et al., 1985, 1986). In the follow-up studies (Leshchinsky and Huang, 1992), they proposed a generalized approach for the 3-D analysis of general but symmetrical slopes. It seems that this method may be used for a layered soil, although no examples for inhomogeneous problems have been given in their paper. It should be noted, that all the solutions provided by the variational limit equilibrium analysis, up to the present, are only restricted to problems possessing a plane of symmetry. The method based on the variational analysis has not yet been applied to slopes of any geometry as well as comparatively complex distributions of soil horizons, including actual slope failures.

In 1980, Baker successfully introduced DP into slope stability analyses to develop a minimization procedure for 2-D problems, where the Spencer method was utilized. The first author applied Baker's idea to the 2-D Janbu method (Yamagami and Ueta, 1986), and/or extended it to finite element stress fields for determining the overall factors of safety (Yamagami and Ueta, 1988). These studies, with application of DP, indicated its validity for handling a variety of problems in 2-D slope stability analysis. It is the purpose of this paper to extend this efficient method into 3-D analysis. This extension employs an existing 3-D method to calculate the factor of safety for a given slip surface and combines it with DP and the method of Random Number Generation to construct a minimization procedure. In this procedure, both the critical slip surface and the minimum factor of safety are provided at the same time. No restrictions are made with respect to the shape of the potential slip surfaces. The proposed method is available for either inhomogeneous problems or cases in which slope failures occur unsymmetrically.

It should be pointed out that the method presented in this paper is based on the factor of safety equations which are valid for an arbitrary slip surface. In the past two decades, several such methods have been proposed in the literature through the limit equilibrium analysis of columns. Three-dimensional slope stability analysis using the method of columns was first made by Hovland (1977). Chen and Chameau (1983) derived a 3-D stability

analysis method for a general but symmetrical slip surface on the basis of the 2-D Spencer method. Ugai (1988) and Ugai and Hosobori (1988) extended the simplified Janbu method and Spencer method in 2-D problems to three-dimensions. A generalized limit equilibrium model for 3-D analysis was developed by Lam and Fredlund (1993). Their 3-D model can be considered partly as an extension of the 2-D Morgenstern and Price method (1965).

Among the aforementioned methods a both simple and comparatively rigorous one is the 3-D simplified Janbu method proposed by Ugai et al. This method possesses a definite advantage in that when the slope under consideration is gentle (less than 45°), a factor of safety can be determined by only one formula rather than by simultaneous equations in general cases, and therefore iterative procedures will be significantly simplified. In this paper, the 3-D simplified Janbu method proposed by Ugai et al. is selected to couple with the minimization approach based on DP and the method of Random Number Generation, for convenience.

OUTLINE OF APPLICATION OF DP TO SLOPE STABILITY ANALYSES

The DP algorithm has been described in detail (e.g. Larson and Casti, 1978) and used for 2-D slope stability analysis (Baker, 1980; Yamagami and Ueta, 1986). One of the authors (Yamagami and Ueta, 1988) introduced DP to finite element stress fields for determining the overall minimum factor of safety. Referring to these papers, an outline of application of DP to locating the 2-D critical slip surface is presented in order to help understand the extension of DP to the 3-D analysis.

DP is a numerical algorithm in mathematical programming, developed by Richard Bellman in the early 1950s. It has been used to solve sequential multistage optimization (maximum or minimum value) problems. Such problems are characterized by two properties (Baker, 1980):

1. At any stage (time) the system may exist in any one of a finite number of states, so that its evolution in stages describes a trajectory in state-stage plane as shown in Fig. 1;
2. It is required to find a minimum (or maximum)

Fig. 1. The description of DP problems

value G_s of some quantity G (called the 'total return function' in DP) which depends on the trajectory and is defined as a sum of returns obtained on passing between two adjacent stages:

$$G_s = \min \sum_{i=1}^m DG_i(j, k) \quad (1)$$

where $DG_i(j, k)$ is called the 'return function' in DP and signifies the change in G on passing from the state j at the stage i to the state k at the stage $i+1$.

The central concept of DP is the principle of optimality that was first stated formally by Bellman as follows: "an optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision." It is rather difficult to understand the implication of Bellman's principle of optimality in the literal sense of the statement. With reference to Fig. 2(a), however, the essence of the principle can be easily understood. It can be assumed as shown in Fig. 2(a) that the path $ABCD$ is the optimal trajectory from point A to D , then, according to the principle of optimality, the path ABC must be the optimal trajectory from point A to C (C is an arbitrary point in the path AD). Application of this principle in slope stability analysis may be illustrated by using Fig. 2(b), in which the arc $ABCD$ is assumed to be the critical slip surface. The arc ABC rather than AEC therefore forms part of the critical slip surface.

The major aspect of DP is to construct a recurrence relation based on the principle of optimality. By solving the recurrence relation the optimal solution i.e. the minimum value G_s of G and the corresponding optimal trajectory can be obtained. The construction of the recurrence relation with regard to slope stability analyses is discussed in this section.

In order to illustrate how to introduce DP into slope stability analyses, Fig. 3 shows a 2-D geometry in stage-state plane. The vertical lines, dividing the profile into

Fig. 2. (a) Illustration of the principle of optimality; (b) The principle of optimality in slope stability analyses

Fig. 3. Stage and state in a 2-D slope

many vertical slices, signify 'stages', and the points at each stage represent 'states' as called in Fig. 1. A possible slip surface can be described by a trajectory AB in the stage-state plane, which is obtained by connecting in sequence a point (i.e. one state) at every stage.

Assuming that the two quantities R_i and T_i shown in Fig. 3 correspond to the driving and resisting forces on the base jk of a slice, respectively, the factor of safety for the 2-D analysis within the framework of limit equilibrium can be generalized as follows:

$$F = \frac{\sum_{i=1}^m R_i(j, k)}{\sum_{i=1}^m T_i(j, k)} \quad (2)$$

It is clear that DP is applicable only to an 'additive function' (Baker, 1980). The safety factor F in Eq. (2) is the ratio of two such functions, and therefore not additive as required. In order to change F into an additive function, the new auxiliary function G (Baker, 1980) is defined as

$$G = \sum_{i=1}^m [R_i(j, k) - FT_i(j, k)] = \sum_{i=1}^m DG_i(j, k) \quad (3)$$

where $DG_i(j, k)$ is the change of G on the base jk , corresponding to the return function mentioned above.

The purpose of this analysis is to determine the minimal value of F (factor of safety) and the associate optimal trajectory i.e. critical slip surface. For the aforementioned reason, however, minimization of Eq. (2) cannot be made directly as long as DP is used. It has been confirmed, through variational analysis (Petrov, 1968; Baker, 1980), that minimizing the function F in Eq. (2) is equivalent to minimizing the new function G in Eq. (3); namely the minimum value of the function G corresponds to the minimum value of the function F . The minimization procedure may therefore be carried out through Eq. (3) over all admissible slip surfaces:

$$\begin{aligned} G_s &= \min G = \min \sum_{i=1}^m [R_i(j, k) - FT_i(j, k)] \\ &= \min \sum_{i=1}^m DG_i(j, k) \end{aligned} \quad (4)$$

in which G_s is the minimum value of the function G , yielding the minimum value of F .

It now becomes evident that the present problem shown by Eq. (4) may be included in the same category as that described by Eq. (1). Such problems can be solved by employing DP. When the DP minimization procedure is carried out, a value for F is first assumed. The assumed value of F is updated and the calculation procedure is continued until some convergence criterion (such as the difference between the factors of safety from the final two iterations being less than 0.001) is satisfied. A number of case studies have shown that different initial values of F do not change the final results, and the convergence is always reached within 2 to 6 iterations. No further explanations are given here, as the details of the iterative procedure are described in reference (Baker, 1980; Yamagami and Ueta, 1986, 1988).

In order to construct the recurrence relation in DP, it is convenient to define the function $H_i(j)$ which signifies the minimum value of the function G between the initial stage and the state j at the stage i , i.e.

$$H_i(j) = \min \left(\sum_{K=1}^{i-1} DG_K \right)_{j=1 \sim S_i} \quad (5)$$

Where $H_i(j)$ is called the 'optimal value function' in DP, and S_i shows the number of states at the stage i . Based on Bellman's principle of optimality, it may be shown that the minimum value of G between the initial stage and the state k at the stage $i+1$, that will be denoted by $H_{i+1}(k)$ is obtained from the sum of the optimal value function $H_i(j)$ and the return function $DG_i(j, k)$ between the two states j and k . This description is the so-called recurrence relation in DP (e.g. Baker, 1980), which can be expressed as

$$H_{i+1}(k) = \min_{j=1 \sim S_i} (H_i(j) + DG_i(j, k))_{k=1 \sim S_{i+1}} \quad (6)$$

The optimal value functions are taken as zero for each state of the initial stage because there are no preceding stages for those states, i.e.

$$H_1(j) = 0; \quad j = 1 \sim S_1. \quad (7)$$

Application of the recurrence relation (6) to each state of each stage, starting from the initial stage and finishing with the final stage $m+1$, will eventually furnish the optimal value functions $H_{m+1}(j)$ ($j = 1 \sim S_{m+1}$), i.e. the minimum values of the function G for each state at the final stage $m+1$. The minimum one of them yields the required minimum value G_s of G ; i.e.

$$G_s = \min G = \min_{j=1 \sim S_{m+1}} [H_{m+1}(j)] \quad (8)$$

Eqs. (7)–(8) may be considered as 'boundary conditions' in DP.

It is important to point out that when solving the recurrence relation Eq. (6) with boundary conditions Eqs. (7)–(8), the solution process records not only the values of the optimal value functions for each state at each stage, but also the corresponding state numbers at each stage i.e. the values of j in Eq. (6). After the minimum value G_s has

been obtained from the above minimization procedure, therefore, the optimal trajectory, i.e. the critical slip surface in the present problem can be traced back from the final stage $m+1$ to the initial stage 1.

It should be noted, referring to Fig. 3, that the search region contains the parts that are outside the actual slope. This is actually a method for determining the points of entrance and exit of potential slip surfaces. As shown in Fig. 3, the actual points of entrance and exit (points E and F) can emerge at any point on the slope surface, namely they are determined as a part of solutions. R_i and T_i for the parts outside the slope are assigned to equal zero in the actual computations.

From the above description, the solution procedure of DP in slope stability analysis can be summarized as follows:

(1) In a given search region, set up the appropriate number of stages and the appropriate number of states for each stage. The segment jk connecting two arbitrary states j and k at two adjacent stages therefore defines a slice, and forms a part of the potential slip surface. A possible slip surface can be obtained by connecting one state at each stage.

(2) Based on the factor of safety equation, define the return function $DG_i(j, k)$ on the segment jk in which the auxiliary function G is employed.

(3) The optimal value function as shown by Eq. (5) is introduced and the recurrence relation constructed as described by Eq. (6).

(4) By solving the recurrence relation with boundary conditions, the solutions i.e. the critical slip surface and the corresponding minimum factor of safety can be obtained.

The problem of determining the critical slip surface can also be regarded as a combinatorial optimization problem. This point of view is helpful for understanding the application of DP to locating 3-D critical slip surfaces which will be described in the next section. The slip surface shown in Fig. 3, in fact, is a combination of the $(m+1)$ state points located respectively at $(m+1)$ stages. In other words, the combination of $(m+1)$ points obtained from picking out one state at each stage represents a slip surface, and corresponds to a factor of safety. It is obvious that the optimal combination of state points at each stage means the critical slip surface having the lowest factor of safety. The problem of determining the critical slip surface consequently is identical with the problem of finding the optimal combination of state points. This type of combinatorial optimization problem may be solved using a number of techniques and DP is one of the most effective algorithms for solving such combinatorial optimization problems.

EXTENSION OF DP TO 3-D SLOPE STABILITY ANALYSIS

The 3-D Simplified Janbu Method

Consider a general slope and an arbitrary slip surface with 3-D shape shown in Fig. 4. The earth mass is divided

Fig. 4. The dividing scheme of a 3-D slope

into many vertical columns, and its direction of down-slope movement is assumed to be in the xz plane (no movement in the y -direction). The analysis will be carried out in terms of total unit weight and boundary water pressure (Yamagami and Ueta, 1982). Figure 5(a) illustrates the forces acting on a typical column. These forces are: W_{ij} —the weight of a column; P_{ij} —the vertical external force acting on the top of the column; Q_{ij} —the resultant of intercolumn forces acting on the column sides, and its direction is shown in Fig. 5(b) (Ugai and Hosobori, 1988) where η is an unknown constant; E_{ij} —an equivalent seismic loading, $E_{ij} = K_h(W_{ij} + P_{ij})$ where K_h is the horizontal seismic coefficient; T_{ij} and N_{ij} —the shear force and total normal force acting on the base of the column, respectively; $XW_{ij}^R, XW_{ij}^L, YW_{ij}^R, YW_{ij}^L$ —the resultant of water pressure acting on the four side faces of the column, respectively. In addition, the quantity signs in Fig. 5 include: $\Delta x, \Delta y$ —the discretization width of the column in the x - and y -directions; $\alpha_{xzij}, \alpha_{yzij}$ —the inclination angles of the base of the column in the xz and yz planes to the horizontal direction.

In Ugai's analysis a simpler case was considered where

Fig. 5. (a) Forces acting on a column; (b) Assumption of Q_{ij}

$P_{ij} = XW_{ij}^R = XW_{ij}^L = YW_{ij}^R = YW_{ij}^L = 0$. This paper treats water pressure as an external force. The derivation of factor of safety, however, is quite similar and therefore only the final results are presented here [Eqs. (9)–(10)].

$$F_{sh} = \frac{\sum_{i=1}^m \sum_{j=1}^n [\lambda_{ij} \Delta x \Delta y + \mu_{ij} (W_{ij} + P_{ij} + \eta \Delta V_{ij}^y \tan \alpha_{yzij})] / (\cos \alpha_{xzij} m_\alpha)}{\sum_{i=1}^m \sum_{j=1}^n [(K_h + \tan \alpha_{xzij})(W_{ij} + P_{ij}) + \eta \Delta V_{ij}^y \tan \alpha_{xzij} \tan \alpha_{yzij} - \Delta V_{ij}^x]} \quad (9)$$

where

$$\lambda_{ij} = (c' - u \tan \phi') (1 + \eta \cos^2 \alpha_{xzij} \tan^2 \alpha_{yzij}); \quad \mu_{ij} = \tan \phi' + \eta F_{sh} \sin \alpha_{xzij} \tan^2 \alpha_{yzij} / J$$

and

$$F_{sv} = \frac{\sum_{i=1}^m \sum_{j=1}^n [\varepsilon_{ij} \Delta x \Delta y + v_{ij} (W_{ij} + P_{ij} + \eta \Delta V_{ij}^y \tan \alpha_{yzij})] / m_\alpha}{\sum_{i=1}^m \sum_{j=1}^n (W_{ij} + P_{ij})} \quad (10)$$

where

$$\varepsilon_{ij} = (c' - u \tan \phi') \eta \sin \alpha_{xzij} \tan^2 \alpha_{yzij}; \quad v_{ij} = \sin \alpha_{xzij} \tan \phi' + F_{sv} / J$$

In Eqs. (9)–(10), u is the average pore-water pressure acting on the base center of the column; c' is the effective cohesion of the soil; ϕ' is the effective angle of internal friction;

$$\Delta V_{ij}^x = XW_{ij}^L - XW_{ij}^R, \quad \Delta V_{ij}^y = YW_{ij}^R - YW_{ij}^L; \quad J = (1 + \tan^2 \alpha_{xzij} + \tan^2 \alpha_{yzij})^{1/2};$$

the symbol m_α is given by

$$m_\alpha = (1 + \eta \tan^2 \alpha_{yzij}) / J + \sin \alpha_{xzij} \tan \phi' / F.$$

F_{sh} in Eq. (9) is the factor of safety with respect to horizontal force equilibrium, while F_{sv} in Eq. (10) represents the factor of safety with respect to vertical force equilibrium. There are only two unknowns $F (=F_{sh}=F_{sv})$ and η in Eqs. (9)–(10), and therefore the factor of safety and may be obtained by solving the simultaneous equations.

If assuming η to be zero, Eq. (9) may be simplified as:

$$F = \frac{\sum_{i=1}^m \sum_{j=1}^n [(c' - u \tan \phi') \Delta x \Delta y + (W_{ij} + P_{ij}) \tan \phi'] / (\cos \alpha_{xzij} m_\alpha)}{\sum_{i=1}^m \sum_{j=1}^n [(K_h + \tan \alpha_{xzij})(W_{ij} + P_{ij}) - \Delta V_{ij}^x]} \quad (11)$$

where

$$m_\alpha = 1 / J + \sin \alpha_{xzij} \tan \phi' / F.$$

Equation (11) is an implicit function of only F and yields directly the required factor of safety when solved using an iterative procedure. It is indicated (Ugai and Hosobori, 1988) that the simplified formula Eq. (11) can provide sufficiently accurate results when the slope is gentle (less than 45°). It is clear, in fact, that most of natural and manmade earth slopes possess a gentle inclination that is not larger than 45° . Therefore, the simplified formula Eq. (11) is adopted in this paper, for convenience.

Stages and States in the 3-D Slope Stability Analysis

For the sake of convenience and for illustration, Eq. (11) is rewritten as

$$F = \sum_{i=1}^m \sum_{j=1}^n A_{ij} / \left[\sum_{i=1}^m \sum_{j=1}^n B_{ij} = \sum_{i=1}^m R_i / \sum_{i=1}^m T_i \right] \quad (12)$$

where

$$\begin{aligned} R_i &= \sum_{j=1}^n A_{ij} \\ &= \sum_{j=1}^n \frac{(c' - u \tan \phi') \Delta x \Delta y (W_{ij} + P_{ij}) \tan \phi'}{\cos \alpha_{xzij} (1 / J + \sin \alpha_{xzij} \tan \phi' / F)} \\ T_i &= \sum_{j=1}^n B_{ij} \\ &= \sum_{j=1}^n [(K_h + \tan \alpha_{xzij})(W_{ij} + P_{ij}) - \Delta V_{ij}^x]. \end{aligned}$$

By contrasting Eq. (12) with Eq. (2), it can be seen that the safety factor equation in the 3-D analysis may also be expressed as an additive function as follows, by introducing the auxiliary function G similar to that in the 2-D problem, signifying applicability of DP to the present 3-D analysis.

$$G = \sum_{i=1}^m [R_i - F T_i] = \sum_{i=1}^m D G_i. \quad (13)$$

The character of this problem is essentially the same as that in 2-D analysis. The only difficulty in applying DP to 3-D analysis is the fact that the two quantities R_i and T_i in Eqs. (12)–(13) are also shown as a sum. The following will clarify how this difficulty is overcome.

In order to apply DP to 3-D analysis, it is initially

necessary to construct a stage-state system, as done in 2-D analysis. Figure 6 illustrates such a stage-state system for the 3-D slope shown in Fig. 4. 'Stages' are a sequence of vertical sections perpendicular to the sliding direction (i.e. the x -axis), corresponding to vertical lines in the 2-D problem (Fig. 3), and a 'state' in a stage is represented by one curve in the stage plane. Only one state curve for each stage is shown in Fig. 6 for convenience, although, in practice, there are a number of state curves in every

Fig. 6. Stage and state in the slope shown in Fig. 4: (a) A part of the 3-D slip surface between state jj in stage i and state kk in stage $i+1$, (b) Vertical section (as stage) perpendicular to x -direction and intersection curve (as state) of the slip surface and each stage plane

stage plane. If a state jj in a stage i is selected and a state kk in the stage $i+1$, as shown in Fig. 6(a), the shaded concave segment sandwiched between these two state curves constitutes a part of the 3-D slip surface. The slip surface will therefore be generated when one state curve in each stage plane is connected.

In the foregoing 2-D analysis, the return function $DG_i(j, k)$ in Eq. (3) was determined by two arbitrary state points at two adjacent stages. The return function DG_i in Eq. (13) for the present 3-D analysis can also be defined in terms of two state curves in two adjacent stage

planes. Referring to Fig. 4 and Fig. 6, Fig. 7 can be easily drawn, in which the earth mass between the two adjacent stage planes i and $i+1$ has been divided into (n) columns by two groups of $(n+1)$ vertical dividing lines located respectively in the stage planes i and $i+1$. Figure 7 shows that when the coordinates of the two states jj and kk are known, the locations of the (n) columns between the two stages i and $i+1$ are determined. Therefore, the quantities R_i and T_i in Eqs. (12)–(13) are computable; namely, the R_i and T_i can be determined for the shaded segment shown in Fig. 6(a). Consequently, Eq. (13) can be further written as

$$G = \sum_{i=1}^m (R_i - FT_i) = \sum_{i=1}^m DG_i = \sum_{i=1}^m DG_i(jj, kk). \quad (14)$$

Obviously, $DG_i(jj, kk)$ in Eq. (14) is just the return function in three-dimensions, corresponding to $DG_i(j, k)$ in Eq. (3) for 2-D problem. Since the return function has been defined, the very same recurrence relation and boundary conditions for 3-D analysis as those in 2-D problem can be constructed through introducing the optimal value function (Eq. (5)). By solving the recurrence relation, the minimum factor of safety and the corresponding critical slip surface in three-dimensions can be obtained at the same time.

In the 2-D problem shown in Fig. 3, the appropriate number of states were set up for each stage. Similarly, a certain number of states (i.e. curves) in each stage plane are necessary in order to obtain a sufficiently smooth 3-D slip surface. With regard to the generation of states, a detailed explanation is given in the next section.

It should be emphasized that construction of the above solution procedure can also be made through the simultaneous equations Eqs. (9)–(10). This is because both F_{sh} and F_{sv} can be expressed as the form of an additive function by virtue of the auxiliary function G , and the equations similar to Eq. (14) can be derived. Only, a comparatively complex iterative procedure is necessary to obtain the solution for the two unknowns F and η . Furthermore, the proposed approach may be incorporated with any other 3-D factor of safety equation for slip surfaces of arbitrary shape as long as the equation can be written in the form of a ratio of two additive functions like Eq. (12).

As pointed out before, determining the critical slip surface is equivalent to finding the optimal combination of states. The same concept is applicable to the present 3-D analysis. In fact, the 3-D slip surface shown in Fig. 6(a) can be considered as a combination of the curves obtained from picking out one state in each stage plane. The optimal combination of the state curves in each stage represents the 3-D critical slip surface giving the minimum factor of safety. DP is utilized to solve such combinatorial optimization problems in order to determine the optimal combination of state curves.

Generation of States

As mentioned earlier, states in three-dimensions are indicated by curves in each stage plane (Fig. 6). Figure 7 il-

Fig. 7. (a) Columns between two adjacent stages i and $i+1$, (b) Two states jj and kk shown by two broken lines

illustrates the state jj in the stage i , the state kk in the stage $i+1$ and the (n) columns determined by them. It should be noted that the two state curves jj and kk are approximated by two groups of coordinates $(y_{i,j}, z_{i,j})$ and $(y_{i+1,j}, z_{i+1,j})$ ($j=1 \sim n+1$) as shown in Fig. 7(b). This signifies that a state in a stage is represented by a broken line obtained by connecting one point for each dividing line (total $n+1$) in the stage plane. In this section, therefore, the emphasis of discussions is placed on how to generate such a broken line (as a state) in a stage plane that will be called a 'broken line state' in the following descriptions.

In order to generate broken line states, the method of Random Number Generation has been employed in this paper. Figure 8(a) shows the discretization scheme of a stage plane. There are $(n+1)$ vertical dividing lines in the stage plane, and each line is discretized by the appropriate number of points that are given by integer numbers. The generation process of one broken line state is explained as follows: first, a random integer, which represents one point at a dividing line as shown in Fig. 8(a), is generated by means of the following equation:

$$N_j = \text{Inter}(S_j r) + 1 \quad (j=1, 2, \dots, n+1) \quad (15)$$

where: S_j is the number of the points at the j th dividing line (see Fig. 8(a)); r denotes even pseudo-random num-

bers between 0 and 1 which can be produced by a mathematical method; the symbol 'Inter' means that only part of the integer is taken.

Application of Eq. (15) to each dividing line in the stage plane, from the leftmost to the rightmost, yields one point at each dividing line. Connecting in sequence those points indicated by the random integers generates a broken line. A large number of broken lines so generated are recorded and will be used for states in the stage plane. This procedure must be repeated over all the stage planes.

In general, most actual sliding surfaces are smooth curved surfaces, except for those in which sliding takes place due to some particular geological conditions. Among the broken line states generated simply by the method of Random Number Generation, however, there are a number of such broken lines that result in oscillatory slip surfaces. In order to ensure the smoothness of potential slip surfaces, it is therefore necessary to make a constraint of $\partial^2 z / \partial^2 y \geq 0$ on the broken line states to be generated. This condition, which restricts the generated broken lines states to be concave, leads to smooth potential slip surfaces in the y -direction and excludes broken line states that oscillate violently.

After states on each stage plane have been determined by the method of Random Number Generation, the minimization procedure based on DP may be carried out in the same way as that in the 2-D problem. However, the following point should be noted: in order to determine two exits of the slip surface in every vertical section (i.e. stage plane), the parts that are outside the slope must be fixed in each stage plane [see Fig. 8(b)]. From Fig. 8(b) it can be seen, due to the parts outside the slope, that the left and right exits of the slip surface in the stage plane can emerge at any point on the slope surface; namely the states of all admissible extent in the search region can be generated.

Discussions about the Number of States

The preceding analysis indicates that the precision of solutions based on the proposed approach depends greatly on the number of stages and the number of states in each stage plane. The number of stages can easily be determined because it corresponds to the number of columns in the sliding direction, and therefore the number of states becomes the main factor influencing the precision of solutions. We cannot determine, in a theoretical manner, however, the number of states which make the solution achieve the required precision. The alternative practical way is to compare results obtained using different numbers of states. The computation results for a number of example problems indicated when states in each stage plane are more than a certain number, the shape of the critical slip surfaces and the value of the corresponding minimum factors of safety are almost unchanged (see, for instance, example 1 in the next section). The slip surface and the associated factor of safety obtained using this number of states are therefore considered to be a sufficiently accurate solution.

Fig. 8. (a) The discretization scheme and generation process of a state, (b) A method of determining two exits of state

The number of states necessary for a sufficiently accurate solution depends on many parameters such as, the number of columns between two adjacent stage planes, the magnitude of the search region, etc. It is rather difficult therefore to give a definite number of states for different problems. Nevertheless, a rough range of the number of states can be described based on the example problems that have been analyzed by the program 3D-DYRANUT as described in the next section. In order to obtain the sufficient solutions, about 100~200 states in each stage plane are required for typical symmetrical problems, while for typical unsymmetrical slopes, about 400~500 states must be generated. Discussion about the effect of the number of states on computation results will be made later, together with the example problems in the next section.

It should be noted, that in the proposed method, the computer time is almost exponentially dependent on the number of states in each stage plane. In order to make the computation procedure more efficient, we have made an attempt that includes the following two steps. The first step is called 'preliminary analysis' in which the calculation procedure is carried out once with small S_j i.e. large discretization width Δz (S_j and Δz are shown in Fig. 8(a)) and small number of states in order to determine the approximate location of the slip surface. The second step is called 'refined analysis' in which the calculation procedure is run close to the approximate slip surface, using large S_j i.e. small Δz but restricting the magnitude of the search region. The actual computations demonstrated that the analysis using two steps makes the proposed approach significantly more efficient in terms of computer time, especially, for such problems with large search regions (see Example 5 in the next section).

RESULTS

The proposed 3-D slope stability analysis approach has been implemented into a computer program called 3D-DYRANUT. This program can be run in two modes: (i). determination of the 3-D critical slip surface and the associated factor of safety through the DP minimization procedure, where a specified number of states in each stage plane are generated in advance; (ii). calculation of the simplified Janbu's factor of safety for a given 3-D slip surface. To demonstrate the capabilities of this program, several example problems, including failures of a natural slope, were analyzed in this section.

Example 1

Example 1 is a simple and homogeneous slope shown in Fig. 9(a). In addition to gravity, the slope is subjected to a rectangular load of width 4.0 m (half), length 8.0 m, and intensity $\Delta q = 50.0$ kPa. This example has been calculated by the first author using the method of random generation of surfaces (Yamagami et al., 1991). The results of computation are shown in Fig. 9(b). This paper reanalyzed the example for two purposes: (1) to compare the results between the method of random generation of

Fig. 9. Example 1: (a) Plan view and cross section, (b) and (c) Critical slip surfaces obtained by the method of random generation of surfaces and the proposed approach, respectively

surfaces and the presented approach; and (2) to observe the effect of the number of states in each stage plane on factors of safety.

Figure 9(c) illustrates the critical slip surface and the corresponding minimum factor of safety obtained by the program 3D-DYRANUT. The number of states in each stage plane is 100. The solution converged after three iterations when the initial factor of safety is assumed to be 2.0. The results shown in Fig. 9(b)–(c) showed that the proposed 3-D approach found the critical slip surface to have a lower factor of safety. The minimum factor of safety of 3D-DYRANUT is 1.026 as compared with 1.141 by the method of random generation of surfaces. (Note that on the slip surface shown in Fig. 9(b) the factor of safety calculated by 3D-DYRANUT is also 1.141).

It is significant to note the difference of computer time between the two methods. The computer time (CPU time) necessary to obtain the results shown in Fig. 9(b) was 4 hours. In marked contrast with this, the computer time using the program 3D-DYRANUT was only 12 minutes for the solution shown in Fig. 9(c) on the same personal computer *NEC PC-9801 FA* (CPU: 80486SX, 16 MHz, and *NDP FORTRAN Compiler*). These results indicate that the proposed approach is more effective and efficient than the method of random generation of surfaces not only in terms of accuracy of solutions but also computer time.

In order to investigate the effect of the number of states in each stage plane on computation results, Table 1 shows the number of states in each stage plane, the corresponding factors of safety and computer times for the example. From Table 1 it can be seen that if the number of states in each stage plane is less than a certain limit, sufficiently precise results cannot be obtained. When the number of states exceeds 100, however, the value of safe-

ty factors is kept unchanged and this unchanged value means the required solution.

Example 2

Example 2 refers to the stability of conical heaps which has been analyzed by Baker and Leshchinsky (1987) through the variational limit equilibrium method. In this example, the slope surface is taken as the surface of a cone inclined at 30°. Figures 10(a)–(b) illustrate half of the extended log-spiral surfaces generated by the variational calculus, in which Fig. 10(a) is the critical slip surface for a c - ϕ soil while Fig. 10(b) represents the surface for a cohesive soil.

A search for the critical slip surface was carried out by 3D-DYRANUT using the same parameters. The results are illustrated in Figs. 11(a)–(b). It should be noted in Fig. 10 that the coordinates are indicated by the non-dimensional parameters x/H , y/H and z/H (H is the height of the cones), while the height of the cones is taken as 5.0 m in Fig. 11.

Using different combinations of strength parameters, Figure 12 shows the traces of the critical slip surfaces on the plane of symmetry obtained by the variational calculus and 3D-DYRANUT. The nondimensional parameter λ in the figure is quoted from the literature (Baker and Leshchinsky, 1987), and is defined as $\lambda = c/(\gamma H \tan \phi)$ where H is the height of the slope. The parameter λ is considered to have a major effect on the critical slip surface. A slope has been analyzed using 3D-DYRANUT by changing combinations of c and ϕ but keeping λ unchanged. As a result, the critical slip surfaces obtained from different values of λ (i.e. different combinations of c and ϕ) were identical in shape and location, but the corresponding factors of safety were different. This result is consistent with the analytical solutions determined from the variational computations.

From Figs. 10–12 it is indicated that the results obtained using the two methods agree well in the extent of failure masses in the x - and y -directions. However, two marked differences between the two solutions should be noted; one is the depth of the failure masses, and the other is the value of the minimum factors of safety. The first discrepancy is evidently reflected in Fig. 12, in which for all combinations of soil parameters, the failure masses obtained using 3D-DYRANUT occur more deeply

Table 1. Effect of the number of state on factor of safety and CPU time

The number of states	50 ~ 90	100	150	200
Factor of safety	1.211 ~ 1.027	1.026	1.026	1.026
CPU time	3.5 ~ 10.5 min.	12 min.	28 min.	68 min.

Fig. 10. Extended log-spiral critical slip surface for 30° conical heaps (from Baker and Leshchinsky, 1987): (a) c - ϕ soil, (b) Cohesive soil

Fig. 11. Critical slip surface obtained by 3D-DYRANUT for the conical heaps shown in Fig. 10: (a) c - ϕ soil, (b) Cohesive soil

Fig. 12. Effect of soil parameters on failure depth (Comparison between the variational analysis and the proposed approach)

Fig. 13. An approximate critical log-spiral surface inferred from Baker and Leshchinsky (1987)

than the log-spiral ones. There is such a tendency in addition that the discrepancy tends to increase with the value of (i.e. approach to cohesive soil, see Fig. 12). The second discrepancy can be found from Figs. 10–11, in which 3D-DYRANUT provided lower factors of safety than the variational method. In particular, a larger discrepancy is demonstrated for the cone comprised of cohesive soils, where the minimum factor of safety obtained using 3D-DYRANUT is 0.89, whereas the variational factor of safety is 1.00.

In an attempt to investigate the difference of factors of safety between the two methods, the factor of safety of Ugai's 3-D simplified Janbu method was calculated on the variational slip surface. Since insufficient information was provided by Baker and Leshchinsky (1987) to calculate exactly the coordinates of the log-spiral surface shown in Fig. 10(b), an approximate slip surface was inferred on the basis of this log-spiral surface, as shown in Fig. 13. On the approximate surface the safety factor of 3D-DYRANUT is 0.98, compared with 1.00 from the variational analysis. This discrepancy is rather small, considering the differences between (1) the variational method and the 3-D simplified Janbu method, and (2) the accurate log-spiral surface and the approximate one.

This result means that for the given log-spiral surface the 3D-DYRANUT provided the factor of safety very close to that of the variational method, while the critical slip surface located by 3D-DYRANUT yielded a lower factor of safety than that of the critical extended log-spiral surface. This is not conclusive proof, however, that the approach presented herein provided critical slip surfaces that are closer to actual situations. In order to confirm which is more accurate, additional methods or further studies, such as centrifugal model tests, are necessary.

Example 3

Example 3 is a homogeneous and symmetrical concave slope in plan view whose geometry is defined in Fig. 14. The results using 3D-DYRANUT are presented here to evaluate the influence of soil strength parameters on critical slip surfaces.

This example considered four different combinations of soil parameters (see Fig. 16). The different combinations of soil parameters can determine the different critical slip surfaces in shape and extent. Three typical categories

Fig. 14. The definition of a concave slope in plan view

ries of critical slip surfaces are shown in Figs. 15(a)–(c). The effect of soil parameters on critical slip surfaces, via cross section, are illustrated in Fig. 16. From Figs. 15–16 it can be concluded that the critical slip surface for concave slope in plan view may pass through the toe, or emerge above the toe or beyond it. In addition, there is a tendency that the extent of sliding masses becomes large (see Fig. 15) and the critical slip surfaces tend to deepen (see Fig. 16) as ϕ increases relative to $c/\gamma H$ (H is the height of the concave slope).

Example 4

In order to demonstrate the capability of 3D-DYRANUT, a convex slope in plan view defined in Fig. 17 was selected as Example 4. This example has two profiles: a homogeneous one, and a multilayered case where a thin weak layer is included. For the homogeneous profile, two different pore pressure distributions were considered. In case 1 there is no pore pressure, and case 2 has a pore pressure defined by a phreatic surface. For the sake of simplicity, this phreatic surface is taken as the surface of a cone to have a vertical-to-horizontal inclination of 1:4.

Figure 18 illustrates the critical slip surfaces and the corresponding minimum factors of safety for the three cases under consideration. From the comparison of the results shown in Fig. 18, it can be determined that (1) with a weak layer in the multilayered slope, the length of the sliding mass in y -direction is somewhat larger than the one without a weak layer, but the sliding still occurs within a limited length; (2) for the case where there is a phreatic surface, the sliding mass has a larger extent in x -, y - and z -directions than the one without phreatic surface.

Example 5

The last example is a case history of large landslides of natural slopes that occurred during the 1984 Nagano-seibu earthquake in Japan. The slope geometry and the failed slip surface obtained from the topographic drawing before and after failure are illustrated in Fig. 19. In

order to analyze the seismic stability for this case history, the method of pseudo-static stability analysis were used, in which the earthquake loading is treated as a horizontal inertia force as described in the Ugai's 3-D simplified Janbu method. Due to a lack of information about the dynamic properties of soils, the strength parameters identified from the back analysis (Yamagami and Ueta, 1994) are adopted. The back analysis were made on the basis of the failed slip surface shown in Fig. 19(b) whose factor of safety was assumed to be 0.95 for both $K_h=0.1$ and $K_h=0.2$ cases.

The computation was first carried out using a two-step analysis i.e. 'preliminary and refined analyses' mentioned in the preceding section. The preliminary analysis specifically selected a larger search region where Δz , S_j and the number of states were taken as 50 m, 20 and 200, respectively. The computer time was then about 1.4 hours for determining the approximate location of the critical slip surface. In the follow-up refined analysis, the search region was restricted in the vicinity of the 'critical' slip surface determined by the preliminary analysis, and Δz , S_j and the number of states were respectively assigned to be 20 m, 20 and 300. The computer time was then about 3.1 hours. Thus, the computer time necessary for a complete analysis was about 4.5 hours.

An additional computation for the case history was performed using 3D-DYRANUT, in which Δz and S_j were taken as 20 m and 50. About 500 states for each stage plane were then required in order to obtain a solution with the same precision. As a result, the time of execution was about 11.5 hours. The comparison of computer times indicated that the analysis in two steps makes the program 3D-DYRANUT much more efficient in terms of computer time.

The results of analysis for this case history are presented here. Fig. 20(a)–(b) illustrates the critical slip surfaces determined using 3D-DYRANUT, corresponding to $K_h=0.1$ and $K_h=0.2$ cases. Since the same factor of safety (0.95) of the failure for both $K_h=0.1$ and $K_h=0.2$ cases was utilized to back-analyze the soil parameters,

Fig. 15. Critical slip surface for Example 3: (a) Emerging above the toe, (b) Passing through the toe, (c) Emerging beyond the toe

Fig. 16. Depth of sliding masses for various combinations of c and ϕ

Fig. 17. The definition of a multilayered convex slope in plan view

the critical slip surfaces located by using these parameters are about the same for the two cases (Fig. 20). The minimum factors of safety by 3D-DYRANUT, however, are slightly lower than the ones on the actual slip surface. This probably implies that the actual failure occurred under conditions of a lower factor of safety than the assumed value of 0.95. Comparing the case record (Fig.

(a)

(b)

(c)

Fig. 18. Critical slip surface for Example 4: (a) Homogeneous slope, (b) Same as (a) with phreatic surface, (c) Multilayered slope

(a)

(b)

Fig. 19. Large landslide of a natural slope: (a) before failure, (b) after failure

19(b)) with the computed results (Fig. 20) indicates that there are divergences between the actual and predicted slip surfaces. In order to thoroughly clarify the sources of the divergences, a number of additional studies should be made, such as seismic coefficients, dynamic properties of the soil, as well as analyses under inhomogeneous conditions, that, obviously, are beyond the scope of this paper.

Fig. 20. Critical slip surface for Example 5: (a) $K_h = 0.1$, (b) $K_h = 0.2$. F_s = minimum factor of safety obtained by 3D-DYRANUT; (F) = factor of safety calculated by 3D-DYRANUT on the actual slip surface shown in Fig. 19(b)

CONCLUSIONS

An efficient and practical 3-D slope stability analysis approach, capable of determining the minimum factor of safety and the corresponding critical slip surface simultaneously, has been presented. This approach is based on

Dynamic Programming and Random Number Generation. No arbitrary restrictions are placed on the shape of the potential slip surfaces. The developed analysis approach has been implemented into the computer program 3D-DYRANUT, which can successfully determine the location and shape of the 3-D critical slip surface as well as the associated factor of safety for a slope of arbitrary shape including layered soils and/or pore pressure (phreatic surface). The method proposed is considered to be useful in practical situations due to its simplicity in theory and capability of handling a variety of 3-D problems.

DP is a numerical algorithm for optimization of sequential multistage decision problems. It was originally used to solve 2-D problems as reported in the literature. Actually, 3-D slope stability analysis cannot directly be performed using a DP algorithm only. In order to apply DP to a 3-D analysis, this paper presented an important idea in which Random Number Generation has been employed to generate states, transforming a 3-D DP problem into a 2-D one. States in 3-D analysis are the intersection curves of a potential slip surface and stage planes and represent the shape of the slip surface in a direction perpendicular to the sliding direction. A larger number of states are necessary for the 3-D slope stability analysis to obtain sufficiently accurate solutions, than those in the 2-D problems.

A comparison of the results computed by 3D-DYRANUT with the results obtained from the method of random generation of surfaces has shown that the proposed approach is much more efficient than the method of random generation of surfaces in terms of not only computer time but also precision of solutions.

The proposed approach combined the 3-D simplified Janbu method with DP and Random Number Generation, to determine the critical slip surface and the corresponding minimum factor of safety. Any factor of safety equation available for a 3-D slip surface of arbitrary shape, however, can be incorporated into this approach as long as the equation is expressed with a ratio of two summations as described by Eq. (12). All the existing 3-D slope stability analyses based on the method of columns have, in fact, derived factor of safety formulae like Eq. (12).

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