A TWO-SURFACE MODEL WITH ANISOTROPIC HARDENING AND NONASSOCIATED FLOW RULE FOR GEOMATERIALS

SHOICHI KIYAMA and TAKASHI HASEGAWA

ABSTRACT

A two-surface model is proposed to describe various volumetric behavior with change in the direction of a stress-controlled path. The model is formulated using the following concept of the isotropic-kinematic hardening. For overconsolidated soils, the nonassociated flow rule is used because of the occurrence of volumetric change, which is beyond the descriptive capabilities of the associated flow rule, during the loading with an abrupt change of stress path. The proposed model consists of nine material parameters that can be determined with a few triaxial tests. The model is verified by comparing model predictions with experimental results. In addition, the shear strength increases for the material that has the characteristic of stress path dependency are evaluated. Experimentally stabilized behavior which contradicts the theoretical condition for material stability by Drucker is discussed.

Key words: anisotropy, dilatancy, (isotropic-kinematic hardening), (nonassociated flow rule), overconsolidation, (silty clay), (stability postulate), (stress path dependency), (two-surface model) (IGC: E2/D6)

INTRODUCTION

Anisotropy of geomaterials, such as soil deposits and reconstituted soils, is brought about in the course of natural and artificial processes memorized as stress history. The relationship, between stress history and anisotropic properties of various natural and reconstituted soils is not well known. The reason for this is the difficulty to have data acquisition which is mostly influenced by geometric constitution such as the distribution of grain size and of particle shape, and other factors such as mineralogy and electricity. Laboratory tests are important tools for the investigation of this relationship, and in this paper we will explore anisotropic states as functions of fundamental soil characteristics.

The reconstituted soils to be considered here can also show anisotropy induced by stress history, even though they are initially isotropic materials. Lade and Duncan (1976) indicated that the stress-strain behavior of soils depends upon the previous stress path due to inherent inelasticity caused by sliding among soil particles. Fleming and Duncan (1990) showed that there is no unique undrained shear strength for a soil, which can be easily disturbed. Kiyama et al. (1995a) pointed out that the undrained shear behavior of silty clay changed depending on the employed stress path including anisotropically overconsolidated paths.

The constitutive model expressing the fundamental behavior to be observed in the experiments is indispensable for numerical analyses examining practical geotechnical scenarios. Simulations by stress path analysis to control construction were conducted by D’Appolonia and Lambe (1971), Simpson et al. (1979), Lambe et al. (1981), Watson et al. (1984) and Cinicioglu and Togrol (1991). The constitutive modelling must be studied sufficiently to insure reliable estimates of soil behavior.

A variety of elasto-plastic constitutive models for soils have been proposed since the Cam clay model proposed by Roscoe and Burland (1968). This model and that of Sekiguchi and Ohta (1977), the latter was used to describe stress-induced anisotropy by Yatomi and Nishihara (1984), are classified as a simple surface model.

This paper will focus on the description of soils that show inelastic behavior within the region of overconsolidation. Pender (1977) suggested that shear hardening occurred when the principal stress ratio increased within the overconsolidated region. The two-surface model which describes this inelastic behavior was proposed by many authors: Mroz et al. (1979), Adachi and Oka (1982), Pietruszczak and Stolle (1987), Hashiguchi (1988), di Prisco, C. et al. (1993) and Whittle (1994). The two surfaces considered are a normal yield surface and a sub-yield surface. The isotropic-kinematic hardening type of model for the sub-yield surface and the isotropic harden-

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ing one and the stress-induced anisotropy for the normal yield surface are adopted. In addition, the nonassociated flow rule is applied to describe the volumetric plastic deformation of the overconsolidated soil. Modelling of granular materials, which show volumetric contraction or dilation depending upon the density, can be performed with the nonassociated flow rule for precise prediction, as proposed, e.g., by Maier and Hueckel (1979), Nova and Wood (1979), Pietruszczak and Stolle (1987) and Cristescu (1994) etc. Kiyama and Hasegawa (1996) found that the stress-strain relationship of soils subjected to an abrupt change in the stress path could not be described by the associated flow rule even with the lower stress ratio far from the failure state. It was indicated that either volumetric contraction or dilation could occur according to the accumulated plastic strain, despite the equal stress state.

The use of the nonassociated flow rule, however, has occasionally been criticized because of experimental results contradicting the postulate of stability proposed by Drucker (1959). Lade et al. (1987) demonstrated the necessity of the nonassociated flow rule and Lade (1993) showed that stable behavior was observed in the region in which Drucker’s postulate was violated. This kind of investigation was limited to the stress path-controlled test.

The results of a series of triaxial test series conducted on a reconstituted silty clay with anisotropic characteristics induced by stress history are presented. The model proposed can account for fundamental mechanisms in the mechanical behavior of the soil. The verification of the model is accomplished by comparing model predictions with experimental results.

REMARKS ON MODELLING

Soil’s Characteristics

The silty clay shows complex behavior as a result of comparatively broad grain sizes (clay 32%, silt 40%, sand 28%) (Kiyama et al., 1995a). Various soil characteristics, therefore, may emerge depending on the deformation history.

It is desirable that the model is relatively simple and convenient to use. A promising simple model might be formulated only using the concepts of isotropic hardening and the associated flow rule. This type of model can describe the following behavior (C1):

1. The volumetric dilation is shown to have a higher stress ratio after reaching the transformed state during shear.
2. The direction of the strain rate vector changes depend on the accumulated strain associated with the controlled stress path. The shape change of the yield surface is required when the associated flow rule is adopted.

It is necessary, however, to develop modelling in order to describe significant characteristics beyond the scope of such a simple modelling. After detailed triaxial testing, we found that the following features are particularly fundamental at the overconsolidation to a useful model:

C2. Dilatancy

The overconsolidated soil in undrained shear never shows a constant condition of $p'$ being the effective mean principal stress, which implies the occurrence of dilatancy. This requires the assumption of a two-surface model.

C3. The kinematic hardening

The Bauschinger effect appears in the stress-strain relationship of soils during the loading-reversal loading. The kinematic hardening can describe this anisotropic effect induced by the plastic straining.

C4. The nonassociated flow rule

After applying compression at a constant $q$ as a deviatoric stress, the soil can emerge with either volumetric contraction or dilation, even if it has passed the equal stress point where it is far below the failure state. This phenomenon reveals the negative increment of the second work, i.e., $d\sigma^2:de < 0$ (cf. Lade et al., 1987; Kiyama and Hasegawa, 1996) and cannot be interpreted without the effect of dilatancy. There is a contradictory volumetric elastic swelling when the associated flow rule is applied, despite the actual volumetric contraction.

In order to interpret this behavior correctly, several hypotheses are proposed within the theoretical framework of elasto-plasticity.

Approach to Formulation

The condition C1-1 can be seen in the distinction between the transformed state and the failure state with the assumption of deviatoric strain hardening proposed by Nova and Wood (1979). The condition C1-2 reveals the shape change of the plastic potential surface. The shape variation can be estimated by the stress-dilatancy relationship of Rowe (1969). This information is employed to acknowledge the influence macroscopically exerted upon the soil behavior with the change of its internal structure. Such an influence has also been cited by Nova and Wood (1979) and Ikeura and Mitachi (1985). The proposed model extends the study of the influence to evaluate the effectiveness of the normally consolidated stress history. The associated flow rule will be adopted in order to simplify the modelling, especially for the normally consolidated soil; such an approximation seems reasonable because the stress rate changes in a geometrically more gentle manner at transitions within the normally consolidated region than from the normally to the overconsolidated region.

It is useful for C2 to introduce an indication by Pender (1977). He discovered from experimental evidence that overconsolidated clays showed dilatancy related to shear hardening when the stress ratio increased. This conclusion can also be reached through the two-surface models proposed by Adachi and Oka (1982), Dafalias (1986), Pietruszczak and Stolle (1987), Hashiguchi (1988) and Whittle (1994). The strain hardening, in addition, also makes the area of the normal yield surface greater or smaller, leading to variations in the shear strength.

The anisotropy induced by an overconsolidated stress history as mentioned for the condition C3, may translate to the sub-yield surface. The anisotropic state is effective in preventing overestimation of the elastic response with
the change in the controlled stress path. This consideration determines the magnitude of the shear strength, which is governed by the corresponding accumulated plastic strain with the stress path history. Phyllips and Weng (1975) indicated that the stress path dependency could be related to the motion and the distortion of the yield surface for metals, and they (1977) determined the shape of this surface. Eisenberg and Chen (1981), Wu and Yeh (1987), Ellyin and Xia (1989) and Boucher and Cordebois (1994) formulated the kinetic process for each yield surface with its distortion. The detailed nature of the anisotropy is reviewed in the next section in this paper.

Although some of the two-surface models belong to the family of similitu-shape types (Daflas and Popov, 1976; Hashiguchi, 1988; Whittle, 1994), a non-similitude-shaped type such as that proposed by Adachi and Oka (1982), Pietruszczyk and Stolle (1987) and di Prisco et al. (1993) has been adopted. The applicability of this method was confirmed by the triaxial test results mentioned above.

The condition of C4 can be described with the independent assignment of the sub-yield surface and the plastic potential surface. The plastic potential function can be derived from the measurement of the plastic strain vector. The sub-yield surface’s function is not directly known because of the difficulty to determine the shape at yielding state, but determines the quality of the constitutive model. It seems that the need for the sub-yield surface points to the nonassociated flow rule rather than the associated one for the correspond geomaterial. The concept of the nonassociated flow rule was used by Maier and Hueckel (1979), Nova and Wood (1979), Adachi and Oka (1982), Pietruszczyk and Stolle (1987) and di Prisco et al. (1993). The sub-yield function in the proposed model is chosen to ensure the proper plastic deformation as seen in the triaxial test. The distinction between the proposed model and others is presented in Table 1.

**Table 1. Classification of constitutive models**

<table>
<thead>
<tr>
<th>Type of model</th>
<th>Normal consolidated state</th>
<th>Overconsolidated state</th>
<th>Num. of parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( f=g ) Isotropy (I) &amp;</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Anisotropy (A)</td>
<td>Hardening function</td>
<td>Plasticity</td>
</tr>
<tr>
<td>Simple surface model</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Roscoe &amp; Burland (1968)</td>
<td>( I )</td>
<td>( e_{p} )</td>
<td>( x )</td>
</tr>
<tr>
<td>Sekiguchi &amp; Ohita (1977)</td>
<td>( I )</td>
<td>( e_{p} )</td>
<td>( x )</td>
</tr>
<tr>
<td>Nova &amp; Wood (1979)</td>
<td>( f \neq g )</td>
<td>( I ) &amp; ( A_{x} )</td>
<td>( e_{p} )</td>
</tr>
<tr>
<td>Cristescu (1994)</td>
<td>( f \neq g )</td>
<td>( I ) &amp; ( A_{y} )</td>
<td>( e_{p} )</td>
</tr>
<tr>
<td>Two-surface model (Bounding surface model, Sub-loading surface model)</td>
<td>( f \neq g )</td>
<td>( I ) &amp; ( A_{x} )</td>
<td>( e_{p} )</td>
</tr>
<tr>
<td>Mroz et al. (1979)</td>
<td>( f \neq g )</td>
<td>( I ) &amp; ( A_{y} )</td>
<td>( e_{p} )</td>
</tr>
<tr>
<td>Adachi &amp; Oka (1982)</td>
<td>( f \neq g )</td>
<td>( I ) &amp; ( A_{y} )</td>
<td>( e_{p} )</td>
</tr>
<tr>
<td>Pietruszczyk &amp; Stolle (1987)</td>
<td>( f \neq g )</td>
<td>( I ) &amp; ( A_{y} )</td>
<td>( e_{p} )</td>
</tr>
<tr>
<td>Hashiguchi (1988)</td>
<td>( f \neq g )</td>
<td>( I ) &amp; ( A_{y} )</td>
<td>( e_{p} )</td>
</tr>
<tr>
<td>di Prisco et al. (1993)</td>
<td>( f \neq g )</td>
<td>( I ) &amp; ( A_{y} )</td>
<td>( e_{p} )</td>
</tr>
<tr>
<td>Whittle (1994)</td>
<td>( f \neq g )</td>
<td>( I ) &amp; ( A_{y} )</td>
<td>( e_{p} )</td>
</tr>
<tr>
<td>Kiyama &amp; Hasegawa (1996)</td>
<td>( f \neq g )</td>
<td>( I ) &amp; ( A_{y} )</td>
<td>( e_{p} )</td>
</tr>
</tbody>
</table>

\( A_{x} \): Anisotropy by rotation at the origin of the coordinate \((p', q)\).

\( A_{y} \): Anisotropy by translational kinetic.

\( A_{z} \): Anisotropy on the octahedral plane with Lode’s angle.
the objectivity of a scalar, a vector and a tensor, as indicated in the introduction of the stress-induced anisotropy by Yatomi and Nishihara (1984) and a fabric tensor by Tobita (1989). The objectivity of a scalar-valued function $X$, composed of polynomial of the scalar and traces of the vector and the tensor, is attained when the following equation is satisfied for any orthogonal transformation $Q$.

$$X(\alpha, \varepsilon, a, W) = \hat{X}(Q \cdot \alpha \cdot Q^T, Q \cdot \varepsilon \cdot Q^T, Q \cdot a, W)$$

where the superscript $T$ means transpose. The objectivity rule is of great importance in introducing a yield function and a plastic potential function.

MODEL FORMULATION

Stress-Controlled and Strain-Controlled Path

In an elasto-plastic model proposed here, the total strain can be given as a sum of the elastic and the plastic components in the rate form.

$$\dot{\varepsilon} = \dot{\varepsilon}^e + \dot{\varepsilon}^p$$

where superscripts $e$ and $p$ express elastic component and plastic one, respectively. The elastic strain rate is related to the effective stress rate with an isotropic fourth-order elastic tensor $E$, while regarding compressive strains and stresses as positive.

$$\dot{\varepsilon}^e = E^{-1} : \dot{\sigma}^e.$$  

Two plastic strain rates can be expressed through the current stress state and the corresponding stress rate. When the current stress point is on the normal yield surface and the corresponding stress rate is outward from the surface, the plastic strain rate occurs in the normal consolidated region. This plastic strain rate is related to the unit normal tensor $m$, to the plastic potential surface: $g_\varepsilon = 0,$

$$\dot{\varepsilon}^p = \dot{\varepsilon}^p, m_n$$

where $A_\varepsilon$ is the scalar-valued plastic multiplier for the normal consolidated soil and its first-order rate is positive. The plastic strain rate can also occur within the over-consolidated region.

$$\dot{\varepsilon}^p = \dot{\varepsilon}^p, m_s$$

where $A_\varepsilon$ is the scalar-valued plastic multiplier for the over-consolidated soil and its first-order rate is also positive; and $m_s$ is the unit normal tensor to a plastic potential surface $g_\varepsilon = 0.$ The effective stress rate is related to the total strain rate:

$$\dot{\varepsilon} = D^{-1} : \dot{\sigma}^e$$

where $D$ is a fourth-order elasto-plastic tensor which is derived, after some calculations, as:

$$D = \hat{D}^{-1} \cdot E(\hat{H} \otimes I - m \otimes E : m)$$

where $I$ is a second-order identity tensor, $\sigma_p^e$ is hardening parameter, $m$ and $n$ are unit normal vectors to the plastic potential surface and the yield surface, respectively. Either $m$ or $n$ is used in place of $m$ in conformity to the loading condition mentioned above.

Two types of controlled processes: either a totally strain-controlled or a totally stress-controlled, as cited by Klisinski et al. (1992) and Mróz and Rodzik (1994). The totally strain-controlled process which is the inverse transformation of Eq. (10) is considered first; then, stress rate can be determined when the denominator part $\hat{H}$ of Eq. (11a) remains positive. This implies the ability to examine the post-peak behavior of the stress-strain curve to a limited degree. Second, considering the totally stress-controlled process, then $D^{-1}$ can be expressed as

$$D^{-1} = H^{-1} (HE + m \otimes n).$$

It is obvious that strain rate can be uniquely obtained from stress when $H$ is positive. The post-peak stress-strain curve cannot be determined because of the zero condition of $H$ at peak strength. This distinction between two types of control, the strain- and the stress-controlled path is significant in performing the strain hardening or softening regime.

The following invariant forms which describe the hydrostatic axis and the corresponding octahedral plane may be defined. The effective mean principal stress $p'$, deviatoric stress $q$, the stress ratio $\eta = q/p'$ and the stress ratio with the stress induced anisotropy $\eta^*$ are defined as

$$p' = \frac{1}{3} \sigma^e : I,$$  

$$q = \left(\frac{3}{2} \varepsilon : s\right)^{1/2},$$  

$$\eta = \frac{3}{2} \varepsilon : \eta,$$  

$$\eta^* = \left(\frac{3}{2} \varepsilon : \eta^*\right)^{1/2},$$  

$$\eta^* = \frac{s}{\eta^*} = \frac{s_0}{\eta^*},$$

The corresponding volumetric strain $\varepsilon_v$ and deviatoric strain $\varepsilon_d$ are given as

$$\varepsilon_v = \varepsilon : I,$$  

$$\varepsilon_d = \left(\frac{2}{3} \varepsilon : \varepsilon\right)^{1/2},$$  

$$\varepsilon_d = \varepsilon - \frac{1}{3} \varepsilon I.$$  

Elastic Relationship

The elastic strain rate related to the effective stress rate with the following $D^e$ as adopted by Sekiguchi and Ohta’s model (1977),

$$E = (\lambda \delta_{ij} \delta_{kl} + 2 \mu \delta_{ij} \delta_{kl}) e_{ik} \otimes e_{jk} \otimes e_{kl} = \mu (1 + v) / (\kappa (1 + v)),$$

$$\mu = 3v(1 - 2v) p' / (2\kappa (1 + v))$$

where, $\delta_i$ = the Kronecker delta, $v$ = void ratio, $\kappa$ = swelling slope of $v - \ln p'$ (Roscoe and Burland, 1968), $v$ = Poisson’s ratio.

Plastic Deformation on the Normal Yield Surface

Yield function and plastic function:

The plastic strain rate from Eq. (8) occurs when the fol-
lowing loading criteria is satisfied:
\[
\frac{\partial F}{\partial \sigma'}: \sigma' > 0 \quad \text{and} \quad F(\sigma', \sigma_p) = f(\sigma') - h(\sigma_p) = 0 \quad (16)
\]
where \( F \) is a normal yield function and \( h \) is its hardening part. The plastic potential function at the normal consolidated state \( g_0 \) can be derived by the stress-dilatancy relationship found in laboratory tests. The curvature of the potential surface governs the plastic strain rate vector.
\[
g_a(\sigma') = \ln \left( \frac{p'}{p_i} \right) + \frac{\xi}{\xi - 1} \ln \left( \frac{\xi M + (\xi - 1)\eta}{\xi M} \right) \quad (\xi \neq 1)
\]
\[
g_a(\sigma') = \ln \left( \frac{p'}{p_i} \right) + \frac{\eta}{M} \quad (\xi = 1)
\]
where the parameter \( \xi \) is simply determined by the stress-dilatancy relation as
\[
M - \eta = \xi \frac{\dot{\varepsilon}_p}{\dot{\varepsilon}^p} \quad (18)
\]
in which \( M \) is the value of the stress ratio at the transformed state. The associated flow rule will be used in the plastic description for the normally consolidated soils, thus
\[
F(\sigma', \sigma_p) = g_a(\sigma'). \quad (19)
\]
The normal yield function can also be defined considering the stress-induced anisotropy:
\[
F(\sigma', \sigma_p) = \ln \frac{p'}{p_i} + \frac{\xi}{\xi - 1} \ln \left( \frac{\xi M + (\xi - 1)\eta}{\xi M} \right) - h(\sigma_p) \quad (\xi \neq 1)
\]
\[
F(\sigma', \sigma_p) = \ln \frac{p'}{p_i} + \frac{\eta}{M} - h(\sigma_p) \quad (\xi = 1)
\]
\[
h(\sigma_p) = \ln p_i \quad (20a)
\]
where the hardening parameter \( p_i \) is a function of the plastic strain (this parameter will be examined in detail in the next section). Figure 1 presents the normal yield surface with the variation of the parameter \( \xi \) on the \( q-p' \) plane. The change in configuration of this surface implies that the volumetric plastic strain hardening rate increases when the parameter \( \xi \) is smaller. The first-order consistency condition of the normal yield surface (\( F = 0 \)) is confirmed with this equation, from the classical theory of plasticity.
\[
\frac{\partial F}{\partial \sigma'} : \sigma' - \dot{A}_n H_n = 0, \quad H_n = \frac{\partial h}{\partial \sigma_p'} \frac{\partial \sigma_p'}{\partial \sigma'} \frac{\partial \sigma_p}{\partial \sigma'}.
\]
\[
(21)
\]
Hardening rule:
The hardening parameter \( p_i \) is related to the volumetric plastic strain \( \dot{\varepsilon}^p \) based on the Cam clay model (1968).
\[
\dot{\varepsilon}^p = \lambda - \kappa \frac{\rho_i}{\rho_p} \quad (22)
\]
where \( \lambda \) is the compressive slope of the isotropic consolidation line. In addition, the deviatoric plastic strain hardening rule, proposed by Nova and Wood (1979), can be used to describe the hardening behavior with volumetric dilation having a higher stress ratio than the one in the transformed state:
\[
\dot{\varepsilon}^p = \phi(\lambda - \kappa) \frac{\rho_i}{\rho_p} \quad (23)
\]
where \( \phi \) is a parameter which can be determined once the stress ratio for the failure state is known. The parameter \( \phi \) has two characteristics. First, the additional deviatoric strain hardening produces a larger normal yield surface than the case where the volumetric strain hardening is only assumed such as in the Cam clay model. The higher stress state is consequently attained to increase the shear strength at the failure state. Second, the volume dilation can be conducted at a higher stress ratio than the one for the transformed state by distinguishing the two states: the transformed state and the failure one. The detailed determination of the parameter \( \phi \) is shown in the next section (See Fig. 2).

The partial differentiation of the plastic potential functions of Eqs. (17a, b) and the normal yield function of Eqs. (20a-c) with respect to the effective stress tensor yields:
\[
\frac{\partial g_n}{\partial \sigma'} = \frac{1}{3p'} \left( \xi M + (\xi - 1)\eta \right) I + \frac{3\xi}{\eta} \quad \{ (\xi M - \eta) I + \frac{3\xi}{\eta} \}
\]
\[
\frac{\partial f}{\partial \sigma'} = \frac{1}{3p'} \left( \xi M + (\xi - 1)\eta \right) I + \frac{3\xi}{\eta} \quad \{ (\xi M + (\xi - 1)\eta) I + \frac{3\xi}{\eta} \}
\]
\[
\times \left[ (\xi M - \eta) I + \frac{3\xi}{\eta} \right].
\]
\[
(24)
\]
\[
(25)
\]
So \( H_n \) of Eq. (21b) is calculated as
\[
H_n = -\frac{\nu(\xi M + \phi - \eta)}{\rho_p(\lambda - \kappa)(\xi M + (\xi - 1)\eta)}. \quad (26)
\]
Substituting Eqs. (25), (26) into Eq. (21), a plastic multiplier is determined to be
\[
\dot{\sigma} = \frac{(\lambda - \kappa)
\left[(\xi M + (\xi - 1)\eta^* - \xi\eta^*)I + \frac{3\xi}{\eta^*} I\right] \sigma'}{\nu(\xi(M + \phi) - \eta)} .
\]

(27)

The plastic deformation can be implemented as long as the denominator of Eq. (27) is positive; therefore, the post-transformed state may be simulated by introducing the parameter \( \varphi \).

**Plastic Deformation on the Sub-Yield Surface**

Sub-yield function and its plastic potential function:

It is assumed that the plastic deformation area must exist within the normal yield surface, which has the following loading condition:

\[
\frac{\partial F_s}{\partial \sigma'}: \sigma' > 0
\]

and

\[
F_s(\sigma', \sigma_{\text{fr}}, \sigma_{\text{up}}) = f_s(\sigma', \sigma_{\text{fr}}) - h_s(\sigma_{\text{up}}) = 0
\]

(28)

where the subscript \( s \) indicates the sub-yielding quantity, \( \sigma_{\text{fr}} \) is a hardening parameter of the function \( f_s \), and \( \sigma_{\text{up}} \) belongs to another hardening part of Eq. (28). The isotropic-kinematic hardening of the sub-yield surface may be formulated with a unit structure tensor \( \alpha \) which describes the anisotropic stress state.

\[
F_s(\sigma', \sigma_{\text{fr}}, \sigma_{\text{up}}) = \ln \tilde{\rho}' + \frac{\eta}{\tilde{\rho}'} - h_s(\sigma_{\text{up}}),
\]

(29a)

\[
h_s(\sigma_{\text{up}}) = \ln \tilde{\rho}^c
\]

(29b)

where a series of invariants of stress and strain are introduced as

\[
\tilde{\rho}' = \sigma': \alpha, \quad \tilde{q} = \left( \frac{3}{2} \tilde{\rho}' \tilde{s} : \tilde{s} \right)^{1/2}, \quad \eta = \tilde{\rho}^c \tilde{\rho}^{1/2}, \quad \tilde{s} = \sigma' - \tilde{\rho}' \alpha,
\]

(30)

\[
\tilde{e} = \varepsilon : \alpha, \quad \tilde{\varepsilon} = \left( \frac{2}{3} \tilde{e} : \tilde{e} \right)^{1/2}, \quad \tilde{\varepsilon} = \varepsilon - \tilde{\varepsilon} : \alpha.
\]

(31)

Figure 3 illustrates this decomposition (\( \tilde{\rho}', \tilde{q} \)) in contrast with the one (\( \rho', q \)) followed by the isotropic structure tensor. The circle with a radius of \( q \) is not geometrically on the octahedral plane with a circle with a radius \( q \). It is assumed that the normal yield surface geometrically contains the sub-yield surface, so that the kinetics of this sub-yield surface are restricted under this condition.

Plastic potential functions for compressive load and extensive one are formulated based on the idea with reference to Adachi and Oka (1982). For a compressive loading process as shown in Fig. 4(a), the plastic potential function is given as

\[
g_s(\sigma') = \ln \left( \rho' \rho' \right) + \left( \frac{\xi}{\xi - 1} \right) \ln \left( \frac{\xi M_r + (\xi - 1)\eta}{\xi M_r} \right) \quad (\eta \neq 1)
\]

(32a)

\[
g_s(\sigma') = \ln \left( \rho' \rho' \right) + \frac{\eta}{M_r} \quad (\xi = 1)
\]

(32b)

where the parameter \( M_r \) under the triaxial compression is automatically determined once the current stress state is known.

\[
M_r = \eta \left( \frac{\xi - 1}{\xi} \right) \left( \frac{\rho'}{\rho'} \right)^{-(\xi - 1)} \quad (\eta \neq 1)
\]

(33a)

\[
M_r = -\eta \ln \left( \frac{\rho'}{\rho'} \right) \quad (\xi = 1).
\]

(33b)

For the triaxial extension state as shown in Fig. 4(b), the plastic potential function is then expressed with stress-induced anisotropy by Yatomi and Nishihara (1984) as

\[
g_s(\sigma') = \ln \left( \rho' \rho' \right) + \left( \frac{\xi}{\xi - 1} \right) \ln \left( \frac{\xi M_r + (\xi - 1)\eta}{\xi M_r} \right) \quad (\eta \neq 1)
\]

(34a)

\[
g_s(\sigma') = \ln \left( \rho' \rho' \right) + \frac{\eta}{M_r} \quad (\xi = 1)
\]

(34b)

where the parameter \( M_r \) for the loading in extension can also be determined with \( \eta^* \) in place of \( \eta \).

Fig. 2. Undrained stress path with variation of \( \varphi \)

Fig. 3. Schematic diagram of transformation of stress invariants for different structure tensor
\[ M_s = \eta^* \left( \frac{\xi - 1}{\xi} \right) \left( \frac{p'}{p'_f} \right)^{\frac{1}{(\xi - 1)}} - 1 \right)^{-1} \quad (\xi \neq 1) \] (35a)

\[ M_s = -\eta^* \ln \left( \frac{p'}{p'_f} \right) \quad (\xi = 1) \] (35b)

Hardening rule:

The size and the orientation of the sub-yield surface given by Eqs. (28a), (29b) are controlled by the three hardening parameters \( \tilde{m}_f, \tilde{p}_f, \) and \( \alpha, \) which are functions of plastic strains. The former two parameters express the isotropic hardening and the latter one is for the kinematic hardening.

A parameter \( \tilde{m}_f \) is given from a postulation of a hyperbolic relation of \( \tilde{m} \) versus \( \tilde{e}^p. \)

\[ \dot{\tilde{m}} = \frac{\tilde{G}(\tilde{m}_f - \tilde{m})^3}{\tilde{m}_f^2} \dot{\tilde{e}}^p \] (36)

where \( \tilde{G} \) is a shear modulus, and \( \tilde{m}_f \) assumed to be the maximum value of \( \tilde{\eta} \) which does not exceed the maximum stress ratio \( \eta_f \) at the failure state on the \( (p', q) \) coordinate. The other parameter \( p'_f \) is introduced in the same manner as \( p'_c \) of Eq. (22) for the normal consolidated state with a positive parameter \( b. \)

\[ \dot{p}'_c = bp'_c \dot{\tilde{e}}^p. \] (37)

A structure tensor \( \alpha \) gives the expression for the axial direction of the sub-yield surface, whose orientation is altered by geometrical restriction: the sub-yield surface is enclosed within the normal yield surface. A kinematic rate, which is related to this \( \alpha, \) may be given with the effective stress rate as,

\[ \dot{\alpha} = k(\tilde{e}^p, \tilde{e}^p)^{1/2} \chi \] (38)

where

\[ \chi = \gamma - (\gamma, \alpha)\alpha, \quad \gamma = \tilde{\sigma}' - \tilde{\sigma}' \] (39)

and where Fig. 5 presents stress tensors \( \chi \) and \( \gamma \) which conform with the first consistency condition of the unit structure tensor: i.e.

\[ \alpha : \dot{\alpha} = 0. \] (40)

The first-order consistency condition of Eq. (28) for the plastic deformation at overconsolidated state can be expressed as,

\[ \frac{\partial f_s}{\partial \sigma'} \dot{\sigma}' - \Lambda \dot{H}_s = 0, \]

\[ H_s = \left[ \begin{array}{cccc} \partial \sigma_{sp} & \partial \sigma_{sp} & \partial \sigma_{sp} & \partial \sigma_{sp} \\ \partial \sigma_{sp} & \partial \sigma_{sp} & \partial \sigma_{sp} & \partial \sigma_{sp} \\ \partial \sigma_{sp} & \partial \sigma_{sp} & \partial \sigma_{sp} & \partial \sigma_{sp} \end{array} \right] \left( \begin{array}{c} \partial g_s \\ \partial g_s \\ \partial g_s \\ \partial g_s \end{array} \right). \] (41)

Furthermore, \( H_s \) is expressed in the case of proposed plastic potential functions of Eqs. (29a), (29b).

\[ H_s(\alpha, \tilde{m}, \tilde{p}_c) = \frac{\partial h_s}{\partial \alpha} \frac{\partial h_s}{\partial \tilde{m}} \frac{\partial h_s}{\partial \tilde{p}_c} + \frac{\partial f_s}{\partial \alpha} B + \frac{\partial h_s}{\partial \tilde{p}_c} C \] (42)
where

\[ A = \left( \frac{\partial g_x}{\partial \sigma^y} \frac{\partial g_y}{\partial \sigma^z} \right)^{1/2} \lambda, \quad B = \frac{\partial m}{\partial \tilde{e}} \left( \frac{\partial g_x}{\partial \sigma^y} + \frac{\partial g_y}{\partial \sigma^z} \right) \xi, \quad C = \frac{\partial \sigma}{\partial \tilde{e}} \frac{\partial g_x}{\partial \sigma^y} + \frac{\partial \sigma}{\partial \tilde{e}} \frac{\partial g_y}{\partial \sigma^z} \chi. \]

The partial differentiation of the sub-yield function with respect to the current effective stress tensor \( \sigma' \) and the structure tensor \( \alpha \) are given by

\[
\frac{\partial f_s}{\partial \sigma'} = \frac{1}{\bar{p}} \left( (\bar{m} - \eta') \sigma' + \frac{1}{\eta} \bar{q} \right),
\]

\[
\frac{\partial F_s}{\partial \alpha} = \left( \frac{\partial h_s}{\partial \alpha} - \frac{\partial f_s}{\partial \alpha} \right) = (\bar{m} - \eta') \sigma' + (\bar{m} - \eta) \eta.
\]

The gradient of the plastic potential function of Eqs. (32a, b) with respect to the effective stress tensor \( \sigma' \) are calculated by substituting \( M_4 \) for \( M \). In addition, gradients of Eqs. (34a, b) are also determined with \( M_4 \) instead of \( M \) in Eq. (24). The rate of the plastic multiplier \( A \) for the overconsolidated state is calculated by substituting Eqs. (41b), (43a-c) and (44a, b) into Eq. (41a) and taking the stress derivative of the plastic potential function into consideration.

**ANALYSIS OF MODEL PREDICTION**

**Identification of Parameters**

Before simulating the material behavior, we need to

![Diagram of material parameters](image-url)

**Fig. 6(a)**. Determination of material parameters
make careful identifications of material parameters for the desired precision to numerical predictions. The number of parameter is nine, five of which belong to the normal yield surface and four of which belong to the sub-yield surface. The determination of all parameters is shown in Fig. 6(a).

Parameters for the normal yield surface:

Three of these parameters, \( \kappa, \lambda \) and \( M \) are a series from the Cam clay model, determined from an isotropic consolidation test and a triaxial compression to be: \( \kappa = 0.009114, \lambda = 0.07812 \) and \( M = 1.38 \) (see Fig. 6(a)). An additional parameter \( \varphi \) can be obtained once the stress ratio at failure state is known from an undrained triaxial compression test. First, substitute Eq. (27) into Eq. (8); add a deviatoric plastic strain increment \( d\varepsilon^p \), which can be integrated from the isotropic stress state to the failure state \( \eta_f \).

\[
\int_{\eta_f}^{+\infty} d\varepsilon^p = \int_0^{\eta_f} \frac{(\lambda - \kappa)\xi^2}{(M + (\xi - 1)\eta)(\lambda(M - \eta) + \varphi\kappa\xi)} d\eta.
\]

Finally, we define the parameter \( \varphi \) as

\[
\varphi = \frac{\lambda(\eta_f - M)}{\kappa\xi}.
\]

Figure 2 shows an example of controllable, effective stress paths during the undrained shear. It is clear that as the parameter \( \varphi \) increases, the material is stronger with high sensitivity of volumetric dilation. The parameter \( \xi \) is taken from Eq. (18) by measuring the stress state and the plastic strain rate at any loading condition in the following. The plastic strain can become the different vector by the rate of development of the plastic hardening even while the stress ratio remains the same. The difference between \( \varphi \) and \( \xi \) should be emphasized: the former, \( \varphi \), controls the size of the normal yield surface without its distortion, and the latter, \( \xi \), distorts the shape of the normal yield surface. The values \( \varphi = 0.51 \) while \( \xi = 1.0 \) from the stress-dilatancy relation of the silty clay are determined (see Fig. 6(a)).

Parameters for the sub-yield surface:

If the sub-yield surface is rotated around the origin \( O \) to describe the anisotropy of materials which is a consequence of the anisotropic consolidation or shearing process, the four parameters \( \tilde{G}, \tilde{m}, \tilde{p}^*, \kappa \) and \( \beta \) cannot be determined from the fixed plane \( (p', q) \). We can, however, determine these parameters from the infinitesimal deformation in the oriented plane \( (\tilde{p}^*, \tilde{q}) \) by an approximation procedure. We can assign the value of \( \eta_f \) to \( \tilde{m} \), noting that the tip angle of the cone is the maximum for the isotropic stress state. The shear modulus \( \tilde{G} \) in Eq. (35) is determined from the relationship of \( \tilde{m} \) and \( \tilde{e}^p \) with unloading, as shown in Figs. 6(a), (b). A kinematic hardening parameter, \( \kappa \), is evaluated from the effective stress path during the unloading-reloading process in the undrained shear as shown in Fig. 6(a). The parameter \( \beta \) controls the location of the central axis of the oriented sub-yield surface which assumes coincidence with the central axis of the elastic region, which is determined from the constant \( p' \) stress path for the unloading during the undrained shear. Finally, a volumetric plastic hardening parameter \( b \) in Eq. (36) can be obtained from an anisotropic consolidation test as seen in Fig. 6(a), (c). Consequently, these parameters for the silty clay were determined to be: \( \tilde{m} = 1.45, \tilde{G} = 368, \kappa = 0.5 \) and \( b = 48.7 \).

Comparison of Numerical Behaviors with Experimental Ones

In what follows, we shall restrict examination to the triaxial testing condition; that is, when two effective principal stress values are equal, \( \sigma_1^e = \sigma_3^e \), and directions of principal stresses are fixed for the soil element. The soil used is silty clay. A series of test procedures and experimental results is shown in a separate paper (Kiyama and Hasegawa, 1996).

Undrained compression test on the isotropically consolidated silty clay:

The representative behavior of some soil conditions un-
under the undrained compression test were examined. Experimental results of the soil which was saturated (B value > 0.95 at the back pressure of 196 kPa) and was subjected at the strain rate of 0.01%/min were used. Figures 7(a)–(c) present numerical data on the response of soils initially given the isotropic consolidation at \( p' = 200 \) kPa, or after partial unloading to \( p' = 150, 100, 73, 50, 20 \) and \( 10 \) kPa, respectively. The initial values of the overconsolidation ratio, OCR, for each condition are 1, 1.3, 2, 2.7, 4, 10 and 20. As shown in Fig. 7(a), all the undrained stress paths describe volumetric dilation at the higher
stress ratio than at the transformed state. In addition, volumetric dilation appeared at any stress state for OCR > 2.7 within the overconsolidated region, whereas volumetric contractions appeared only for OCR < 2.7. This characteristic variation appears reasonable because numerical stress paths actually produce similar undrained stress paths obtained as laboratory tests as shown in Fig. 8(a). Figure 7(b) shows q-e curves, which indicate q has a greater reduction at any equal deviatoric strain, e, as the value of OCR increases. Excess pore water pressure p_e-e relationships are shown in Fig. 7(c), where p_e gradually degrades at the post-peak condition for all the numerical simulations. It was concluded that numerical analysis with the proposed model can describe soil characteristics under conditions C1-1 to C1-3 correctly, by comparison with Figs. 8(b), (c) describing experimental results. It is difficult, however, to maintain an appropriate description in the model as the deviatoric strain increases.

Influence of stress history on shear strength:

Undrained shear tests beginning at the same stress state after different controlled stress histories are considered. Figure 9 presents these schematic effective stress paths for two specimens, TAS1 and TAS2. Referring to another paper (Kiyama et al., 1995(a)) for more detailed experimental content, there is a significant difference between the two in that TAS2 is subjected to the overconsolidated process, whereas the entire stress history of TAS1 is a normal consolidation process. The predicted material responses are shown in Figs. 10(a), (b) where relations of ε_p, p' and ε_p', respectively, are presented for the path CDB during reloading process. Both numerical representations can fairly well describe the experimental behavior of TAS2.

For the undrained behavior of TAS1 and TAS2: TAS1 has ε = 1; TAS2 has ε = 1.54 at the same stress point B from the stress-dilatancy relation of Eq. (18). Figure 10(c) compares derived stress paths with experimental ones numerically. The predicted values show a difference about 15% for q at the transformed state from the actual experimental values. Figures 10(d), (e) present predicted q-e and p_e-e relation with experimental results, respectively. The distinction of predicted behavior between TAS1 and TAS2 is significant, in particular during the initial stage of the undrained shear, because of the employment of the overconsolidated plastic deformation and the stress-dilatancy relationship.

![Fig. 10(a). ε_p versus p' during const. q/p' (=0.5) reloading test](image1)

![Fig. 10(b). ε_p versus p' during const. q/p' (=0.5) reloading test](image2)

![Fig. 10(c). Numerical and experimental undrained stress paths](image3)
A more important addition to this discussion is a generalization of changes of shear strength. Examining the controlled stress paths beginning after the unloading stress path OA (see Fig. 11(a)), the two cases of the kinematic hardening parameter \( k = 0.1 \) and \( k = 0.5 \) are noted. Solid curves express boundaries between the states of normal consolidation and overconsolidation. It was shown that increasing kinematic hardening restricts the amount of plastic hardening. The kinematic hardening rate, as shown in Fig. 11(b), results in a variation of shear strength for the \( \zeta = 0 \) material when undrained shears are conducted at any stress state on the dotted curve which expresses the initial normal yield surface (see Fig. 11(a)). The incremental increase of the shear strength therefore is the maximum in proportion to the enlarged area of the overconsolidation shown in Fig. 11(a).

Cyclic loading plasticity:

A circulatory stress history was examined as is seen in Fig. 12(a) and its behavior in the numerical model and by experiment. It is important to account for the description either of a volume contraction or dilation, as mentioned in condition C3 from Chapter "REMARKS ON MODELLING" particularly as the volume dilation occurs immediately after the abrupt change of the stress path such as OAB. The nonassociated flow rule is helpful in expressing this volumetric behavior precisely because the associated flow rule requires the unloading process, i.e., volumetric elastic swelling. The detailed explanation is presented in separate papers (Kiyama and Hasegawa, 1996). Figure 12(b) presents the numerical and experimental \( \epsilon_p - p' \) relations, in which the volume change can be estimated reasonably for the path OAB. Figures 12(c), (d) present the predicted behavior of the \( q/p' - \epsilon \) and the \( q/p' - \epsilon_p \) relations, respectively. It has been concluded by comparison with experimental results in Figs. 12(c), (f), the \( q/p' - \epsilon \) curve and the \( q/p' - \epsilon_p \) curve respectively, that the predicted behavior can reasonably describe the experimental one.

Material state in overconsolidated region:

A further inspection is necessary in order to observe whether the material state maintains stability or not when the nonassociated flow rule is applied. This paper suggests that the associated flow rule has a fundamental disadvantage in this situation: i.e., a prediction opposite to the actual material behavior at the initial stage of the stress path AB. The material behavior seems to be stable from the experimental confirmation that this behavior occurs at the lower stress ratio \( (q/p' < 0.5) \) than the one at failure state \( (q/p' = 1.45) \) and the strain is observed to be less than 0.1%. The material stability proposed by Drucker and Hill, however, can remain based on the condition that the minimum eigen value of a constitutive matrix is persistently nonnegative. In the triaxial condition, the stress-strain relation is expressible in terms of
the controlled stress path with Eqs. (10), (12).

\[
\begin{bmatrix}
\dot{\varepsilon} \\
\dot{\varepsilon}_e
\end{bmatrix} = [D] \begin{bmatrix}
\dot{q} \\
p'
\end{bmatrix},
\]

\[
[D] = \frac{\bar{H}}{H G K} \begin{bmatrix}
G(H+Kn_fm_p) & -GK_n_m_q \\
-GKn_nm_p & K(H+G_n_m_q)
\end{bmatrix}
\]

(47)

where

\[
G = 2\bar{\mu}, \quad K = \frac{1}{3}(3\bar{\lambda} + 2\bar{\mu}), \quad \bar{H} = H + G_n_m_q + Kn_p m_p.
\]

(48)
It is obvious that the nonassociated flow rule generally makes the constitutive matrix of Eq. (47) nonsymmetric because of \( m_p \neq n_p \) and \( m_q \neq n_q \).

Eigen values \( \lambda_{ij} ; i, j = 1, 2 \) for the constitutive matrix of Eq. (47) are calculated:

\[
\begin{align*}
\lambda_{ij} &= \frac{b \pm \sqrt{d}}{2H}, \\
b &= (H(K+G)+GK(n_p m_p + n_q m_q)), \\
d &= \left[ (H(K-G)-GK(n_p m_p - n_q m_q))^2 \\
 &\quad + 2G^2 K(n_p m_p^2 + n_q m_q^2) \right]^{1/2} \\
\end{align*}
\]

(49)

where the following relation exists when the strain hardening develops \((H > 0)\):

\[
b - \sqrt{d} > 0.
\]

(50)

This inequality guarantees a positive for the minimum eigen value. Accordingly, one-to-one mapping can be conducted perfectly for the strain hardening process. The minimum eigen value remains positive, indicating stability, regardless of any asymmetrical matrix at any pre-failure state during the path AB. The second increment of work, however, has negative: \( \Delta \sigma' : \Delta \varepsilon < 0 \), which is regarded as unstable by the Drucker and Hill postulations. This contradiction indicates that there is no unique judgment of stable condition for the nonassociated flow rule because Drucker and Hill exclude the case of the non-symmetric matrix from their discussion.

In the eigen value problem for the symmetric part \( D_1 \) of the matrix \( D \) as cited by Bazant and Cedolin (1991), the condition \( \det |D| = 0 \) is defined as a ‘critical state of neutral equilibrium’ and \( \det |D_1| = 0 \) as ‘critical state of stability limit’. Each critical state can be evaluated separately from the minimum eigen value \( \lambda_{\text{min}} \) for matrix \( D \) and \( \lambda_{\text{min}} \) for matrix \( D_1 \):

\[
[D_1] = \frac{\hat{H}}{HKG} \begin{bmatrix} G(H+K m_p) & -GK(n_p m_p + n_q m_q)/2 \\ -GK(n_p m_p + n_q m_q)/2 & 2K(H+G m_q) \end{bmatrix}
\]

(51)

\[
\lambda_{(q)} = \frac{b_1 \pm \sqrt{d_1}}{2H} , \quad b_1 = b , \\
d_1 = d + \hat{H}^2 G^2 K^2 (m_p n_q - m_q n_p)^2.
\]

(52)

The second term of right side in Eq. (52c) indicates that we have a non-symmetric matrix; thus the following inequality also holds true for the strain hardening process.

\[
b_1 - \sqrt{d_1} < 0.
\]

(53)

Figure 13 presents the transition of minimum eigen values \( \lambda_{\text{min}} \) and \( \lambda_{\text{min}} \) during the controlled stress path OAAB. It can be seen that \( \lambda_{\text{min}} \) increases but keeps a positive immediately after passing stress point A. \( \lambda_{\text{min}} \) decreases to a negative during the same stress state. It was concluded that this behavior was considered numerically as an unstable behavior. Figure 13 also shows a phenomenon that the instability disappears both numerically and experimentally, resulting in stability as the controlled loading of path AB continues. The stable condition for the material which is subject to the effect of an abrupt change of stress path must be clarified. This is because classical potential methods by Drucker, Hill and Bazant and Cedolin have not stated physically the discontinuity of the potential curve which may exist at the directional change of applied stress path.

**CONCLUSIONS**

The proposed model meets the reasonable requirement of accuracy for the description of inelastic behavior of normally consolidated and overconsolidated soils. The modelling is made with a combination of isotropic and kinematic hardening, the latter of which relates to the anisotropy induced by the corresponding plastic deformation to the controlled stress history. Although the proposed model consists of as many as nine material parameters, all of which can be determined by a few triaxial tests. The remaining problem is the lack of a smooth transition from the overconsolidated region to the normal consolidated one due mainly to the employment of a non-similitude surface, which may requires further study. The change in shear strength and the corresponding stress-strain behavior observed in experiments can be predicted with reasonable accuracy by using the proposed model with actual material parameters.

**REFERENCES**


