NUMERICAL SIMULATION OF BEARING CAPACITY CHARACTERISTICS OF STRIP FOOTING ON SAND

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ABSTRACT

This study aims at numerically simulating the bearing capacity characteristics of strip footing on sand, and consequently explaining the scale effects observed in a series of plane strain model tests carried out on a particular type of sand (Toyoura sand). The model tests were performed using different sizes of footing, with the largest width being 50 cm, under normal gravity and in a centrifuge. A constitutive model developed for Toyoura sand based on the results from an extensive series of plane strain compression tests is used. A number of factors which affect the strength and deformation of sand are taken into account, including; 1) confining pressure; 2) anisotropy; 3) non-linear strain-hardening and strain-softening; 4) dilatancy; and 5) strain localization into a shear band(s). This material model is coupled with an isotropically hardening, non-associated, elasto-plastic material description. A widely used numerical technique, FEM, is applied to solve the non-linear equations. A parametric study is performed to evaluate the effects of using different assumptions for the material model. The simulation of bearing capacity compares well with the physical model test results. It is shown that the isotropic perfectly plastic modelling of soil property, assumed in most of the classical bearing capacity theories, is an overly simplified approximation to be used in FEM analysis of this issue. It is explained that the scale effect consists of the pressure level effect and the particle size effect.

Key words: bearing capacity, centrifuge, finite element method, model test, scale effect, strain localization (IGC: E3/E14)

INTRODUCTION

Most of the classical theories for the bearing capacity of a strip footing placed on sand are based on the assumption that the stress and strain characteristics of sand are isotropic and perfectly plastic. When following this assumption, for a homogeneous sand deposit, the ultimate footing load is attained as a constant peak friction angle ϕ_p , defined as the arcsine of the peak value of $(\sigma_1 - \sigma_3)/(\sigma_1 + \sigma_3)$, which is mobilized simultaneously along the failure plane(s). In reality, dense sand has a more-or-less strain-softening post-peak property, and therefore, a mass of dense sand fails progressively in the sense that the peak strength ϕ_p is by no means mobilized simultaneously along the potential failure plane(s). A failure plane is in fact a zone in the three-dimensional sense, or a band in the two-dimensional sense, having a certain width (or thickness), which is a function of particle size, among other factors (Vardoulakis et al., 1978; Yoshida et al., 1995; Yoshida and Tatsuoka, 1997). Therefore, strain fields in a failing sand mass are controlled by shear band-

ing pattern, which is a function of the ratio of the shear band width, or the particle size, to the footing size. This factor could result in effects of the relative particle size on the bearing capacity factor N_γ .

The decrease in the bearing capacity coefficient for soil weight N_γ with the increase in the footing size under the normal gravity is called the scale effect. The increase in the size of a footing placed on a given homogeneous sand deposit results not only in a decrease in the ratio of the particle size to the footing width, but also in an increase in the pressure level at the corresponding point in the sand mass (e.g., at a depth of the footing width along the footing center line). The latter factor could be called the pressure level effect (e.g., Tatsuoka et al., 1991, 1995). In the past, the pressure level-dependency of ϕ_p has often been considered as the unique cause for the scale effect. However, the pressure level-dependency of deformability also increases the degree of progressive failure, which in turn decreases the N_γ value. The combined effects of the particle size effect and the pressure level effect lead to a gross decrease in the N_γ value. Accordingly, the scale

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effect can be considered to consist of the particle size effect and the pressure level effect (Tatsuoka et al., 1991, 1995). Solely the pressure level effect can be studied by performing centrifuge tests using a single footing placed on the same type of sand at different acceleration levels.

To study into the scale effect, however, a very comprehensive series of well-controlled physical model tests is necessary. For the purpose of validating the classical bearing capacity theories, a number of plane strain model bearing capacity tests have been performed using a model strip footings on sand (e.g., Ko and Davidson, 1973). In this type of model tests, the effect of the side wall friction should be minimized and, if necessary, this effect should be taken into account when interpreting the model test results (Bransby and Smith, 1975). For a series of plane strain model tests using model footings having widths ranging from 0.5 cm to 10 cm (as summarized in Tatsuoka et al., 1991, 1995), Tani (1986) and Morimoto (1990) attempted to obtain reliable results by carefully lubricating the side walls of the sand box using a thin rubber membrane smeared with a thin layer of silicon grease as used for lubricating the ends of specimen for triaxial compression and plane strain compression tests (Tatsuoka et al., 1984; Goto et al., 1993). In addition, the footing load was measured at the central third of the footing to eliminate the effects of wall friction, if any. In addition, a series of both centrifuge tests with footing widths of 2 cm and 3 cm and large scale 1 g model tests with footing widths of 23 cm and 50 cm were also performed under otherwise the same plane strain conditions, using the same type of sand (Toyoura sand), at the Public Works Research Institute (PWRI), the Ministry of Construction.

For analyzing plane strain model test results, it is also essential to use the strength and deformation parameters of the test sand obtained from relevant plane strain shear tests (e.g., the plane strain compression tests; the PSC tests). The test results from recent PSC tests (Tatsuoka et al., 1986; Tatsuoka and Shibuya, 1992; Park and Tatsuoka, 1994) have re-confirmed the fact that the strength and deformation characteristics of sand re-constituted by pluviating sand particles through air or water, or by vertical compacting, is considerably anisotropic (Oda, 1972). That is, the peak friction angle ϕ_p is a distinct function of the angle δ of the direction of the major principle stress σ_1 relative to the direction of the bedding plane, which is horizontal when preparing the specimen. Such anisotropy as described above is not taken into account in most of the classical bearing capacity theories.

The scale effect should be linked to the progressive failure of ground, which is controlled by the pattern of shear band formation and the associated strain-softening behaviour, and therefore, their detailed and relevant formulation is essential for the present study. Yoshida et al. (1995) and Yoshida and Tatsuoka (1997) obtained, for a large variety of sands including Toyoura sand, the post-peak relationships among shear stress level, shear deformation and dilatancy of shear band by carefully observing the deformation of plane strain surface of each PSC

specimen. In addition, the width and direction of shear band were evaluated.

The objective of this paper is, therefore, to simulate the results of the well-controlled plane strain model tests using a fine quartz-rich sand (Toyouura sand) in terms of load-settlement relationships, bearing capacity factor N_γ , settlement characteristics and strain fields in the ground, based on the detailed strength and pre- and post-peak deformation characteristics of the sand obtained from a series of PSC tests. A widely used numerical technique, FEM, was used, taking into account the relevant factors, including anisotropy, pressure-level dependency and shear banding, among others, in modelling the strength and deformation characteristics of Toyoura sand. In this way, simulation was attempted of; 1) the scale effect, as observed in the 1 g model tests using different footing sizes placed on the same type of stand; 2) the pressure-level dependency of the bearing capacity characteristics, as observed in the centrifuge model tests using the same footing placed on the same type of sand performed at different acceleration levels; and 3) the particle size effect, as observed as differences between the above two.

MATERIAL MODEL AND FEM DETAILS

The material model used in this study is a generalized elasto-plastic, isotropic strain hardening-softening one, described in Tanaka and Kawamoto (1988), Tatsuoka et al. (1993) and Tanaka et al. (1997). The anisotropic material stress-strain property is taken into account by an iteration procedure; at each step of computation, the deformation and strength characteristics are re-defined continuously and repeatedly as a function of the computed angle δ between the instantaneous σ_1 direction and the horizontal bedding plane direction until the computed value of δ converges to the assumed value from which the current tangential deformation and strength characteristics at each point were defined. The isotropic yielding property in the model has no direct bearing on the results of analysis, as only monotonic loading cases are analyzed in the present study. As shown later, a generalized hyperbolic equation, which is described in detail in Tatsuoka et al. (1993), was used as the growth function of the yield surface. The constitutive equation of sand is also summarized in Kotake et al. (1999). The yield surface used is a generalized Mohr-Coulomb one given by;

$$\Phi = -\eta I_1 + \frac{1}{g(\theta)} \sqrt{J_2} - K = 0 \quad (1)$$

where I_1 is the first invariant of stress (i.e. hydrostatic stress component; positive in compression), J_2 is the second invariant (i.e., deviatoric stress) and $g(\theta)$ is the Lode angle function, which is defined as;

$$g(\theta) = \frac{3 - \sin \phi_{\text{mob}}}{2\sqrt{3} \cos \theta - 2 \sin \theta \sin \phi_{\text{mob}}} \quad (2)$$

η is the ratio of the deviatoric stress to I_1 at $\theta = 30^\circ$ on the π -plane, which is related to the mobilized angle of internal friction ϕ_{mob} as;

$$\eta = \frac{2 \sin \phi_{\text{mob}}}{\sqrt{3(3 - \sin \phi_{\text{mob}})}} \quad (3)$$

The plastic potential is defined as;

$$\Psi = -\alpha' I_1 + \sqrt{J_2} - K = 0. \quad (4)$$

This equation is of the Drucker-Prager type and similar to that of the yield surface, except for the difference in that $g(\theta)$ in Eq. (1) is equal to 1.0 in Eq. (4). This equation was employed so as to have differentiability at all the stress states (Tanaka and Kawamoto, 1988; Tanaka et al., 1997). The factor α' depends on the type of analysis. As all the analysis was done under plane strain conditions in the present study, the following factor α' was used;

$$\alpha' = \frac{\tan \psi}{\sqrt{9 + 12 \tan^2 \psi}} \quad (5)$$

where Ψ is the mobilized angle of dilatancy, which is given by;

$$\psi = \arcsin \left(-\frac{d\varepsilon_1^p + d\varepsilon_3^p}{d\varepsilon_1^p - d\varepsilon_3^p} \right). \quad (6)$$

Here $d\varepsilon_1^p$ and $d\varepsilon_3^p$ are the major and minor principal strain increments (positive in compression). In this study, the value of ψ was determined based on Rowe's stress dilatancy relationship (Rowe, 1962);

$$\frac{\sigma_1}{\sigma_3} = -K \left(\frac{d\varepsilon_1^p}{d\varepsilon_3^p} \right) \quad (7)$$

where K is a material constant. As the model has the yield function and plastic potential surface in different forms, it is a non-normal plasticity model, or a non-associated flow model (Vermeer and de Borst, 1984).

Generally four-noded quadrilateral elements show a stiff response. In the present study, to obtain an accurate solution, reduced integration (Zienkiewicz et al., 1971) was used. As reduced integration in a four-noded isoparametric element can give rise to the zero-energy mode (i.e., the hour-glass mode), an anti-hourglass scheme (Flanagan and Belytschko, 1981) was used to prevent this happening. A careful choice of parameters is required for the hour-glass mode control in a given materially non-linear analysis. In the present study, an elastic stiffness approach was employed, where a very small elastic stiffness was added as an hour-glass resisting nodal force to the non-linear system whenever any element started to form a hour-glass mode. A value of 0.05% of the actual material stiffness was found relevant after some trial runs. In fact, this procedure effectively prevented a hour-glass mode while it did not increase the stiffness of the solution. More details are reported in Siddiquee et al. (1996).

A system of non-linear equations was solved by the dynamic relaxation (DR) technique (Tanaka and Kawamoto, 1988; Siddiquee et al., 1996; Tanaka et al., 1997), which is known to be relevant for solving highly non-linear equations, especially for high friction angle materials, as in the present study. The integration of the elasto-plastic equations was done by the return mapping

Fig. 1. Strain-softening behaviour of three different granular materials from special PSC tests (Yoshida et al., 1995; Yoshida and Tatsuoka, 1997)

scheme (Ortiz and Simo, 1981), which is a first-order-approximated Euler backward integration. Strain localization is captured by using a mesh size-dependent hardening modulus, as described in Tanaka and Sakai (1993) and Siddiquee et al. (1996), and also in the later part of this paper. The strain-softening parameters were obtained from the results of an extensive series of PSC tests with the measurement of local shear strains inside each shear band (Yoshida et al., 1995; Yoshida and Tatsuoka, 1997). Figure 1 shows the post-peak relationships between the shear stress level R_n and the shear deformation u_s obtained from special PSC tests on dense specimens of Toyoura sand, Silver Leighton Buzzard sand and Hime gravel. R_n is defined as $\{(\sigma_1/\sigma_3) - (\sigma_1/\sigma_3)_{\text{res}}\} / \{(\sigma_1/\sigma_3)_{\text{peak}} - (\sigma_1/\sigma_3)_{\text{res}}\}$, where $(\sigma_1/\sigma_3)_{\text{peak}}$ and $(\sigma_1/\sigma_3)_{\text{res}}$ are the principal stress ratios at the peak and residual states. In addition, u_s is the average shear deformation, defined zero at the peak stress state, of a shear band which appeared in each specimen (20 cm high, 8 cm wide in the σ_3 direction and 16 cm long for Toyoura and SLB sands, and 57 cm high, 21 cm wide in the σ_3 direction and 24 cm long for Hime gravel). It is obvious from this figure that the shear deformation of shear band to bring a granular material from the peak stress state to the residual state increases as the grain size increases (n.b., this relationship may be affected also by other factors, including grading, grain shape and grain crushability; Yoshida and Tatsuoka, 1997). It can be inferred, therefore, that the failure mechanism in a sand deposit upon footing loading is affected by the grain size of sand.

The solution of boundary value problems involving strain-softening material properties is full of serious difficulties both in modelling of strain localization and

from the view point of numerical analysis. The straightforward use of the strain-softening model in a classical continuum generally does not result in a well-posed problem. The standard finite element solution of strain localization in a rate-independent material results in solutions that are strongly mesh-dependent. Several techniques have been proposed to resolve the mesh-dependent pathology of finite element solutions; Pietruszczak and Mroz (1981) proposed the idea of employing a softening modulus scaled by the element diameter as a technique to regularize the problem.

In the present study, the model takes into account strain localization into shear bands by assuming that;

- shear banding starts at the peak stress state, which is independent of boundary conditions;
- the stress-strain characteristics of shear banding can be defined in an objective way, independent of boundary conditions;
- each shear band has a width that is specific for the sand type; and
- each shear band has a specific post-peak relationship among stress level, shear deformation and dilatancy.

Following the method proposed by Pietruszczak and Mroz (1981), shear banding was represented in the numerical analysis by introducing a strain localization parameter S in the additive decomposition of total strain increment as follows (Tanaka and Kawamoto, 1988; Siddiquee et al., 1996; Tanaka and Sakai, 1993);

$$d\epsilon_{ij} = d\epsilon_{ij}^e + S d\epsilon_{ij}^p \quad (8)$$

where $S = F_b / F_e$; F_b is the area of the shear band in each element and F_e is the area of the element (n.b., before the start of shear banding, $S = 1$). The plastic strain component $d\epsilon_{ij}^p$ obeys the yield function given by Eq. (1) and the plastic potential given by Eq. (4). In this way of approximation, S actually becomes;

$$S = w / \sqrt{F_e} \quad (9)$$

where w is the width of shear band. So, the parameter S means the ratio of the shear band width to the average width of a given finite element. The average plastic shear strain γ^p was obtained by dividing the shear deformation u_s , as obtained from Fig. 1, with the shear band width w , ignoring the very small contribution of elastic shear strains. Unlike the method used by Pietruszczak and Mroz (1981), the direction of shear banding is not specified in the present study. Rather, it is implicitly assumed that the direction of shear band coincides in a broad sense with the direction of maximum shear strain.

As mentioned above, it is assumed, based on experimental observations in PSC tests (Tatsuoka et al., 1990; Yoshida et al., 1995; Yoshida and Tatsuoka, 1997), that the deformation of a sand mass under uniform boundary stress conditions with constant directions of σ_1 and σ_3 is homogeneous up to the peak stress state and strain localization into a shear band(s) starts at the peak stress state. It is assumed, therefore, that in the post-peak regime, the shear deformation of the whole of each element consists of plastic deformation in the yielding proc-

ess occurring inside the shear band and elastic rebound occurring in the whole element. The volumetric strain ϵ_v also consists of elastic and plastic parts (i.e., $\epsilon_v^e + \epsilon_v^p$). It is also assumed that for a given plastic major principal strain increment $d\epsilon_1^p$, the increment $d\epsilon_v^p$ can be obtained by Rowe's stress-dilatancy equation (Eq. (7)) using $K = 3.5$, obtained by a series of PSC tests (Tatsuoka et al., 1986).

The peak and residual angles of friction ϕ_p and ϕ_r were determined based on the experimental results, as summarized in Tatsuoka (1987) and Tatsuoka et al. (1993); that is, ϕ_p is a function of σ_3 , the initial void ratio and the angle δ of the direction of σ_1 relative to the horizontal bedding plane, as shown in Fig. 2. Figure 2(a) shows the data of ϕ_p from a series of PSC tests by Tatsuoka et al. (1986), from which the following strength model was constructed:

$$\begin{aligned} \phi_p \text{ (in degree)} \\ = \left\{ 59.47 \cdot (1.5 - e) - 10 \cdot (1 - e) \cdot \log_{10} \left\{ \frac{\sigma_3}{(\sigma_3)_0} \right\} \right\} \cdot R(\delta) \end{aligned} \quad (10a)$$

where

$$(\sigma_3)_0 = 4 \cdot (1 - e) p_a \quad (p_a = 98 \text{ kPa}). \quad (10b)$$

The void ratio function shown above was constructed by using the data of ϕ_p at $\delta = 90^\circ$. Figures 2(b) and 2(c) show summaries of the dependency of ϕ_p on σ_3 and δ , respectively, which are incorporated in Eq. (10). The function $R(\delta)$ is the average curve for dense Toyoura sand, shown in Fig. 2(c).

A close examination of all the test results from the PSC tests on Toyoura sand and SLB sand which have been obtained in the authors' laboratory showed that the residual angle of friction ϕ_r is not constant for a wide range of confining pressure, but decreases noticeably with the increase in the pressure level (Fig. 3). Based on the above, ϕ_r is assumed to be either constantly equal to 34° , or a variable as a function of σ_3 as;

$$\begin{aligned} \phi_r \text{ (in degree)} &= 35.7 \quad (\text{for } \sigma_3 \leq 10 \text{ kPa}) \\ &= 35.7 - 3.0 \cdot \log(\sigma_3 / p_a) \quad (\text{for } \sigma_3 > 10 \text{ kPa}). \end{aligned} \quad (11)$$

The elastic shear modulus G^e was obtained by substituting a void ratio $e = 0.66$ and a current value of p into the empirical relation;

$$G^e = 900 \cdot \frac{(2.17 - e)}{(1 + e)^2} \cdot \left(\frac{p}{p_a} \right)^{0.4} \cdot p_a \quad (12)$$

where p is the current mean principal stress $= (\sigma_1 + \sigma_2 + \sigma_3) / 3$. The elastic Poisson's ratio was assumed to be constant, equal to 0.3.

Four typical average stress and average strain relationships in PSC tests on specimens with a height of 10.5 cm obtained based on this model are presented in Fig. 2(d) (Tatsuoka et al., 1993); 'isotropic' and 'anisotropic'

Fig. 2(a). ϕ_p in PSC as a function of σ_3 , e and δ : 1) $\sigma_3 = 4.9, 9.8$ and 49 kPa and 2) $\sigma_3 = 98$ and 492 kPa (Tatsuoka, 1987; Tatsuoka et al., 1991, 1993)

Fig. 2(b). Dependency of ϕ_p on σ_3 of Toyoura sand in PSC (Tatsuoka, 1987; Tatsuoka et al., 1991, 1993)

Fig. 2(c). Dependency of ϕ_p on δ of Toyoura sand in PSC (Tatsuoka, 1987; Tatsuoka et al., 1990, 1991, 1993)

means that the specimen was first consolidated, respectively, isotropically and anisotropically, and then sheared to failure in PSC at a constant confining pressure. The post-peak modelling will be explained in the later part of this paper.

In this numerical analysis, convergence was confirmed by checking; a) the global residual force norm as:

$$\frac{(\|F - P + P^{\text{init}}\|)^2}{(\|F\|)^2} \leq t_1 \quad (13)$$

and; b) the differential residual force norm between two successive iterations as;

$$^n(\|F - P + P^{\text{init}}\|)^2 - ^{n+1}(\|F - P + P^{\text{init}}\|)^2 \leq t_2 \quad (14)$$

where F is the external applied force, P is the internally developed forces, P^{init} is the initially existing forces, n and $n+1$ mean successive iterations, and t_1 and t_2 are tolerance of convergence. The details of the above are described in Siddiquee et al. (1996). In this analysis, most computations were made by using $t_1 = t_2 = 10^{-6}$. Setting of such differential tolerance as stated above is necessary to make the solutions independent of the number and size of finite elements. This specific value was chosen as a compromise between computation time and desired ac-

Fig. 2(d). Model simulation of PSC tests (the shear strain has been averaged for the whole specimen) (Tatsuoka et al., 1993)

Fig. 3. Variation of ϕ_r with σ_3 for Toyoura sand and SLB sand

curacy. It has been established that one order of reduction in tolerance from the value shown above doubles the computation time, although results do not change significantly. The results of some analysis done with other tolerance values will be described where needed.

The test conditions which were mainly analyzed are as follows;

- the strip footing has a physical width B_0 of 50 cm, the footing base is rough without an initial footing depth, and the footing load is vertical and central; and
- the sand bed is under plane strain conditions, made

of normally consolidated air-dried Toyoura sand, prepared by air-pluviation with an initial void ratio = 0.66 and placed under normal gravitational acceleration (i.e., $n = 1$ (g)).

The other cases are;

- for the 1 g conditions, $B_0 = 0.5$ cm, 1.0 cm, 3.0 cm, 5.0 cm, 10 cm and 23 cm, for which physical model tests were performed, and $B_0 = 20$ cm, 100 cm, 500 cm and 1,000 cm, for which physical model tests were not performed; and
- for the centrifuge test conditions; a) for $B_0 = 3.0$ cm, the equivalent footing width $B = B_0 \times n$ (the acceleration level in terms of gravitational acceleration, g) = 23 cm and 50 cm, for which physical model tests were performed, and $B = 90$ cm, 300 cm, 500 cm and 1,000 cm, for which physical model tests were not performed; and b) for $B_0 = 2.0$ cm, $B = 50$ cm, for which physical model tests were performed.

PARAMETRIC STUDY

In order to perform FEM analysis incorporating the effects of strain localization with results being independent of the mesh size used, the strain-hardening modulus should be made properly dependent on the mesh size used in the analysis (as by Eq. (8) or Eq. (9)). To make the solution objective, differential tolerance checking was also introduced. In addition, the hour-glass modes due to the effect of bifurcation were controlled by detecting and adding anti-hour glass resistance. Failure of sand mass usually involves large deformation, which in turn involves rigid body rotations. It has been confirmed, however, that in the case of very high material non-linearity, the large deformation phenomenon, like rigid body rotations or such, has little effect on the behaviour

Fig. 4. Load-displacement curves for 201 and 851 FEM element meshes ($B_0=50$ cm)

of a footing up to the peak footing load (Siddiquee, 1994). So, all subsequent analyses were carried out based on the small strain theory. The parametric study was performed for $B_0=50$ cm in 1 g.

Effects of Mesh Number: Figure 4 compares the relationships between the normalized footing pressure $N=2q/(\gamma B)$ ($B=B_0 \times n$) and the relative footing settlement (S/B_0) obtained by using ordinary mesh (element number=201) and finer mesh (element number=851). In this case, $B=B_0$ with $n=1.0$. A half domain of 70 cm wide $\times$ 80 cm deep was used. The results were obtained by displacement control computation with a settlement increment of 0.0025 cm. Only in the case presented in Fig. 4 was a force tolerance of 10^{-2} used, and the hour-glass mode was controlled by applying 0.5% of the initial elastic modulus as the anti-hour glass mode stiffness only for this analysis. In this figure, the results from two physical model tests for $e_0=0.66$ (Tatsuoka et al., 1991) are also presented. Compared with the model test results, these FEM solutions provided considerably larger bearing capacity values. The difference is principally due to the fact that the analysis domain employed was too small, compared with $B_0=50$ cm. It may be seen that the solution is affected by the element number, as, with the increase in the number of degrees of freedom, the solution approaches the exact one (provided there is no singularity). Despite the above, it may also be seen that the peak footing load does not change much by this large change in the element number.

Effects of the Softening Parameter: The post-peak strain-softening characteristics have large effects on the ultimate footing load and settlement property of foundation. Different geomaterials have different strain-softening responses, depending on grain size, grain angularity, grain crushability, grain deformability, particle packing, contact-point distribution and so on. For the present

Fig. 5. Normalized footing load against the relative displacement for different softening parameter, ϵ_r

Fig. 6. Mesh used in the FEM analysis

study, the strain-softening parameters for Toyoura sand were determined by exponential fitting of the experimental data shown in Fig. 1, as:

$$R = R_{\text{res.}} + (R_{\text{peak}} - R_{\text{res.}}) \exp \left[-\frac{(\gamma - \gamma_f)^2}{\epsilon_r} \right] \quad (15)$$

where $R = \sigma_1 / \sigma_3$, γ is the total shear strain ($=\epsilon_1 - \epsilon_3$) in a shear band, and ϵ_r is the parameter which controls the strain-softening characteristics. As ϵ_r increases, the rate of the decrease in the shear stress in the strain-softening regime becomes slower for a given increase in the shear strain in the shear band. The experimentally determined value of ϵ_r for Toyoura sand is equal to 0.513 with a

Fig. 7(a). Variation of the bearing capacity factor with the strain softening parameter ε_r

Fig. 7(b). Variation of the relative footing settlement at peak with the softening parameter ε_r

shear band thickness of 3 mm. In this study, the effect of ε_r on the calculated bearing capacity was assessed by performing a parametric study artificially changing the value of ε_r . Figure 5 shows the normalized load-displacement curves for $\varepsilon_r = 0.05 \sim 2.0$. For this analysis, a wider FEM domain, 300 cm wide $\times$ 400 cm deep for a half domain considering symmetry (Fig. 6), was used with the number of elements equal to 222. The figure inserted in Fig. 6 is the zoom-up of the sub-domain immediately below the footing. The result shows a general tendency of the increase in the bearing capacity with the increase in ε_r , as summarized in Fig. 7(a). Figure 7(b) shows that the relative footing settlement at peak increases with the increase

Fig. 8. Variation of the inverse of the coefficient of vertical subgrade reaction with the softening parameter ε_r

in the softening parameter, ε_r . Figure 8 shows the variation of the inverse of the coefficient of secant vertical subgrade reaction defined at a half of the peak footing load. It is seen that this value decreases with the increase in ε_r for a range of $\varepsilon_r < 0.4$ and remains constant for ε_r values exceeding 0.4. The variation is, however, very small. In summary, the effect of ε_r is largest on N_y , intermediate on S_f/B_0 and smallest on $(S/q)_{50}$.

Effects of Different Assumptions for the Constitutive Relationship: In the classical theories for the bearing capacity of footing on soil, a number of assumptions are used to simplify the highly complicated deformation and strength properties of soil. Almost all of the classical theories are based on the following two assumptions;

1. The angle of internal friction ϕ_p of sand is unique within a given homogeneous ground (i.e. ϕ_p is independent of pressure level without anisotropy).
2. The failure of soil mass is simultaneous, associated with a sudden occurrence of distinct and complete failure surface(s), along which the same peak frictional angle ϕ_p is mobilized and its value is maintained with further loading (i.e., perfect plasticity is assumed and the failure is assumed not to be progressive).

In the classical type of FEM analysis, it is assumed that the directions of stress vector and plastic strain increment vector are the same on the stress and plastic strain increment plane (i.e., associated flow rule).

In the present study, all of the above mentioned assumptions were relaxed one by one, while approaching the realistic material behaviour, as shown in Figs. 9 and 10. The 1 g test using a strip footing with a width $B_0 = 50$ cm placed on air-dried Toyoura sand ($e_0 = 0.66$) under 1 g ($n = 1.0$) was simulated using the mesh shown in Fig. 6.

Fig. 9. Relationships between normalized footing load and relative footing settlement from linear and nonlinear elastic analysis and simple elasto-plastic analyses ($e_0=0.66$; $B_0=50$ cm and $n=1$)

The results shown in Fig. 9 were obtained for the following four different cases (Table 1) with different idealizations of soil properties. The meshing and other conditions, other than described below, are the same as those described in the preceding section (Fig. 6):

- Case a is the linear elastic solution, based on assumption 1 (Table 1), with a constant Young's modulus, $E=125.6$ MPa, and a constant Poisson's ratio, $\nu=0.3$. That E value is the one calculated from the G^e value obtained by substituting $e=0.66$ and $\sigma_3=p_a=98$ kPa into Eq. (12).
- Case b is the non-linear elastic solution, based on assumption 2, with the shear modulus being a function of mean pressure p (Eq. (12)).
- Case c is the elasto-perfectly plastic solution, based on assumptions 2 and 3, with the associated flow rule (assumption 4). In cases c and d, the peak friction angle ϕ_p equal to 49 degrees was used, which is the value of ϕ_0 obtained from the PSC tests at $\delta=90^\circ$ performed at low confining pressures on Toyoura sand with $e_0=0.66$.
- Case d is the elasto-perfectly plastic solution, also based on assumptions 2 and 3, but using a non-associated flow rule, based on Rowe's stress-dilatancy relation (Eq. (7); assumption 5).

The following trend is seen from Fig. 9;

- The results of the two elastic analyses (cases a and b) show no peak footing load, while the stiffness is unrealistically high. Of course, an equivalent elastic Young's modulus E can be obtained so that the solution fits the apparently linear part of the measured relationship.
- Although case c is based on an elasto-plastic model, the use of the associated flow rule results in too high peak footing load. The pre-peak stiffness also becomes larger when compared with that obtained based on the non-associated flow rule (case d).

Table 1. Analysis cases and their respective assumptions

Analysis cases	Assumptions
a	1
b	2
c	2, 3, 4, $\phi=49^\circ$
d	2, 3, 5, $\phi=49^\circ$
e	2, 5, 6, $\phi=49^\circ$
f	2, 5, 6, 7
g	2, 5, 6, 7, 8
h	2, 5, 6, 7, 8, 9
i	2, 5, 6, 7, 8, 9, 10
j	2, 5, 6, 7, 8, 9, 10, 11

No.	Assumptions
1	Linear elastic; $E=125.6$ MPa, $\nu=0.3$
2	Non-linear elastic; $E=2(1+\nu)G$, $G=f(p, e)$
3	Elasto-perfectly plastic
4	Associated flow rule
5	Non-associated flow rule
6	Isotropic hardening
7	Pressure level dependency of ϕ
8	Strength anisotropy
9	Strain softening
10	Shear banding
11	Double yield surface model

- The use of the non-associated flow rule in the elasto-perfectly plastic analysis (case d) gives a peak footing load which is very close to the value obtained by the Terzaghi solution with $\phi_p=\phi_0=49^\circ$ (Terzaghi, 1943). Note, however, that this peak footing load is considerably higher than the measured value, as shown below. The agreement between the solution for case d and the Terzaghi solution results from the fact that both solutions ignore both the effects of pressure level and anisotropy on the ϕ_p value and the effects of progressive failure on the peak footing load.

The results shown in Fig. 10 are for the following six cases, listed in Table 1, from more sophisticated elasto-plastic analyses. In all the cases, the pre-peak stress-strain relationship of sand is non-linear, and determined based on the measured sand property model (Tatsuoka et al., 1993). For the first three cases e, f and g, the model material property is elasto-plastic (isotropic strain-hardening, followed by perfectly plastic behavior without strain softening), while for the last three cases h, i and j (and the other case k presented in Fig. 11), the material exhibits post-peak strain-softening (i.e., Eq. (15)) with $\varepsilon_r=0.513$, which results in post-peak softening also in the N and S/B relationships. Only in cases i and j

Fig. 10. Relationships between normalized footing load N and relative footing settlement S/B_0 from sophisticated elasto-plastic analyses, compared with those from two physical model tests ($e_0=0.66$; $B_0=50$ cm and $n=1$)

Fig. 11. Comparison of normalized footing load N and relative footing settlement S/B_0 from the analyses with a constant residual ϕ_r and variable values of ϕ_r

(and case k), the effects of shear banding were considered by Eq. (8). In Fig. 10, the measured relationships and the Terzaghi solution are also shown for comparison purposes. That is;

- e) Case e; the peak friction angle ϕ_p is isotropic and independent of pressure level, equal to $\phi_0=49^\circ$.
 - f) Case f; ϕ_p is isotropic and equal to the value measured when $\delta=90^\circ$, it decreases from $\phi_0=49^\circ$ to smaller values with pressure level (Eq. (10)).
 - g) Case g; ϕ_p is anisotropic, being a function of δ (Eq. (10)), and depends on pressure level, while ϕ_0 at $\delta=90^\circ$ is equal to 49° .
 - h) Case h; ϕ_p is anisotropic, depending on δ , while it depends on pressure level. The effect of shear banding is not considered, but the post-peak stress and strain relationship in a shear band distributes uniformly in each element, and those obtained by setting the localization factors $S=w/\sqrt{F_e}$ (Eq. (9)) equal to 1.0 are used, irrespective of the area of each element F_e . The residual angle of friction ϕ_r equal to 34° is used in this and the following cases i and j.
 - i) Case i; the stress-strain property of sand is modeled as close as possible to the actual one. That is, ϕ_p is anisotropic and depends on pressure-level (while ϕ_0 at $\delta=90^\circ$ is equal to 49°), and the post-peak stress and strain relationship for each element is the same as the one in the corresponding PSC specimen having the same lateral area as the concerned FEM element. In other words, the post-peak stress and strain relationship is obtained by using the localization factor $S=(0.3 \text{ cm})/\sqrt{F_e}$ (Eq. (9)). As a result, the post-peak stress and strain relationship depends on the ratio of the shear band width w to the footing size B_0 , and thus depends on the ratio D_{50}/B_0 .
 - j) Case j; this case is the same as that of case i, except for the use of double yield surfaces. In this case, the initial anisotropy of plastic-deformation is considered by introducing a separate set of yield function and plastic potential in the isotropic compression direction. The details are explained in Appendix I.
- The following trends may be seen from Figs. 9 and 10.
- 1) The use of isotropic strain-hardening with non-linear pre-peak stress-strain property (case e) has noticeable effects on the pre-peak bearing capacity characteristics of footing, compared with elasto-perfectly plastic case (case d). That is, the stiffness decreases drastically and becomes more realistic. Despite the above, the ultimate bearing capacity remains nearly the same.
 - 2) In the elasto perfectly-plastic analysis (cases d, e and f), the peak footing load is attained at a very large footing settlement. This is because this large settlement is needed to mobilize the peak strength along the whole failure surfaces due to the deformability and compressibility of sand.
 - 3) The differences seen between cases e and f and between cases f and g show the importance of taking into account the effects of, respectively, pressure level and anisotropy on the peak strength of sand.
 - 4) The difference seen between cases g and h indicates that the effect of post-peak strain-softening is also important.
 - 5) The result of case h depends on meshing, because the

degree of strain localization depends on the meshing due to the use of a constant strain localization factor S equal to 1.0. That is, in the FEM analysis, a shear band starts to appear from the zones near the footing edges, as in the physical model tests. As the mesh size in case h is 2.5 cm, which is about 8 times as large as the actual shear band width, the solution becomes different from that of case i, which is much more realistic. Therefore, if, in case h, the mesh size in the zones near the footing edges were set equal to the actual shear band width, equal to 0.3 cm, the result would become similar to that of case i. Rather, the solution of case h would be similar to that of case i if the footing width B_0 were made 6 cm, which is about $1/8$ times as small as the actual value $B_0 = 50$ cm, under $n = 8$ (g) using the actual shear band width (this is indeed the simulation of a centrifuge test). The difference between the solutions of cases h and i is, therefore, due to the effects of the different ratios of B_0 /"shear band width" or B_0/D_{50} (i.e., the particle size effect). This point will be discussed in more detail later.

- 6) Only the solutions of case i and j are reasonably close to the experimental results, at least up to the moment of peak footing load. This result shows that in order to numerically simulate realistically the bearing capacity characteristics of a footing on sand, the following factors should be properly taken into account;
 - a) pre-peak strain-hardening elasto-plastic deformation characteristics;
 - b) non-associated flow characteristics;
 - c) anisotropic strength and deformation characteristics;
 - d) the effect of pressure level and anisotropy on the strength and deformation characteristics;
 - e) post-peak strain-softening; and
 - f) shear banding with a characteristic shear band width (i.e., particle, size effect).
- 7) The results of case j is very close to the experimental results, but the settlement at the peak load is slightly lower than the measured values (see Fig. 10). This is probably due to some specific characteristics of the modeling of variable stress-dilatancy relationships (see Appendix I).

Lastly, in case k, ϕ_r is assumed to be a function σ_3 (i.e., Eq. (11)), while the other assumptions are the same as in case i with a constant residual angle of friction ($\phi_r = 34^\circ$). Figure 11 compares the normalized load-displacement curves for cases i and k. It is seen that the use of the variable residual angle of friction results in a smaller bearing capacity factor (case k), and it becomes eventually closer to the physical model test result. However, further study into this point will be necessary.

DETAILED SIMULATION OF MODEL TEST ($B_0 = 50$ cm and 1 g)

Figure 12 shows one of the pictures of the σ_2 plane of a

Fig. 12. Picture of the model sand at the peak footing load ($S/B_0 = 2.4$ cm/50 cm; $e_0 = 0.69$; and $n = 1$) (Morimoto, 1990)

Fig. 13. Physical model test result ($B_0 = 50$ cm; $e_0 = 0.69$ and $n = 1$) (Morimoto, 1990)

model sand bed taken through a transparent window (Morimoto, 1990; Tatsuoka et al., 1991a). The window was made of a thick Acrylic platen, stiffened with a steel framework. In this model test, the initial void ratio e_0 was 0.69, but the other test conditions were the same as those where $e_0 = 0.66$, the results of which are shown in Figs. 4 and 10. This difference in void ratio has a noticeable effect on the N_γ value, but not on the footing settlement and strain field in the sand at peak footing load (Tani, 1986; Morimoto, 1990). The loading stage for Fig. 12 is near the peak state, as indicated in Fig. 13. The grid of latex rubber was deformable and had a spacing of 1 cm, as seen in Fig. 12. The grid had been printed on a large but thin (0.2 mm-thick) latex rubber membrane. A thin silicon grease layer was smeared on the external surface of the latex membrane, which was in contact with

Fig. 14. Comparison of displacement field between a) physical model test, and b) FEM simulation ($B_0=50$ cm; $e_0=0.69$ and $n=1$)

the internal surface of the Acrylic platen. Figure 14(a) shows the deformed grid pattern obtained by digitizing the picture shown in Fig. 12 (Morimoto, 1990). Local shear strain contours shown in Fig. 15(a) were obtained by comparing the displacements of the nodes of the grids shown in Fig. 14(a) with those before loading. Note that as uniform strain $\gamma = \epsilon_1 - \epsilon_3$ is defined within each $1 \text{ cm} \times 1 \text{ cm}$ element, strain values in largely strained zones, in particular in and near the shear bands, are not very reliable. It may be seen from Fig. 15(a) that strains are considerably non-uniform, showing the progressive failure of the sand bed. Figures 14(b) and 15(b) show the corresponding displacement and strain fields at the peak footing load obtained from the FEM analysis, case i. In Fig. 15(b), the contours are shown for every shear strain increment equal to 2%. It is seen that although it is not perfect, the FEM analysis captures the major characteristic features of the displacement and strain fields observed in the physical model test.

Figures 16(a)–16(j) show shear strain fields at each certain moment of loading obtained from the FEM simulations for, respectively, cases a through j (Figs. 9 and 10).

Fig. 15. Comparison of contours of shear strain between a) physical model test ($B_0=50$ cm; $e_0=0.69$ and $n=1$), and b) FEM simulation (case i)

In each figure, a zone of 50 cm deep and 40 cm wide is indicated by a rectangle of broken line. In case c, very large plastic shear strains were obtained, which is likely due to the use of the associated flow rule (Vermeer, 1984). It may be seen that a realistic strain field can be obtained only in cases i and j.

Figure 17(a) shows the normal pressure distributions at the footing base measured with eleven independent load cells at several loading stages, indicated in Fig. 13, in the test of $B_0=50$ cm, $e_0=0.66$ and $n=1$ (g). The corresponding result from FEM analysis of case j is shown in Fig. 17(b). It may be seen that the pattern of normal pressure distribution observed in the physical test is well simulated by the FEM analysis. Note that the sharp peak at the footing center seen in Fig. 17(a) is not typical of the test results. In the other tests, smoother distributions were observed (Fig. 18); the loading stages A, B, C and D for this are indicated in Fig. 19.

Figure 20 shows the relationships between the normalized footing pressure N and the relative settlement of footing S/B_0 obtained from three physical model tests and their corresponding two FEM simulations. Comparison is carried out for both the 1 g condition with a physical footing width B_0 of 50 cm and an acceleration level n of 1 (g) and centrifuge condition with $B_0=2$ cm and $n=25$ (g), while the equivalent footing width $B=B_0 \times n$ is the same, equal to 50 cm. The FEM analysis of the centrifuge test also used the meshing shown in Fig. 6 by scaling down by a factor of $1/25$. It can be observed from Fig. 20 that the FEM analysis simulates closely the pre-

Fig. 16. Comparison of strain fields among different FEM solutions based on different assumptions ($B_0=50$ cm; $e_0=0.69$ and $n=1$)

Fig. 17. Comparison of contact pressure distribution between a) in the physical model test ($B_0=50$ cm; $e_0=0.66$ and $n=1$) (Morimoto, 1990), and b) FEM simulation (case j)

Fig. 18. Contact pressure distribution in another physical model test ($B_0=50$ cm; $e_0=0.66$ and $n=1$)

peak behaviour and peak load in the physical model tests and the difference in the behaviour between the 1 g tests and the centrifuge tests. It is seen, however, that the FEM analysis under-predicts the footing settlement at the peak. This difference may be due partly to the approximated nature of either the modelling of the stress-strain property of the model sand or the FEM analysis or both,

Fig. 19. Normalized footing load N and relative footing settlement S/B_0 corresponding to Fig. 18 (Morimoto, 1990)

Fig. 20. Comparison of normalized footing load N and relative footing settlement S/B_0 between physical test results (1 g and centrifuge) and the corresponding FEM simulation ($B=50$ cm and $e_0=0.66$)

but is also due partly to the effects of bedding error between the footing base and the top surface of the model sand bed in the physical model tests (particularly in the centrifuge test). It can also be seen that in the FEM simulation, the post-peak relationship has a sharper drop in the post-peak regime, which is not the case with the experimental load-displacement behaviour. A number of factors could be responsible for this, including the approximate nature of the modelling of strain localization and non-homogeneous element size. Further research will have to be carried out to overcome this problem.

SCALE EFFECT

The so-called scale effect could be defined as the variation in the bearing capacity characteristics with the varia-

Fig. 21. Summary of N_γ values from 1 g and centrifuge tests (Tani, 1986; Morimoto, 1990) and the corresponding FEM simulation ($e_0 = 0.66$)

tion in the footing size (or width) under normal gravity $n=1$ (g). The N_γ values obtained from a series of plane strain bearing capacity tests using strip footings having widths B_0 from 0.5 cm to 50 cm placed on air-dried Toyoura sand ($e_0 = 0.66$) performed under $n=1$ (g) (Tani, 1986; Morimoto, 1990) are summarized in Fig. 21 (the data points denoted by solid triangles). Each data point shows the average of a set of data from multiple tests performed under the same test conditions other than small variations in the void ratio of sand. Some of these results have been reported by Tatsuoka et al. (1991; 1995). A very significant scale effect may be seen in the results. Also in this figure, the results from the corresponding series of centrifuge tests using a strip footing with $B_0 = 3$ cm performed changing the acceleration level are shown, denoted by reversed solid triangles. As the change in N_γ with the change in B due to the change in n is caused solely by the change in the pressure level in the sand bed, this change is due to the pressure level effect. On the other hand, the difference in the N_γ values between the 1 g and centrifuge tests for the same $B = B_0 \times n$ are due to the particle size effect. The results of the FEM simulation (case i) for the 1 g and centrifuge tests are also shown in Fig. 21. It is seen that the FEM analysis simulates well both types of physical model tests. It is of particular importance that the FEM analysis can simulate not only the scale effect but also the particle size effect.

CONCLUSIONS

The following conclusions can be drawn from the

results presented above.

- 1) The results of physical model tests could be reasonably simulated by FEM simulations only when based on appropriate modelling of sand properties taking into account the effects of relevant factors, including; a) confining pressure; b) anisotropy; c) non-linear strain-hardening and strain-softening; d) dilatancy; and e) strain localization. In particular, the post-peak strain-softening characteristics of sand have large effects on the bearing capacity of footing on dense sand, as the failure of ground is largely progressive.
- 2) The isotropic perfectly-plastic modelling of the deformation and strength property of sand, which are assumed in most of the classical bearing capacity theories and in some simplified FEM analyses, is an overly simplified approximation.
- 3) The scale effects seen with the increase in the footing size in 1 g consists of; a) the pressure level effect; and b) the particle size effect, as shown by the physical model tests and the FEM simulation. The centrifuge tests using a scaled-down model footing placed on the prototype sand performed under otherwise identical conditions would over-estimate the bearing capacity of the prototype footing and the footing settlement at the peak footing load.

Appendix I: Variable stress-dilatancy relation

The assumptions used are the following (Fig. A-1):

1. For Toyoura sand, the parameter K for the stress-dilatancy relation $R = KD$ (D is defined in Fig. A-1) is constant, independent of δ and σ_3 in PSC at a constant p .
2. At anisotropic compression, the principal strain increment ratio varies with the change in δ linearly from 90° (conventional PSC tests) to $\delta = 0^\circ$. This is to incorporate the effects of inherent anisotropic deformation property during compression.

Formulation

From Fig. A-1, we can write the equation for a straight line at $\delta = 90^\circ$ as a generic equation.

$$\frac{\sigma_1}{\sigma_3} = -m \frac{d\epsilon_3^p}{d\epsilon_1^p} + c \quad (\text{A.1})$$

Fig. A-1. Variable stress-dilatancy relation

Fig. A-2. Plastic potential

Fig. A-3. Variation of parameter a with δ

By using $s=(1/2)(\sigma_1-\sigma_3)$ and $t=(1/2)(\sigma_1+\sigma_3)$, we obtain from Eq. (A.1):

$$\frac{d\varepsilon_3^p}{d\varepsilon_1^p} = \frac{c(s-t)-(s+t)}{m(s-t)}. \quad (\text{A.2})$$

By using $\varepsilon_v^p = \varepsilon_1^p + \varepsilon_3^p$ and $\gamma^p = \varepsilon_1^p - \varepsilon_3^p$, we obtain Eq. (A.3) from Eq. (A.2):

$$\frac{d\varepsilon_v^p}{d\gamma^p} = \frac{t(-1-c-m)-s(1-c-m)}{t(1+c-m)+s(1-c+m)}. \quad (\text{A.3})$$

The equation of plastic potential ψ (Fig. A-2) is obtained from Eq. (A.3):

$$\left(\frac{dt}{ds} \text{ for } \psi \right) = \left(-\frac{d\varepsilon_v^p}{d\gamma^p} \text{ from Eq. (A.3)} \right). \quad (\text{A.4})$$

From Eq. (A.4), we obtain:

$$(tc_1 + sc_2)ds + (tc_3 + sc_4)dt = 0 \quad (\text{A.5})$$

where $c_1 = -1 - c - m$, $c_2 = -1 + c + m$, $c_3 = 1 + c - m$, $c_4 = 1 - c + m$, and $c_5 = c_1 + c_4$.

Substituting $t = vs$ (v : a variable) and $dt = vds + s dv$ into Eq. (A.5) and integrating it, we obtain:

$$\int \frac{1}{s} ds + \int \frac{c_3 v + c_4}{c_3 v^2 + (c_1 + c_4)v + c_2} dv = 0. \quad (\text{A.6})$$

After completing the integration, we obtain:

$$(c_3 t^2 + c_5 s t + c_2 s^2) \left(\frac{t + b_1 s}{t + b_2 s} \right)^{b_4} = X. \quad (\text{A.7})$$

When $t=0$ and $s=s_0$, we obtain:

$$X = c_2 s_0^2 \left(\frac{b_1}{b_2} \right)^{b_4}. \quad (\text{A.8})$$

Hence, the required equation for the plastic potential is:

Fig. A-4. Variable plastic potential surface

$$(c_3 t^2 + c_5 s t + c_2 s^2) = c_2 s_0^2 \left(\frac{b_1 t + \frac{c_2}{c_3} s}{b_2 t + \frac{c_2}{c_3} s} \right)^{b_4} \quad (\text{A.9})$$

where

$$\begin{aligned} b_1 &= \frac{c_5}{2c_3} - \sqrt{\frac{c_5^2}{4c_3^2} - \frac{c_2}{c_3}} \\ b_2 &= \frac{c_5}{2c_3} + \sqrt{\frac{c_5^2}{4c_3^2} - \frac{c_2}{c_3}} \\ b_4 &= \frac{c_4 - c_1}{2c_3 \sqrt{\left(\frac{c_5^2}{4c_3^2} - \frac{c_2}{c_3} \right)}}. \end{aligned} \quad (\text{A.10})$$

Determination of ' m ' and ' c ' depends on the different angle of bedding plane orientation (δ) as:

$$\begin{aligned} m &= \frac{K(R_{\max} - 1)}{\omega K + R_{\max}} \\ c &= \frac{R_{\max}(\omega K + 1)}{\omega K + R_{\max}} \end{aligned} \quad (\text{A.11})$$

where

$$\begin{aligned} \omega &= \left(\frac{1 - a^2}{90a} \right) \delta + a \\ K &= \frac{1 + \sin \varphi_r}{1 - \sin \varphi_r} \\ R_{\max} &= R_{\max}(\sigma_3, e, \delta) \end{aligned} \quad (\text{A.12})$$

where a is the parameter showing inherent anisotropy, defined as the ratio $d\varepsilon_h^p/d\varepsilon_v^p$ (horizontal to vertical strains) during isotropic compression. The three main variables for this model are the confining pressure σ_3 , void ratio e and the angle δ of bedding plane relative to the σ_1 direction. Figure A-4 shows an example of the variable plastic potential surface depending on the parameter a .

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