STRESS HISTORY-DEPENDENT DEFORMATION CHARACTERISTICS
OF DENSE SAND IN PLANE STRAIN

S. J. M. YASIN\textsuperscript{0} and FUMIO TATSUOKA,\textsuperscript{0,1)}

ABSTRACT

A series of drained stress path plane strain tests was performed on saturated dense specimens of Toyoura sand with precise stress and strain measurements. It is shown that all the strain increments (i.e., axial, lateral, shear and volumetric) that occurred by loading between two stress states were dependent on the intermediate stress paths. It is suggested that the use of a strain quantity as the hardening parameter that is independent of stress history in an elasto-plastic model for sand may not be relevant. Based on the test results, one form of energy function is shown to be stress history-independent. This quantity is a state parameter, being a unique function of one form of stress parameter for different stress paths. How this function can be used as a stress history and stress path-independent hardening function in an elasto-plastic model briefly discussed. The effects of stress history and instantaneous stress path on the friction angles at the failure and residual states are negligible. The effects of stress path on stress-dilatancy relationships based on plastic strain increments are found to be small but noticeable.

Key words: elasto-plastic model, plane strain, sand, state parameter, stress-dilatancy relationship, stress path, stress-strain relationship (IGC: D6)

INTRODUCTION

A number of elasto-plastic models have been proposed to describe the time-independent aspects of the deformation characteristics of sand. It is known that the elastic deformation characteristics of uncremented granular materials can be defined only in terms of very small strain increments, and the elastic compliance depends on the instantaneous stress state along with other factors; this property is called "hypo-elasticity". Hoque and Tatsuoka (1998) showed that the Young's modulus defined for the major principal reversible strain increment occurring in a certain direction is a unique function of the normal stress in that direction. All of the test results obtained in the authors' laboratory, including those of Poisson's ratio, suggest that the elastic compliance matrix is symmetric (Tatsuoka et al., 1999a).

In the elasto-plastic models for soils, the hardening function controls the development of yield locus as the soil yields. The hardening parameter involved in the hardening function is usually a state parameter, which is independent of previous stress history and current stress path direction. For soils, a certain plastic strain component or plastic strain energy is often employed as the hardening parameter.

In this paper, a brief review of the strain hardening parameter for soils is first attempted. Then, results from a series of stress path plane strain tests on dense Toyoura sand that were performed to better understand this issue will be presented. The tests were performed using an advanced automated stress path plane strain testing system with precise measurements of stresses and strains. Each test was performed along multiple stress paths, including anisotropic compression at different constant stress ratios. In some of the tests, the stress ratio was very close to that at failure, which is very difficult to perform with conventional test systems. A part of the present research has been reported elsewhere (Yasin and Tatsuoka, 1999).

A BRIEF REVIEW

Henkel (1960) performed a series of drained and undrained triaxial tests (not including unloading paths) on isotropically consolidated specimens of remolded Weald and London clays and anisotropically consolidated specimens of Wiener Tegel and Weald clays. He confirmed the results obtained for Wiener Tegel clay by Rendulic that "there is a unique relationship between the effective stresses and the water content"; that is, the plastic volumetric strain that takes place between any two given stress states is independent of the intermediate stress paths. Critical State Soil Mechanics (Schofield and

\textsuperscript{0} Associate Professor, Department of Civil Engineering, Bangladesh University of Engineering and Technology (BUET) (Formerly Graduate Student, University of Tokyo).
\textsuperscript{1) Professor, Department of Civil Engineering, University of Tokyo.

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Wroth, 1968) is based on these test results, using plastic volumetric strain as the stress history-independent strain hardening parameter within the elasto-plastic framework for soils (particularly for normally consolidated soft clays).

It has been shown by several researchers, however, that this is by no means always the case. Henkel and Sowa (1963) performed a series of undrained triaxial compression tests on isotropically and $K_0$-consolidated Weald clay. Their data showed that the void ratio of a $K_0$ consolidated specimen is noticeably larger than that of another specimen which reached the same $K_0$ stress state by isotropic compression followed by undrained triaxial compression. This result suggests that for clay specimens with the same initial void ratios at the same initial stress states, the void ratio at a given later stress state will be larger when that stress state is reached, with continuous yielding, by tracing stress paths closer to the failure envelope. Lewin and Burland (1970) and Lelièvre and Wang (1970) also performed a series of consolidated undrained and drained triaxial compression tests, similar to the above, and suggested such "non-Rendulic behaviour" of clays. Gens (1986) also obtained similar test results for a low plasticity clay, Lower Cromer Till. Recently, Tatsuoka et al. (1998) reported similar results, clearly showing non-Rendulic behaviour, obtained from a comprehensive series of triaxial compression and extension tests on anisotropically and isotropically consolidated specimens of reconstituted Fujinomori and Kaolinite clays.

Tatsuoka (1972) also observed similar non-Rendulic behaviour in a series of triaxial compression and extension tests on Fuji River sand. Figure 1 shows four effective stress paths for a constant void ratio, obtained from the triaxial compression tests. These effective stress paths are intersections of a constant void ratio plane with state surfaces in $(q, p', e)$ space constructed by using the data from isotropic compression tests and those from undrained tests and drained tests at a constant effective axial or lateral or mean principal stress $p'$. Here, $q$ is the deviator stress $\sigma_i - \sigma_i'$ and $p' = (\sigma_i' + 2\sigma_o)/3$. This plot is equivalent to stress paths on the $q/p', p'/p_o'$ stress plane, where $p_o'$ for any specified state ($p', q, e$) is the $p'$ value delineated by a point on the current consolidation line that corresponds to the current void ratio, $e$. It is seen that the sand did not obey the Rendulic principle, showing that the void ratio at the same stress state in triaxial compression was larger when the stress state was reached through a stress path closer to the failure envelop. Tatsuoka (1972) and Tatsuoka and Ishihara (1974) suggested that instead, shear strains obtained from different stress path tests starting from isotropic stress states are essentially stress history-independent, provided that shear strain increments are zero during isotropic compression. However, they did not study whether this is also the case with tests including anisotropic compression at high axial-to-lateral stress ratios.

Tatsuoka (1972) and Tatsuoka and Ishihara (1974) also showed that the shear strain at the start of yielding during reloading, following unloading in the stress ratio $q/p'$ and changes in the $p'$ value, is nearly the same as that observed immediately before unloading. Based on these observations, Tatsuoka (1980) suggested the use of plastic shear strain as the stress history-independent strain hardening parameter. Later, Siddiquee et al. (1994, 1999), Tatsuoka et al. (1991, 1994a), Yoshida et al. (1994) and Kotake et al. (1999) employed this assumption in modelling the elasto-plastic behaviour of Toyoura sand. They performed a series of numerical simulations by the FEM of the results from plane strain bearing capacity model tests of strip footing on Toyoura sand. The shear function for shear strain was obtained based on results from a series of plane strain compression tests performed at constant $\sigma_i$, while considering anisotropy in the strength and deformation characteristics (Tatsuoka et al., 1991, 1993). The peak footing load could be rather accurately simulated in the analyses (Tatsuoka et al., 1991; Siddiquee et al., 1994, 1999; Kotake et al., 1997, 1999; Kotake, 1999). However, the pre-peak footing settlement was always under-estimated. The authors considered that this discrepancy could be attributed, at least partly, to that the assumption with respect to the hardening parameter was inadequate, and, for that reason, shear strains for stress paths traced in the ground were underestimated. The FEM analysis also revealed that typical stress paths in the ground in the yielding zones below the footing are very similar to anisotropic compression stress paths with high stress ratios, which

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**Fig. 1. Effective stress paths for constant volume conditions from four types of triaxial compression tests on saturated loose Fuji River sand** (Tatsuoka, 1972); $p'_o = 98$ kPa
are considerably different from those in the conventional triaxial and plane strain compression tests at constant confining pressure.

Moroto (1980) performed a series of triaxial compression tests along effective stress paths of constant mean principal stress, constant lateral stress and constant axial stress on isotropically consolidated specimens of glass beads. He showed that the plastic work (or plastic strain energy) $W_p$ which was done between arbitrary two stress states in the continuously yielding test material was considerably stress history-dependent. He proposed the following parameter $S_r$ as a stress history-independent state parameter:

$$S_r = \int dW_p^p / p'$$

where $dW_p^p$ is the plastic strain energy increment $= \varepsilon_p^p d\sigma_p$.

Nakai (1989) performed a series of stress path triaxial compression tests on Toyoura sand. He showed that both plastic shear and volumetric strains are noticeably stress history-dependent, and the plastic work $W_p$ is slightly stress history-dependent when the triaxial compression and extension stress conditions are considered separately. He showed that another plastic work function defined based on the stress parameters introduced in his $t_0$ modelling is essentially stress history-independent. In the framework of his elasto-plastic modelling, Nakai used this plastic work function as the hardening parameter.

Recently, more precise strain measurements and stress path control have become possible, compared with those in the previous studies. Considering that the plane strain condition is one of the typical strain conditions encountered in the field and is employed in many laboratory model tests, Kamegai (1994) performed stress path plane strain compression tests, following anisotropic compression at different stress ratios, on dense Toyoura and SLB sands. It was shown that for both types of sands, shear strains were noticeably larger with more dilative behaviour when tracing stress paths closer to the failure envelope line. The test results of Kamegai were quoted in Figs. 4.8 and 4.9 of Tatsuoka and Kohata (1995). However, the test results were not very conclusive due to a limited number (ten in total) of tests.

**TESTING SYSTEM AND TEST METHOD**

In the present study, therefore, extending the study of Kamegai, a more comprehensive series of stress path plane strain tests were performed on saturated specimens of Toyoura sand to obtain a better insight into this issue. The testing method employed in the present study is essentially the same as that of Kamegai (1994). A conventional plane strain apparatus with rigid-lubricated $\sigma_1$ and $\sigma_2$ boundaries and flexible $\sigma_3$ boundaries (Fig. 2(a)), as described in detail by Park and Tatsuoka (1994) and Kamegai (1994), was used. In a PSE test, the cell pressure, as the major principal stress $\sigma_1$, acts on the two lateral surfaces. As the specimen is elongated in the axial direction, with the conventional confining platen setup shown in Fig. 2(a), the specimen contracts in the $\sigma_3$ direction with separation of the $\sigma_2$ surfaces of the specimen from the confining platens, resulting in deviation from plane strain conditions. To overcome this problem, as shown in Fig. 2(b), two additional platens were placed in-
side the sample membrane and screwed to the outer platens so that the specimen does not contract in the $\sigma_1$ direction in any case. The details are given in Masuda et al. (1999).

The axial load was controlled using a specially designed gear system driven by an analog motor. As described in Tatsuoka et al. (1994b), Yasin et al. (1999) and Santucci de Magistris et al. (1999), this loading system can control displacement to an accuracy of less than 1 $\mu$m and allows a reversing of loading direction with essentially zero backlash. The principal lateral stress $\sigma'_1$ was applied by electro-pneumatically controlled air pressure acting on the cell water surface. By taking advantage of these features, the stress paths can be controlled rather accurately as shown later. By following the methods described in detail by Tatsuoka et al. (1986) and Tatsuoka (1988), the axial load, free from piston friction, was measured with a strain gauge-type load cell placed inside the triaxial cell, while the load in the $\sigma_2$ direction was measured with another load cell placed on the back of one of the confining platens (see Figs. 2(a) and 2(b)). The triaxial cell was partially filled with water so that the specimen was completely submerged. Friction force in the axial direction along the specimen $\sigma_2$ surfaces was measured by using two load cells attached to the bottoms of the two confining platens. The axial stresses presented herein are those defined at the mid-height of specimens that have been corrected for side friction. The axial strains presented are the average of those obtained by using a pair of LDTs (Fig. 3) and the lateral strains presented are the average of those obtained by using four pairs of gap sensors (proximity transducers). The lateral strains are those that have not been corrected for the possible effects of membrane penetration (MP). This decision was made based on the following facts:

(a) Goto (1986) has shown that the effects of MP are mainly due to elastic deformation of the latex rubber membrane, and they are essentially stress history-independent. Therefore, the dependency of strains on intermediate stress paths can be evaluated reasonably even with possible effects of MP.

(b) There have been two independent experimental studies in the authors' laboratory on the effects of MP in tests using Toyoura sand, with inconsistent results. Goto (1986), using solid cylindrical specimens, showed that MP errors were not negligible, while Kamegai (1994), using rectangular prism specimens, showed that they were negligible when compared with the bedding errors at the top and bottom lubricated ends of a specimen. At this stage, therefore, we have no reliable method for MP error correction.

The positioning of the LDTs and gap sensors (GSs) on the specimen's lateral surfaces, along with the specimen dimensions, are shown in Fig. 3. Changes in the specimen horizontal and vertical cross-sectional areas during each test were considered in the calculation of the axial (or vertical) stress $\sigma'_1$ and the intermediate principal stress $\sigma'_i$. All the tests were performed using the same batch of Toyoura sand, designated as 'Batch-E' (Yasin et al., 1999). Specimens were prepared by air-pluviating air-dried sand particles into a mold and subsequent wetting and freezing. After being placed in the apparatus, the specimens were thawed under a suction of 5 kPa (0.05 kgf/cm²), and then made saturated with a back pressure of 49 kPa (0.5 kgf/cm²). Similar to those employed by Kamegai (1994), the stress paths in these tests comprised of three straight lines in the ($\sigma'_1$, $\sigma'_i$) space (Figs. 4(a) through 4(d)). From the initial isotropic stress state of $\sigma'_1 = \sigma'_i = 29$ kPa (0.3 kgf/cm²), denoted as A in Figs. 4(a) through 4(d), the specimens were first sheared to a desired stress ratio, $R = \sigma'_1/\sigma'_i = R_s$, followed by compression along an $R$ (= $R_s$) constant path to either $\sigma'_1/p'_s = 4.0$, i.e. $\sigma'_i = 392$ kPa (4.0 kgf/cm²) (Figs. 4(a) and 4(c)) or $\sigma'_1 + \sigma'_i = 8p'_s$ (Fig. 4(b)). The specimens were then sheared to failure. Depending on the final shearing stress path, the tests will be termed: (i) $\sigma'_1 = \text{constant}$ tests, (ii) ($\sigma'_1 + \sigma'_i$) = constant tests, and (iii) $\sigma'_i = \text{constant}$ tests. In each test, the initial and final shearing stress paths were parallel to each other. The measured stress paths (not the intended ones) in ($\sigma'_1$, $\sigma'_i$) space for these three types of tests are shown in Figs. 4(a), (b) and (c) and together in Fig. 4(d). Here $e_{05}$ means the initial void ratio measured when $\sigma'_1 = \sigma'_i = 5$ kPa (0.05 kgf/cm²). In Fig. 4(d), the intersection points of different stress paths are labeled A, B, C · · · Z, a and b. Strain increments that occurred between two of the same stress points but with different intermediate stress paths will be compared later. In type (i) tests (Fig. 4(a)), small unload-reload cycles were applied at several stress states along the

![Fig. 3. Positioning of LDTs and gap sensors on the specimen $\sigma_2$ surfaces](image-url)
Fig. 4(a). Measured stress paths in type (i) tests; i.e. $R$-constant path followed by shearing at $\sigma'_v = 4, p'_a = 392$ kPa ($p'_s = 98$ kPa)

Fig. 4(b). Measured stress paths in type (ii) tests; i.e. $R$-constant path followed by shearing at $\sigma'_v + \sigma'_h = 8, p'_a = 784$ kPa ($p'_s = 98$ kPa)

Fig. 4(c). Measured stress path in type (iii) tests; i.e. $R$-constant path followed by shearing at $\sigma'_v = 14, p'_a = 1.37$ MPa ($p'_s = 98$ kPa)

Fig. 4(d). Common stress points for different stress paths at which the stress history-dependency of strain increments were examined

$R$-constant path, which are not discussed in this paper.

Using this apparatus, Park (1993) performed a series of $\sigma'_v = \text{constant}$ plane strain compression tests (PSC tests) at an axial (or vertical) strain rate of 0.125%/min on Toyoura sand. So, to keep consistency, initially it was intended to use the same axial strain rate in all the shearing tests. However, it was observed in trial tests performed when developing the control software program that the measured stress paths other than $\sigma'_v = \text{constant}$ paths were not very smooth when the axial strain rate was
faster than 0.0125%/min. This restriction was due to the relatively large volume of the triaxial cell compared to the air flow capacity of the regulators used. On the other hand, as continuous attendance and monitoring was required to reset some of the instrumentation during each test, a considerably long test was intended to be avoided. As a compromise, an axial strain rate of 0.125%/min was selected for the final PSC shearing along \( \sigma_{ii}' = \text{constant} \) paths, while a 10 times slower axial strain rate of 0.0125%/min was applied for all other stress path tests.

Park (1993) confirmed that the effects of ten times difference in the constant axial strain rate on the overall stress-strain and strength behaviour of Toyoura sand is negligible. This point was re-confirmed in the present study. As seen from Figs. 5(a) and 5(b), three specimens, having nearly the same void ratios, reached the same stress state I after the initial PSC phase (path A to A') and anisotropic compression at a stress ratio \( R_e \), equal to 3.0 (path A' to I) at a constant axial strain rate \( \dot{\varepsilon}_e \), equal to 0.0125%/min (\( = \dot{\varepsilon}_a \)). The specimens were loaded from stress state I at a constant \( \sigma_{ii}' = 392 \text{ kPa}; \ 4.0 \text{ kN/m}^2 \) towards peak stress state (P), as the other specimens described in Figs. 4(a) through 4(d), at a constant axial strain rate equal to \( \dot{\varepsilon}_e = 10^{-2} \dot{\varepsilon}_a \) in test T303C and \( \dot{\varepsilon}_a \) in test T307C. It may be seen that the effects of strain rate are negligible on the overall stress-strain behaviour. To further confirm this point, in the third test (test T308C), the constant axial strain rate was changed stepwise by a factor of ten at several stress states between state I and state P, denoted by vertical arrows directing upwards and downwards in Fig. 5(a). It may be noted that the stress level suddenly increased and decreased immediately after each step increase and decrease in the constant axial strain rate. However, the changes in stresses were not large, and with further straining after a step change in stress, the stress-strain relationship tended to rejoin the original stress and strain relationship that would have been obtained by applying the same constant strain rates throughout from state I to state P (i.e., the same as tests T303C and T307C). The effects of step change in constant axial strain rate on the stress-strain behaviour of sands and other geomaterials are discussed in great detail elsewhere (Tatsuoka et al., 1998, 1996 and Matsushita et al., 1998), which is beyond the scope of this paper. Figure 5(c) will be discussed later.

**DEPENDENCY OF STRAIN ON INTERMEDIATE STRESS PATH**

The relationships between the stress ratio \( \sigma_{ii}' / \sigma_{ii}' \) and the vertical strain \( \varepsilon_z \) from type (i) tests with \( R_e = \sigma_{ii}' / \sigma_{ii}' \geq 3.0 \) and those with \( R_e < 3.0 \) are shown in Figs. 6(a) and 6(b). Similar relationships for shear and volumetric strains are shown in Figs. 7 and 8. Stress history-dependency of vertical, shear and volumetric strains can be clearly noted in these figures. It may be seen from Fig. 6 that the vertical strain \( \varepsilon_z \) at the end of R-compression phase with a certain \( R_e \) is larger than those that have been attained at the same stress point in tests with lower values of \( R_e \). For example, starting from the same stress point A, the vertical strain at point F (see Fig. 4(d)) in test TE452 (\( R_e = 4.5 \)) is larger than that in test T303C (\( R_e = 3.0 \)). That is, the larger the difference in \( R_e \) between different stress histories, the larger the stress history-dependency of vertical strain.
Fig. 6(a). Stress ratio $\sigma'_v/\sigma'_i$ vs. vertical strain $\varepsilon_v$ (by LDT measurements) relations from $\sigma'_i=$ constant tests with $R_v=\text{3.0 to 6.1}$

Fig. 6(b). Stress ratio $\sigma'_v/\sigma'_i$ vs. vertical strain $\varepsilon_v$ (by LDT measurements) relations from $\sigma'_i=$ constant tests with $R_v=\text{0.33 to 2.5}$

Fig. 7(a). Stress ratio $\sigma'_v/\sigma'_i$ vs. shear strain $\gamma$ (by LDT and GS measurements) relations from $\sigma'_i=$ constant tests with $R_v=\text{3.0 to 6.1}$

Fig. 7(b). Stress ratio $\sigma'_v/\sigma'_i$ vs. shear strain $\gamma$ (by LDT and GS measurements) relations from $\sigma'_i=$ constant tests with $R_v=\text{0.33 to 2.5}$

Fig. 8(a). Stress ratio $\sigma'_v/\sigma'_i$ vs. volumetric strain, $\varepsilon_{vol}$ (by LDT and GS measurements) relations from $\sigma'_i=$ constant tests with $R_v=\text{3.0 to 6.1}$

Fig. 8(b). Stress ratio $\sigma'_v/\sigma'_i$ vs. volumetric strain, $\varepsilon_{vol}$ (by LDT and GS measurements) relations from $\sigma'_i=$ constant tests with $R_v=\text{0.33 to 2.5}$
The stress history-dependency of shear strain $\gamma = \varepsilon_v - \varepsilon_h$ is even larger, as evident from Figs. 7(a) and 7(b). This behaviour results from the fact that both the vertical and lateral strains are stress history-dependent, being affected in the same direction. Referring to Figs. 8(a) and 8(b), for tests with $R_c = 5.0$ or below, the respective specimens underwent volume contraction during the initial PSC shearing phase as well as during the $R$-constant phase. Volume contraction continued after the start of PSC at $\sigma_1' = 392$ kPa (4.0 kgf/cm$^2$) until the stress ratio $R = \sigma_1'/\sigma_3'$ reached around 4.5, after which the specimens started dilating. For tests with $R_c > 5$, on the other hand, the specimens dilated from the beginning of the final PSC phase. In the two tests with $R_c = 5.9$ and 6.1, the specimens exhibited nearly zero volume change during the $R$-constant phase. These results described above are in good accordance with those obtained by Kamegai (1994), but more conclusive.

As will be analyzed later, the following two factors exist in the stress history-dependency of strains described above: a) the stress history-dependency for stress paths in which the shear strain is always increasing (tests other than T302E); and b) that for stress paths with a decrease and an increase in the shear strain (test T302E). The first factor is the main issue of the present study.

In all the relationships between $\sigma_1'/\sigma_3'$ and $\varepsilon_v$ (Figs. 6(a) and 6(b)), $\gamma = \varepsilon_v - \varepsilon_h$ (Figs. 7(a) and 7(b)) and the volumetric strain $\varepsilon_{vol} = \varepsilon_v + \varepsilon_h$ (Figs. 8(a) and 8(b)), it is observed that at the beginning of the final PSC phase, there is a small quasi-linear zone that ends at a yield point. After yielding, the stress-strain relation resumes the primary stress-strain curve for shearing at $\sigma_1' = 392$ kPa (4.0 kgf/cm$^2$) in other PSC tests starting from the respective lower stress ratio $R = \sigma_1'/\sigma_3'$. The development of such a quasi-linear zone is due to: a) effects of ageing at the end of the $R$-compression phase, which lasts about 5 to 10 minutes; and b) a step increase in the strain rate from the end of the ageing phase to the PSC phase. The effects of ageing observed with Toyoura sand in the present study were positive structural effects by drained creep deformation, not associated with cementation at interparticle contacts, which increased the stiffness of sand when loading was resumed after creep stages. The size of the quasi-linear zone increased with the increase in the ratio $R$ at which the specimen was aged (or subjected to drained creep). Figure 5(c) shows the initial part of the relationships between $\sigma_1'/\sigma_3'$ and $\varepsilon_v$ for the final PSC phase (n.b., in test T309C, the constant strain rate was changed stepwise as in test T308C during the final PSC path). It may be seen that the size of the quasi-linear zone became smaller as the strain rate in the final PSC phase became smaller. These time effects are discussed in detail by Tatsuoka et al. (1998, 1999b).

The shear and volumetric strain increments that occurred between the same two stress states but with different intermediate stress paths are compared in Figs. 9 and 10. In these figures, the strain increments concerned are plotted along the $Y$-axis against the name of the two stress points under consideration indicated along the $X$-axis. For example, referring to Fig. 4(d), in Fig. 9(a), all the data points that lie on the ordinate above AB show the shear strain increments that were observed between the stress points A of $\sigma_1' = 29$ kPa (0.3 kgf/cm$^2$) and $\sigma_3' = 29$ (see Fig. 4d).
kPa (0.3 kgf/cm²) and B of $\sigma_s^I = 392$ kPa (4.0 kgf/cm²) and $\sigma_s^I = 2274$ kPa (23.2 kgf/cm²) in different tests. The values of $\sigma_s^I$ and $\sigma_s^I$ at stress points A, B and so on are listed on the right side of each figure. Figures 9 and 10 reveal clearly the effects of intermediate stress paths between two respective stress states on the resulting strain increments. For example, according to Fig. 9(a), the difference between the shear strain increments that occurred in tests T610C ($R_c=6.1$ during the R-constant path) and test T202C ($R_c=2.0$ during the R-constant path) due to changes in stress state from A to B is more than 2.5%. According to Fig. 10(a), the largest difference between the volumetric strain increments that occurred between the same two stress points is more than 1.0%. Both the shear strain increment and negative volumetric strain increment (n.b., volume contraction is positive) are larger with stress paths that are closer to the failure envelope. In other words, the stress history-dependency becomes larger when strains are compared for stress histories with and without stress paths along which sand exhibits dilating behaviour. In stress path triaxial tests on sand, Lade (1976) observed similar behaviour with axial strain.

The test results from the present study show that the effects of stress history are generally larger on the shear strain than on the axial (vertical) strain, and the effects on the volumetric strain are even larger. In Cam clay type models (Schoeefeld and Wroth, 1968; Muir Wood, 1992), plastic volumetric strain is the stress history-independent state parameter, and is used as the hardening parameter. For sands, plastic shear strain has been used as the hardening parameter in modeling stress-strain relationships of sand by many researchers (e.g., Tatsuoka et al., 1993), assuming that it is stress history-independent. This study shows that all the strain increments, i.e., vertical, lateral, shear and volumetric (and their plastic parts, as shown below), are stress history-dependent, and therefore, they alone are not appropriate to be used as the hardening parameters for sand. Soil elements in the field may be subjected to various stress paths. Therefore, the prediction of soil deformation assuming linear summation of strain increments from conventional tests (isotropic compression, followed by shearing at a constant lateral pressure) without considering the effects of intermediate stress history is likely to be erroneous.

Hoque (1996) and Hoque and Tatsuoka (1998) showed that the incremental elastic deformation characteristics of sands are a specific function of instantaneous stress state, and elastic strains that are obtained by integrating elastic strain increments for a given stress history are stress history-dependent. It is then possible that the observed stress history-dependency of total strain increments is largely due to the stress history-dependency of integrated elastic strains. The plastic shear and volumetric strains were computed by integrating plastic strain increments obtained as "measured total strain increments" - "elastic strain increments". The latter was obtained based on the hypo-elasticity model, which incorporates inherent and stress system-induced anisotropy of elastic deformation (Hoque and Tatsuoka, 1998). The results are plotted in Figs. 11(a) and 11(b), which clearly show that both plastic shear strain $\gamma^p$ and plastic volumetric strain $\varepsilon_{vol}^p$ are considerably stress history-dependent, essentially to the same degree as total strain increments (see
Fig. 11(a). Stress ratio $\sigma'_i/\sigma'_s$ vs. plastic shear strain $\gamma^p$ relations in type (i) tests

Fig. 11(b). Stress ratio $\sigma'_i/\sigma'_s$ vs. plastic volumetric strain $\varepsilon^p_{vol}$ relations in type (i) tests

Fig. 12(a). $\phi_{peak}$ vs. initial void ratio $e_{05}$ relations in stress path PSC tests

Fig. 12(b). $\phi_{residual}$ vs. initial void ratio $e_{05}$ relations in stress path PSC tests

Figs. 7(a) and 8(a).

**PEAK AND RESIDUAL STRENGTHS**

The angles of internal friction at peak $\phi_{peak} = \arcsin \left( \left( \sigma_1 - \sigma_3 \right) / \left( \sigma_1 + \sigma_3 \right) \right)_{\text{max}}$ and those at the residual stage $\phi_{residual}$ obtained from the different stress path tests are plotted against the initial void ratio $e_{05}$ in Figs. 12(a) and 12(b), respectively. The confining pressure $\sigma'_s$ at failure, $(\sigma'_s)_f$, for each data point is indicated in these figures. No appreciable trend by different intermediate stress paths can be observed in the data points with $(\sigma'_s)_f = 4.0$ from type (i) tests. For a given void ratio, the $\phi_{peak}$ values scatter less than $1^\circ$, which can be considered as a normal scatter of such data. One may consider, however, that this scatter is due to different $\sigma'_i$ values at the peak stress state among these different tests. However, these $\sigma'_i$ values were very similar, not depending on the intermediate stress history, as shown later. Therefore, within the limit of stress range covered in the present study, the stress history appears to have no significant effects on the peak angle of internal friction for Toyoura sand. Bishop and Eldin (1953), Haythornthwaite (1960) and Lade (1976) have also
shown similar results from triaxial compression tests on sand employing a variety of stress paths. A similar conclusion can be drawn for $\phi_{\text{residual}}$ as seen from Fig. 12(b).

The $(\sigma_3')_T$ values are different among type (i), (ii) and (iii) tests. The pressure level-dependency of both the peak and residual friction angles are noticeable, which is perhaps due to effects of larger deformability and crushability of Toyoura sand particles at higher pressure levels. The pressure level-dependency of $\phi_{\text{residual}}$ is discussed more in detail by Siddique et al. (1999).

**INTERMEDIATE PRINCIPAL STRESS**

For type (i) tests, the ratio of the intermediate principal stress $\sigma_2'$ to the effective cell pressure $\sigma_3'$ is plotted against the stress ratio $\sigma_2'/\sigma_3'$ in Figs. 13(a) and 13(b). Note that $\sigma_2'$ was equal to the intermediate principal stress $\sigma_2'$ at loaded conditions in PSC, but it was not the case in the initial stress states in nearly half of the tests. The intermediate principal stress $\sigma_2'$ more-or-less drifted from the respective initial setting value during saturation of specimen and other operations. However, the values of $\sigma_2'$ could not be controlled from outside the triaxial cell. Despite these variations in the initial $\sigma_2'/\sigma_3'$ ratios, the relationships between $\sigma_2'/\sigma_3'$ and $\sigma_2'/\sigma_3'$ remained parallel for the initial PSC paths. However, during the $R$-compression phase, they joined an essentially unique line either from above (for initial $\sigma_2'/\sigma_3'$ values greater than 1.0) or from below (for initial $\sigma_2'/\sigma_3'$ values less than 1.0) and then followed the unique line during the subsequent final $\sigma_2'$=constant PSC phase. In particular, the stress states ($\sigma_2'/\sigma_3'$, $\sigma_2'/\sigma_3'$) at the failure are essentially unique, being independent of the different initial $\sigma_2'/\sigma_3'$ ratios and the different intermediate stress paths. In addition, the scatter in the initial $\sigma_2'/\sigma_3'$ ratios was not systematic, while the stress history-dependency of strains was very systematic, as shown before. It is unlikely, therefore, that the scatter in the initial $\sigma_2'/\sigma_3'$ ratios had meaningful effects on the observed large stress history-dependency of strains. On the other hand, the comparison between Fig. 13(a) and 13(b) shows that the $\sigma_2'/\sigma_3'$ and $\sigma_2'/\sigma_3'$ relationships during $R$-compression phase and the final PSC phase are nearly the same whether the inner confining platens (shown in Fig. 2(b)) are used (Fig. 13(b)) or not (Fig. 13(a)). These results ensure that potential effects of using different confining platens are negligible. The data that support this point are presented in Masuda et al. (1999).

**FLOW RULE**

One of the simple and relevant flow rules for elasto-plastic models of soils is the relationship between stress ratio and plastic strain increment ratio, or the stress-dilatancy relationship, first proposed by Rowe (1962). Various theoretical considerations on stress-dilatancy relationships for monotonic loading and their experimental verifications have been provided by Rowe et al. (1964), Rowe (1969), Horne (1965a, b), Matsuoka (1974), Oda (1975), Tatsuoka (1976), Tokue (1978), and others. Pradhan et al. (1989) examined the stress-dilatancy relationship of sand subjected to cyclic loading. However, potential effects of stress path on the stress-dilatancy relationship under plane strain conditions have not yet been reported in the literature.

The original stress-dilatancy relationship for plane strain conditions is:

$$R(=\sigma_1'/\sigma_3')=K\cdot D(=\frac{-d\varepsilon_s}{d\varepsilon_t}) \quad (2)$$

where $K$ is the material constant. The data obtained from type (i) tests are plotted in Fig. 14(a). Except test T302E, $\sigma_2'/\sigma_3'=\sigma_2'/\sigma_3'$ and $-d\varepsilon_s/d\varepsilon_t=0$. The data points for PSC tests at $\sigma_1'=\text{constant} (=29 \text{ kPa} \text{ and } 392 \text{ kPa})$ paths were obtained by fitting second degree polynomials on suitable segments of $\varepsilon_s$ and $\varepsilon_t$ relationships, as shown in Figs. 15(a) and 15(b), and taking the derivative of the fitted function. From Fig. 14(a), it can be observed that the stress-dilatancy relationship for the initial and final PSC paths at $\sigma_3'/p_3'=0.3$ and 4.0 are essentially the same.
Fig. 14. Stress-dilatancy relationships: (a) and (b): total and plastic strain increments for $R = \text{constant}$ and (i) $\sigma^*_p$-constant stress paths, (c) and (d): total and plastic strain increments for $R = \text{constant}$, (ii) $\sigma^*_e = \text{constant}$ and (iii) $(\sigma^*_p + \sigma^*_e) = \text{constant}$ stress path, and, (e) and (f): total and plastic strain increments for all the stress paths

On the other hand, the relationship between $\varepsilon_2$ and $\varepsilon_3$ is virtually a straight line during the $R$-compression phase (see Figs. 15(a) and (b)). The values of $D$ for these paths were computed from the increments $\Delta e_2$ and $\Delta e_3$ between the starting and ending points of each $R$-compression phase. Thus, in Fig. 14(a), data for each $R$-compression phase is represented by a single point. It may be seen that the relationships for the $\sigma^*_p = \text{constant}$ and the $R$-compression stress paths (except the one at $R$ close to 1.0) appear to be better fitted by two equations in the form of $R = K \cdot D + C$, where $C$ is a constant. Park (1993) obtained similar empirical relationships from a large number of PSC tests on a wide variety of sands. This form is different from Eq. (2). Although $K$ is nearly the same between the two types of stress paths, the constant $C$ appears to be different between them. Figure 14(c) shows
similar results from type (ii) and (iii) tests to those from the associated R-compression phase, and Fig. 14(e) compares those from all the tests. The effects of stress path direction are noticeable.

Figure 14(b) shows the stress-dilatancy relations based on plastic strain increments, obtained by the same method used to obtain the results presented in Figs. 11(a) and 11(b). Although it is not totally negligible, particularly at R close to 1.0, the stress path-dependency in Fig. 14(b) is noticeably smaller than that in Fig. 14(a). Figure 14(d) shows similar results from type (ii) and (iii) tests, and Fig. 14(f) compares those from all the tests. It may be seen that the effects of stress path direction are smaller in the plots with plastic strain increments (Fig. 14(f)) than in those with total strain increments (Fig. 14(e)), but still noticeable. More discussions on this issue are beyond the scope of this paper.

STRESS HISTORY-INDEPENDENT STATE PARAMETER

Following Nakai (1989), the possibility of plastic work, or its function, to be stress history-independent was explored. Figure 16 shows the relationships between the plastic work defined as follows and the mean stress $s=(\sigma_1^\prime+\sigma_3^\prime)/2$ in type (i) tests:

$$W^p = \int \left( t \cdot dp^p + s \cdot ds_{vol} \right)$$

(3)

where $t=(\sigma_1^\prime-\sigma_3^\prime)/2$ and $s=(\sigma_1^\prime+\sigma_3^\prime)/2$ and $W^p$ is equal to 0 at the stress state A of $\sigma_1^\prime/p'_s=\sigma_3^\prime/p'_s=0.3$ (Fig. 4(d)). In the calculation of $W^p$, the elastic and plastic strains were separated as before. In this and the following figures, the plastic work $W^p$ and the stress $s$ have been made non-dimensional by being divided by $p'_s=98$ kPa. The data points at a certain $s/p'_s$ along the final PSC phase from different tests represent the values of $W^p$ at the same stress state. Clearly, the relationships along the final PSC stress path are not unique, showing that $W^p$ is not a state quantity for sand. The total work defined as
W = \int (t \cdot dy + s \cdot de_{vol}) was also obtained, as shown in Fig. 17 for type (i) tests. Noticeable dependency of the total work \( W \) on stress history may be seen. It can be easily shown that when based on the anisotropic elasticity model (Hoque and Tatsuoka, 1998), the elastic strain energy between two stress points is stress history-independent (Puzrin and Tatsuoka, 1998). Therefore, the stress history-dependency of the total work \( W \) (Fig. 17) is solely due to the stress history-dependency of plastic work \( W^p \).

Based on his triaxial compression test results mentioned earlier, Moroto (1980) considered that the plastic work increment done by the external stress increments consists of: (i) a plastic work increment done by shearing and (ii) a plastic work increment done by consolidation (or compression). The plastic work increment by shearing in triaxial tests was defined as:

\[
dW^p = p' \cdot (de_{vol})_a + q \cdot \frac{2}{3} \cdot dy^p
\]  

where \((de_{vol})_a\) is the increment of plastic volumetric strain due to dilatancy, which is obtained from tests at constant \( p' \) values. Moroto observed that the relationship between the integrated shear plastic work \( dW^p \) and the stress ratio \( q/p' \) was considerably dependent on the studied intermediate stress histories. However, a rather unique relationship between the parameter \( S_r = \| dW^p / p' \) and \( q/p' \) was obtained. So he proposed \( S_r \) as a stress history-independent state parameter for specifying the shear deformation of sand in triaxial tests on granular media.

Following this work by Moroto, the plastic work increment \( dW^p \) was normalized by being divided by the stress variable \( s = (\sigma'_1 + \sigma'_2) / 2 \), or by some power of \( s \), in the present study to obtain the following quantity:

\[
W^p = \int [(t \cdot dy^p + s \cdot de_{vol}) / (s / p'_c)]^n .
\]  

Inclusion of \( p'_c = 98 \) kPa in Eq. (5) is to maintain the unit of the quantity \( W^p \) to the same as the plastic work \( W^p \) (Eq. (3)). Different values of \( n \) were tried. Figure 18(a) shows the case for \( n = 1.0 \). It was found that the relationships between \( W^p / p'_c \) and \( s / p'_c \) for the final PSC shearing paths from type (i) tests became very unique for \( n = 0.9 \) (Fig. 18(b)), essentially not influenced by stress history. That is, \( W^p / p'_c \) with \( n = 0.9 \) is a state parameter for Toyoura sand in plane strain shear.

DISCUSSIONS ON THE PROPOSED STATE PARAMETER

In the theory of hardening plasticity, a yield function or loading function \( f(\sigma_0, k) = 0 \) characterizes the yielding of the material, where \( \sigma_0 \) is the stress tensor, and \( k \) is the hardening parameter, usually a function of plastic strain or plastic work. Changes in plastic deformation occur only when \( f = 0 \) (i.e. when the stress state is on the yield surface) and \( df > 0 \). No meaning is associated with \( f > 0 \). The growth of the yield surface is associated with an increase in the hardening parameter \( k \). In this study, attempts were not made to delineate the entire yield surfaces by performing stress probe tests. Rather, advantage was taken of the established forms of yield locus for sands reported in the literature to seek a stress history-independent state parameter.

In Cam clay model, it is assumed that the yield locus expands with a constant shape, controlled by the tip \( p' \) value, which is a unique function of plastic volumetric strain. Therefore, Cam clay model is classified as a volumetric hardening model. On the other hand, Pooorooshab et al. (1967) suggested that yield loci for sand are lines of constant stress ratio, \( q/p' = \text{constant} \). A modified form of yield loci, \( q/p' + m \cdot \ln (p') = \text{constant} \), was later proposed by Pooorooshab (1971), where \( m \) is a positive constant. A similar observation has also been reported by Stroud (1971) from simple shear test results on sand. A comprehensive series of triaxial tests was performed to investigate the yielding of sand by Tatsuoka (1972) and the results were summarized by Tatsuoka and Ishihara (1974). They found that a complete picture of yield loci in \((q, p')\) space can be described in a more general form \( q/p' = F(p') + \eta_0 \). It has also been shown that these yield loci represent equi-shear strain lines, forming the basis of a shear strain hardening model for sand (Tatsuoka and Ishihara, 1974; Ishihara et al., 1975).

On the basis of the aforementioned form for yield loci, \( q/p' + m \cdot \ln (p') = \text{constant} \) in the triaxial stress space \((p', q)\), a stress parameter of the form \( \eta + r \cdot \ln (s/p'_c) = \text{constant} \) in the plane strain stress space \((t, s)\) was conceived, where \( \eta = t / s, t = (\sigma'_1 - \sigma'_2) / 2, s = (\sigma'_1 + \sigma'_2) / 2 \) and \( p'_c = 98 \) kPa.

In Fig. 19, the quantity \( W^p / p'_c \) with \( n = 0.9 \) from type (i) test results (excluding the result from test T302E,
which consists of stress paths in both PSC and PSE) is plotted against $\eta + r \cdot \ln (s/p_r^*)$ with $r=0.09$. Among different values of $r$ tried, the most unique relationship was obtained for $r=0.09$ between $W^p/\sigma_r$ and $\eta + r \cdot \ln (s/p_r^*)$ for different R-compression tests and the initial and final $\sigma_r^* = constant$ PSC shearing phases. The value $r=0.09$ may be linked to the curvature of the failure envelope, which may be controlled by the deformability and crushability of sand particles, among other factors. Further study will be required to understand the meaning of this quantity in physical terms. The results shown in Fig. 19 suggest that the quantity $W^p = \{ (s \cdot \dot{\gamma}/s \cdot d \varepsilon_{\text{a}}) / (s/p_r^*) \}$ as a function of $X = \eta + r \cdot \ln (s/p_r^*)$ can be used as a stress history-independent as well as stress path-independent hardening parameter for combined (or total) yielding in shear and compression. An outline as to how the $W^p \sim X$ relationship as the hardening function can be applied to determine the plastic shear and volumetric
strain increments is presented in the appendix.

The relationship between \( Y = W^{p'} / p'_s \) and \( X = \eta + r \ln (s/p'_s) \) was fitted by a function \( Y = A \cdot \exp (B \cdot X) - C \) (see Fig. 20), where \( A, B \) and \( C \) are non-dimensional constants. Note that \( Y = 0.0 \) at stress state A (Fig. 4(d)) where \( t/p'_s = 0.0 \) and \( s/p'_s = 0.3 \) with \( X_0 = -0.1084 \), giving \( C = A \cdot \exp (B \cdot X_0) \). Note also that as the yield loci for shearing and compression should be different from each other, they would not be the same as this function for \( W^{p'} \). The quantities \( W^{p'} / p'_s \) and \( W^{p'} / p'_s \) with \( n = 0.9 \) from all the tests are plotted against \( s/p'_s \) in Figs. 21 and 22 respectively. The quantity \( Y = W^{p'} / p'_s \) is also plotted against \( X = \eta + r \ln (s/p'_s) \) with \( r = 0.09 \) for all the three types of tests in Fig. 23. It is clear from Fig. 23 that the proposed \( W^{p'} - X \) relationship is suitable also for \( \sigma'_c = \text{constant} \) stress paths (test T350V, Fig. 4(c)) and \( \sigma'_c + \sigma''_c = \text{constant} \) stress paths (tests T101C, T351S and T392S, Fig. 4(b)), except for test T302E.

**MODIFICATION TO INCORPORATE TESTS WITH PSE AND PSC STRESS PATHS**

In the preceding section, discussions were limited to the \( R \)-constant compression followed by monotonic PSC at a constant or varying \( \sigma'_c \). In this section, the effectiveness of the stated function is examined by using data from test T302E. Test T302E consists of shear loading stress paths in plane strain compression and extension (PSC and PSC). The quantity \( Y = W^{p'} / p'_s \) with \( n = 0.9 \) from test T302E is plotted against \( X = \eta + r \ln (s/p'_s) \) with \( r = 0.09 \) in Fig. 23 along with those from all the previously considered tests and separately in Fig. 24. It is seen that \( W^{p'} / p'_s \) from test T302E is not in accordance with those from the other tests with monotonic shear loading paths. It is obvious that for cyclic shear loading, the energy function \( W^{p'} \) continues to increase during both loading and unloading along the same stress path, and a unique \( Y - X \) relationship cannot be obtained.

The results shown in Fig. 23 suggest that the slope of the \( Y = W^{p'} / p'_s \) and \( X = \eta + 0.09 \ln (s/p'_s) \) relationships for loading in PSC might be unique. So, the quantity \( dy/dx = Y_i \) was calculated. It should be remembered here that the stress and strain relationships, shown in Figs. 6, 7, 8 and 11, have small pseudo-elastic zones with
very small plastic strains at the beginning of the final PSC shearing path (see Fig. 5(c)). In the \( Y = W^p/\sigma' \) and \( X = \eta + 0.09 \cdot \ln (s/p'_s) \) curves (Figs. 19 and 23), this pseudo-elastic zone turned out as a kink and a relatively flat zone at the beginning of the final PSC shearing path. Eventually, at the beginning of the final PSC shearing path for each test, the quantity \( Y_t = dY/dX \) dropped and then sharply increased to rejoin the normal relationship, as can be observed from typical plots of \( Y_t \) against \( X \) in Fig. 25. In Fig. 26, the quantity \( Y_t \) is plotted against the quantity \( X \) for a range from the initial isotropic stress states to the peak stress state in the final PSC shearing path for all the tests. In this figure, the expected uniqueness of the relationships between \( Y_t \) and \( X \) are masked by data points in the stated pseudo-elastic zones. The data points in the pseudo-elastic zones have been removed in the plots shown in Fig. 27. Then, the relationships between \( Y_t \) and \( X \) become unique. This \( Y_t \sim X \) relationship is the tangential form of a generalized hardening function for loading in PSC. A similar but different hardening relationship for loading in plane strain extension would be obtained for loading from point \( a \) to point \( b \) in Fig. 24. The behaviour inside the stated pseudo-elastic zones and that after these zones are discussed in detail in Tatsuoka et al. (1998); this issue is beyond the scope of this paper.

CONCLUSIONS

From the results of a series of drained stress path plane strain tests on Toyoura sand presented above, the following conclusions can be derived:

1) All the strain increments (axial, lateral, shear and volumetric) are noticeably stress history-dependent even for stress paths where shear strains are consistently increasing. In particular, when tracing stress paths closer to the failure envelop, larger shear strain increments with larger volume expansion due to dilatancy were observed. This fact means that none of these strain quantities are a state parameter.

2) The effects of stress history and stress path on the friction angles at the peak and residual states were found negligible.

3) The effects of stress path on the stress-dilatancy relationship based on total strain increments were noticeable. On the other hand, the relationships based on plastic strain increments were less stress path-dependent, but still noticeable.

4) The plastic work defined as \( W^p = \int (t \cdot dy^p + s \cdot de_{vol}^p) \) depends on the stress history, where \( t = (\sigma'_s - \sigma'_i) / 2 \) and \( s = (\sigma'_i + \sigma'_d) / 2 \), and \( dy^p = de_{e}^p + de_{h}^p \) and \( de_{vol}^p = de_{e}^v + de_{h}^v \). The quantity \( W^p = \int (t \cdot dy^p + s \cdot de_{vol}^p) (s/p'_s)^{0.9} \) (\( p'_s = 98 \) kPa) is essen-
Fig. 26. Relationships between $Y_i = dY_i/dX$ ($Y = W^o_{s}/p'_a$) and $X = t/s + r \cdot \ln (s/p'_a)$ for all the stress path tests [n = 0.9 and r = 0.09]

Fig. 27. Relationships between $Y_i = dY_i/dX$ ($Y = W^o_{s}/p'_a$) against $X = t/s + r \cdot \ln (s/p'_a)$ from all the stress path tests after removing the quasi-elastic zone at the start of PSC at $\sigma_s = 392$ kPa (4.0 kgf/cm$^2$) [n = 0.9 and r = 0.09]

Partially stress history-independent, while $W^o_{s}$ is a rather unique function of the stress parameter $X = t/s + 0.09 \cdot \ln (s/p'_a)$ for different stress histories and stress paths. That is, the quantity $W^o_{s}$ is a state parameter, which is unique for a given stress state with stress histories including loading only. It is suggested that the quantity $W^o_{s}$ as a function of $X$ can be used as a hardening parameter in a double hardening elasto-plastic model. It is suggested that the yield functions and plastic potential functions for shear and volumetric yielding have to be established to make use of it.

5) The relationship between the rate $Y_i = dY_i/dX$ ($Y = W^o_{s}/p'_a$) and $X$ is shown to be applicable to more general stress paths including those in plane strain extension and compression.

This study is empirical in nature. Further study will be required to understand in physical terms the meanings of the empirical relationships obtained by the present study.
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REFERENCES


**APPENDIX**

Plastic shear strain increment $d\gamma^p$ and plastic volumetric strain $d\varepsilon^v_{vol}$ are considered to consist of sub-components as follows:

\[
d\gamma^p = d\gamma^s + d\gamma^c \tag{A-1}
\]

\[
d\varepsilon^v_{vol} = (d\varepsilon^w_{vol})_s + (d\varepsilon^w_{vol})_c \tag{A-2}
\]

where strain components with subscripts ‘s’ and ‘c’ refer to those caused by shear and compression yielding, respectively. In the present discussion, time effects on the stress-strain behaviour of sand are ignored.

Defining $\phi_s$ and $\phi_c$ as the plastic potential functions corresponding to, respectively, the shear yield function $f_s$ and the volumetric yield function $f_c$, we obtain:

\[
d\gamma_s^2 = \frac{\partial \phi_s}{\partial \tau} \cdot d\tau_s \tag{A-3}
\]

\[
(d\varepsilon^w_{vol})_s = \frac{\partial \phi_s}{\partial \varepsilon_s} \cdot d\varepsilon_s \tag{A-4}
\]

\[
d\gamma_c^2 = \frac{\partial \phi_c}{\partial \tau} \cdot d\tau_c \tag{A-5}
\]

\[
(d\varepsilon^w_{vol})_c = \frac{\partial \phi_c}{\partial \varepsilon_c} \cdot d\varepsilon_c \tag{A-6}
\]

where $d\tau_s$ and $d\tau_c$ are scalar quantities, which are functions of $dt$ and $ds$. Equations (A-1) through (A-6) can be combined to form a set of equations as follows:

\[
\begin{bmatrix}
 d\gamma^p \\
 d\varepsilon^v_{vol}
\end{bmatrix}
= \begin{bmatrix}
 c^p \\
 c^v
\end{bmatrix}
\begin{bmatrix}
 dt \\
 ds
\end{bmatrix} \tag{A-7}
\]

where $c^p$ is the plastic compliance matrix, which is a function of current stress state.

The change in the function $W^* = W^* + s^* \cdot (d\varepsilon^w_{vol})_s / (s^*/p^*)^n$.

By substituting Eqs. (A-1) and (A-2) into Eq. (A-8) and by rearranging, we obtain:

\[
W^* = (t \cdot d\gamma^s + s \cdot (d\varepsilon^w_{vol})_s) / (s^*/p^*)^n + (t \cdot d\gamma^c + s \cdot (d\varepsilon^w_{vol})_c) / (s^*/p^*)^n \tag{A-9}
\]

As $W^*$ is functions of both shear and compression yield functions $f_s$ and $f_c$, we obtain:

\[
W^* = \frac{\partial W^*}{\partial f_s} \cdot df_s + \frac{\partial W^*}{\partial f_c} \cdot df_c \tag{A-10}
\]

Equating Eqs. (A-9) and (A-10) and using Eqs. (A-3) through (A-6) yields:

\[
d\lambda_s \cdot \left[ t + \frac{\partial \phi_s}{\partial \gamma_s} + s \cdot \frac{\partial \phi_s}{\partial \varepsilon_s} \right] (s^*/p^*)^n + d\lambda_c \cdot \left[ t + \frac{\partial \phi_c}{\partial \gamma_c} + s \cdot \frac{\partial \phi_c}{\partial \varepsilon_c} \right] (s^*/p^*)^n
\]

\[
= \frac{\partial W^*}{\partial f_s} \cdot df_s + \frac{\partial W^*}{\partial f_c} \cdot df_c \tag{A-11}
\]

When $df_c = 0$, then the second term on the left hand side of Eq. (A-11) is zero and when $df_s = 0$, the first term on the left hand side of Eq. (A-11) is zero. Thus,
\[ d\lambda_s = \frac{(\partial W^\ast)}{(\partial \phi_c)} \bigg|_{df_c=0} \quad df_c. \quad (A-13) \]

Let us tentatively assume that the yield functions in shear and compression be as follows:

\[ f_s = \eta + m_s \cdot \ln (s) \quad (A-14) \]
\[ f_c = s. \quad (A-15) \]

Qualitative plots of these yield loci are shown in Figs. A-1(a) and (b). The coefficient \( m_s \) controls the shape of the yield locus \( f_s = \eta + m_s \cdot \ln (s) \) for shear yielding. The shear yield locus becomes more curved with the increase in \( m_s \), as shown in Fig. A-1(a). Because the quantity \( W^\ast \), as a function of \( \eta + r \cdot \ln (s) \), should increase due to compression yielding even for loading along a constant \( f_s \), the coefficient \( m_s \) should be larger (perhaps only slightly) than \( r \) (see Fig. A-2). When based on Eqs. (A-14) and (A-15), we obtain

\[ \left( \frac{\partial W^\ast}{\partial f_s} \right)_{df_s=0} = \left( \frac{\partial W^\ast}{\partial \phi_c} \right)_{df_c=0} = \left( \frac{\partial W^\ast}{\partial \eta} \right)_{df_s=0} \quad (A-16) \]

because we get \( (\partial \eta / \partial f_s)_{df_s=0} = 1 \) from \( \eta = f_s - m_s \cdot \ln (s) \).

We also obtain:

\[ \left( \frac{\partial W^\ast}{\partial f_c} \right)_{df_c=0} = \left( \frac{\partial W^\ast}{\partial s} \right)_{df_c=0}. \quad (A-17) \]

Referring to Fig. A-3, we obtain

\[ \left( \frac{\partial W^\ast}{\partial s} \right)_{df_s=0} = \frac{1}{ds} \left[ W^\ast_s - W^\ast_a \right] \]

\[ = \frac{1}{ds} \left[ W^\ast_s - W^\ast_c + W^\ast_c - W^\ast_a \right] \]

\[ = \frac{1}{ds} \left[ W^\ast_s - W^\ast_a \right] + \frac{1}{ds} \left[ W^\ast_s - W^\ast_c \right] \]

\[ = \frac{1}{ds} \left[ W^\ast_s (\eta + ds, s) - W^\ast_s (\eta + ds, s + ds) \right] \]

\[ + \frac{1}{ds} \left[ W^\ast_c (\eta + ds, s + ds) - W^\ast_c (\eta, s) \right] \]

\[ = \frac{1}{ds} \left[ \frac{\partial W^\ast}{\partial \eta} \right] \bigg|_{d\eta=0} \quad ds + \frac{1}{ds} \left[ \frac{\partial W^\ast}{\partial s} \right] \bigg|_{ds=0} \quad ds. \quad (A-18) \]

As \( df_c=0 \) means that \( d\eta = -m_s (ds/s) \), we obtain:

\[ \left( \frac{\partial W^\ast}{\partial s} \right)_{df_c=0} = -m_s \left( \frac{\partial W^\ast}{\partial \eta} \right)_{d\eta=0} + \left( \frac{\partial W^\ast}{\partial s} \right)_{ds=0}. \quad (A-19) \]

On the other hand,

\[ t \frac{d\phi_s}{dt} = t \left( \frac{\partial \phi_s}{\partial \phi_c} \right) \left( \frac{\partial \phi_c}{\partial \eta} \right) t \left( \frac{1}{s} \right) = t \left( \frac{\partial \phi_s}{\partial \eta} \right) \left( \frac{1}{s} \right) = \eta \left( \frac{\partial \phi_s}{\partial \eta} \right). \quad (A-20) \]

By substituting Eqs. (A-15) through (A-19) into Eqs. (A-12) and (A-13), we get:

\[ d\lambda_s = \frac{\left( \frac{\partial W^\ast}{\partial \eta} \right)_{d\eta=0}}{\eta \left( \frac{\partial \phi_s}{\partial \eta} \right) + \frac{s}{\partial \phi_s}} \left( \frac{s}{p^\prime_s} \right)^n \left[ \frac{d\eta}{s} + \frac{-m_s}{s} ds \right] \quad (A-21) \]
Eqs. (A-21) and (A-22) together with Eqs. (A-3) through (A-6) can be used, with a known function $W^p = W^p(s, t)$, to find the plastic components of shear and volumetric strain increments, provided the plastic potential functions $\phi_s$ and $\phi_v$ and the coefficient $m_s$ have been defined.