STRESS-STRAIN RELATIONSHIP OF FROZEN SAND AS A COMPOSITE GEOMATERIAL

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ABSTRACT

The present paper aims at developing a simplified model for a stress-strain relationship at small strains up to an initial yielding of homogeneous saturated frozen sand. The modelling of elastic moduli for an ice-sand mixture as a composite geomaterial is first introduced based on previous studies, where a saturated frozen sand is treated as the ice-sand mixture. The applicability of modelling the elastic moduli under constant temperature is then discussed from the theoretical and experimental points of view. In addition, the hyperbolic stress-strain modelling of saturated frozen sand at small strain is presented. The effect of the volume fraction of sand, stiffness of sand inclusion and temperature changes on the elastic moduli and the stress-strain behaviour up to about 1-2% axial strain, which is defined as an upper yield stress, are investigated through the model, whose applicability is verified in comparison with the experimental data.

Key words: composite geomaterial, constitutive equation, elasticity, frozen soil, modulus of deformation, stress-strain relation (IGC: D6/E9/S2)

INTRODUCTION

Perennially frozen ground can be found in different places around the world. It underlies nearly 20% of the land surface, and is usually known as permafrost. Various types of permafrost exist due to the range of climatic and geographical conditions. An important distinction has to be made between Alpine and Arctic permafrost, and another exists between frozen rocks and frozen soils. However, all types of permafrost have something in common. Their structure is a matrix of mineral particles with different size, shape, pore ice, unfrozen water and air voids. The mechanical behaviour is dependent on the interaction of these components, which is strongly influenced by the existing temperature.

Since climatic change appears to be causing a gradual rise in ambient temperatures, the permafrost is melting and reducing in size, with some associated problems. Slope instability in Alpine permafrost is to be expected as well as excessive ground deformation under structures on the thawing ground. Research has begun recently in Switzerland to investigate the behaviour of Alpine permafrost (in the form of a rock glacier) from the geotechnical, geophysical and glaciological viewpoints. The contribution from the geotechnical engineering team is to clarify

fy and model the mechanical properties of the frozen ground, in order to evaluate the changes in the mechanical behaviour of Alpine permafrost with increasing temperature (Springman and Arenson, 1998). In addition, it is believed that there will be useful information transfer to and from methods used in artificial soil freezing, e.g. for ground improvement for underpinning foundations, or strengthening soil around excavations.

In this study, the modelling of elastic moduli for an ice-sand mixture as a composite geomaterial is first introduced based on the previous work presented by Voigt (1889), Reuss (1929) and Omine et al. (1993). A homogeneous saturated frozen sand, for which the sand inclusions are more or less uniformly distributed, is then treated as an ice-sand mixture. The applicability of using elastic moduli to model the ice-sand mixture at a constant temperature is then discussed from the theoretical and experimental points of view. In addition, simple modelling of a stress-strain relationship at small strain is presented for saturated frozen sand, reflecting the features of a composite material in the mechanical properties. This study is mainly focused on the effect on the elastic modulus in the small strain region of the volume fraction of sand, the stiffness of sand inclusion and temperature changes. It is noted that the response up to an

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Table 1. Effects of volume fraction, temperature, confining pressure and strain rate on secant Young's modulus E of ice and frozen sands based on prior data

References	Treated material	Mixture material	Range of E (GPa)	Axial strain corresponding	Dependency on			Measurement technique
					Volume fraction of sands	Temperature	Stress	Strain rate
Baker & Kurfurst (1985)	Frozen sand	Ottawa sand	24-30	$<10^{-5}$	decreases with decreasing volume fraction	not clear	—	—
Anderson et al. (1995)	Frozen sand	Manchester fine sand	21-35	$<10^{-4}$	decreases with decreasing volume fraction	not clear	not clear	independent
Shibata et al. (1985)	Frozen sand	Toyoura sand	0.5-8.0	<0.002	—	increases with reduction of temp.	not clear	not clear
Parameswaran (1980), Parameswaran & Jones (1981), Parameswaran & Roy (1982)	Frozen sand	Ottawa sand	1.2-5.6	0.001-0.005	—	increases with reduction of temp.	decreases with increasing confining pres- sure	decreases with decreasing strain rate
Goughnour & Anders- land (1968)	Frozen sand	Ottawa sand	0.6-7.0	0.005-0.09	not clear	decreases with reduction of temp.	—	—
Kuribayashi et al. (1985)	Frozen sand	Kiso river sand	0.05-1.0	0.015	—	increases with reduction of temp.	—	decreases with decreasing strain rate
Sinha (1989)	Ice alone	Granular ice	8.9-9.4	$<10^{-5}$	—	increases with reduction of temp.	—	—
Hawkes & Mellor (1972)	Ice alone	polycrystalline ice	7-11	$<10^{-4}$	—	—	—	independent of strain rate

Acoustic tests & on-specimen extensometer

Triaxial tests with on-specimen axial strain device

Triaxial tests without special devices for axial strain measurements

Uniaxial and triaxial tests without special devices for axial strain measurements

Triaxial tests without special devices for axial strain measurements

Supersonic pulse-echo method and double pulse interference technique (see Dantl (1969))

Uniaxial tests with dumbbells specimen

upper yield stress, which is defined as the first point on the stress-strain curve where the slope either becomes zero or reaches its minimum positive value before significant strain hardening (Parameswaran, 1980; Andersen et al., 1995), is herein regarded as small strain behaviour, for example, roughly up to 1–2% axial strain. The effect of the structure of the sand skeleton on the Young's modulus of the saturated frozen sand up to the upper yield stress is shown to be relatively small, compared with the effect of the stiffness of sand particles as a composite material (Andersen et al., 1995). For simplicity, the saturated frozen sand is called frozen sand in the following sections.

BRIEF REVIEW ON YOUNG'S MODULI OF ICE AND FROZEN SANDS

Past research conducted to find the initial moduli of ice and frozen sand is summarized in Table 1, together with the measured strain range, material used and main conclusions. E is defined as a secant Young's modulus. Figure 1 shows these moduli of ice and frozen sand schematically, reported against the measured axial strain range. Where not specifically mentioned below, the secant Young's modulus is simply named Young's modulus. The characteristics of the Young's moduli and Poisson's ratio, which have already been reported, are pointed out from this table and figure as follows;

- 1) Young's moduli and Poisson's ratios of granular ice (Sinha, 1989), typically obtained by a supersonic pulse-echo method and a double-pulse interference technique, is in the range from 8.9 to 9.4 GPa and from 0.308 to 0.311 respectively, for temperatures ranging from 0 to -38°C , in which the axial strain range obtained may be less than 0.01%. It has been shown that these values increase with the decreasing temperature; however, the strain rate dependency is very small.
- 2) Young's moduli of frozen sands, with different sizes of sand particles and with various relative densities from 20 to 80% for temperatures between -10 and -25°C , lie between 21 to 35 GPa, and the corresponding Pois-

Table 2. Effect of four factors on elastic moduli of ice-sand mixture at small strain region

	Strain range (Small strain)		
	<0.0001	0.001–0.0001	0.02–0.001
f_s -dependency	?	?	?
T_m -dependency	?	○	○
σ -dependency	?	?	○
Rate-dependency	×	?	○

?: not clear, ×: independent, ○: dependent

son's ratios are in the range from 0.2 to 0.3 (Baker and Kurfurst, 1985; Andersen et al., 1995). All the data are based on the axial strain range less than 0.01% (10^{-4}). Both values tend to increase with increasing relative density, although the influence of temperature and sand particle stiffness are not clear from this data.

3) As already pointed out by Andersen et al. (1995), the Young's modulus for frozen sands is about three times larger than that of granular ice at equivalent small strain ranges, which is also one or two orders of magnitude larger than the stiffness of the sand particle skeleton alone. It is therefore proposed that the frozen sand is treated as a composite material with an ice matrix and sand inclusions, and that the overall stiffness of the frozen sand at small strain is mainly dependent on the stiffness of the two components: the ice and the sand particles.

4) Young's moduli of frozen sands are strongly dependent on the corresponding strain range. However, the dependency of the remaining four factors, volume fraction, temperature, stress level and strain rate, on the Young's modulus in Table 1 remains unclear, and this is summarized in Table 2.

ELASTIC MODULI OF THE ICE-SAND MIXTURE

Elastic Moduli of an Ice-Sand Mixture with Isotropic Composite Materials

A two phase mixture consists of a basic and a supplementary material. These are called a matrix and an in-

Fig. 1. Characteristics of initial secant moduli of ice and frozen sands based on prior data

Fig. 2. Schematic view of two phase mixtures

clusion respectively, as shown schematically in Fig. 2. In this study, the ice is treated as the matrix and the sand particles are treated as the inclusions. In order to clarify the elastic behaviour of two phase mixtures, it is necessary to evaluate the stress and strain distributions in the mixture. A new parameter for estimating the stress distribution in the mixtures has been introduced by Omine et al. (1993), so that:

$$b = \frac{\bar{\sigma}_s}{\bar{\sigma}_*} \quad (1)$$

where, $\bar{\sigma}_s$ and $\bar{\sigma}_*$ are average stresses applied to the sand inclusions and the matrix respectively. Young's modulus E and Poisson's ratio ν of the two-phase mixtures containing isotropic elastic materials were derived using the stress distribution parameter b as follows:

$$E = \frac{(b-1)f_s + 1}{\frac{f_s b}{E_s} + (1-f_s)\frac{1}{E_*}} \quad (2)$$

$$\nu = \frac{f_s b \frac{\nu_s}{E_s} + (1-f_s)\frac{\nu_*}{E_*}}{\frac{f_s b}{E_s} + (1-f_s)\frac{1}{E_*}} \quad (3)$$

where,

$$b = \left(\frac{E_s}{E_*} \right)^{0.5} \quad (4a)$$

$$f_s = \frac{V_s}{V} = \frac{1}{1+e} \quad (4b)$$

and where, E_s , E_* and ν_s and ν_* are the Young's moduli and Poisson's ratios of the inclusion and the matrix respectively, and f_s is volume content of the inclusions, with V_s , V and e describing the volume of sand, total volume and void ratio respectively.

The key assumption for representing the stress distribution parameter as a function of the elastic moduli of inclusion and matrix materials in Eq. (4) is that the strain energy increments of the inclusion and the matrix per unit volume are equal, so that $dW_s = dW_*$. The outline of this derivation process is summarized in Fig. 3, and more detail has already been reported by Omine et al.

(1993). In addition, it is important to point out that, when assuming the mixture elements are subjected to a uniform strain, parameter b becomes $(E_s/E_*)^1$, and when assuming the mixtures are subjected to a uniform stress, parameter b would be derived as $(E_s/E_*)^0 = 1$. These are known as the Voigt (1889) and Reuss (1929) models, respectively (Omine and Ohno, 1997). The three different assumptions are summarized in Table 3, together with the equations which define the Young's moduli E and the Poisson's ratios ν of the mixture.

In this study, Eqs. (2)–(3) are utilized to evaluate the elastic and non-linear behaviour of frozen sand as an ice-sand mixture in the relatively small strain region up to the upper yield stress about 1–2% axial strain level. It is important to emphasize that, when using these equations, the elastic properties of the ice-sand mixture can be evaluated from those of ice and sand particles themselves as the function of the volume fraction of sand particles and a stress distribution parameter, although such equations do not directly represent the effect of the structure of sand skeleton on the Young's modulus of frozen sand.

Prediction of Elastic Moduli as an Ice-Sand Mixture

The Young's modulus was calculated from experimental tests by Andersen et al. (1995) on frozen Manchester fine sand at temperatures ranging from -10 to -25°C , dependent on the relative density, which was directly related to the volume fraction f_s . It is noted that the resultant Young's modulus is much larger than that of either pore ice or the sand skeleton mentioned above. As a first attempt, the applicability of Eq. (2) is investigated through the comparison of the predicted results with the experimental data, which was obtained from triaxial specimens with special small strain measurement devices. Andersen et al. (1995) adopted a Young's modulus for the individual Manchester sand particles of 90 GPa, which is an average value for quartzite (based on Table 12.5 of Lambe and Whitman, 1969). Together with a value of 9 GPa for polycrystalline ice from Sinha (1989), the Young's modulus for frozen Manchester fine sand may be predicted. Figures 4(a) and (b) show the comparison of experimental data with the predicted results based on the three different mixture models. The results in the range of f_s -values from 0 to 0.7 are shown in Fig. 4(a)

Table 3. Elastic moduli of two-phase mixture with various stress distribution parameters

Elastic moduli	$E = \frac{(b-1)f_s + 1}{\frac{f_s b}{E_s} + (1-f_s)\frac{1}{E_*}}, \quad \nu = \frac{f_s b \frac{\nu_s}{E_s} + (1-f_s)\frac{\nu_*}{E_*}}{\frac{f_s b}{E_s} + (1-f_s)\frac{1}{E_*}}$		
Stress distribution parameter b	$b = \left(\frac{E_s}{E_*} \right)^{0.5}$	$b = \left(\frac{E_s}{E_*} \right)^1$	$b = \left(\frac{E_s}{E_*} \right)^0 = 1$
Key assumption related to matrix and inclusion	$dW_s = dW_*$	$d\bar{\epsilon}_s = d\bar{\epsilon}_*$	$d\bar{\sigma}_s = d\bar{\sigma}_*$
References	Omine et al. (1993)	Voigt (1889)	Reuss (1929)
Characteristics	$E_{\text{Voigt}} > E_{\text{Omine}} > E_{\text{Reuss}}$ when $E_s > E_*$

Fig. 3. Outline of derivation process for E and ν

and the results from $f_s=0.5$ to 0.65 , which corresponds to the probable range of relative densities for the minimum and maximum void ratios, are plotted in Fig. 4(b). It can be seen that although the experimental data is quite scattered, most of the data points plot in between the results predicted by the Omine and Reuss models. It is also noted that, based on the model used, the mean value of the Young's modulus increases slightly and nonlinearly with the increasing volume fraction of sand.

The predicted and measured Poisson's ratios are shown in Figs. 5(a) and (b). Similarly, the experimental parameters used to predict the results are also taken from

Lambe and Whitman (1969) and Sinha (1989). The measured values are quoted from Baker and Kurfurst (1985), whose data are obtained by an acoustic wave propagation technique. The experimental data are in fairly good agreement with the results predicted by the Omine model, which shows that the predicted Poisson's ratios of this frozen sand gradually decreases with the increasing volume fraction of sands. It is considered that the assumption of the equal strain energy increments of inclusion and matrix per unit volume is most fitted to predict the elastic moduli of frozen sand as a composite material at an extremely small strain. This assumption reflects indi-

Fig. 4. Prediction of composite Young's modulus by three models

rectly the mechanism mobilized within the ice-sand mixture in the extremely small strain region, which may be represented as purely elastic.

Influences on This Ice-Sand Mixture Model

(a) Effect of sand particle stiffness

In basing the ice-sand mixture model on Eqs. (2) to (4), it is recognized that the Young's modulus of the ice-sand mixture with any volume fraction of sands depends only on the ratio of ice-matrix Young's modulus E_* to sand particle Young's modulus E_s . The corresponding Poisson's ratio is dependent on not only the values of E_s/E_* but also the values of ν_s/ν_* . Figure 6 shows the normalized Young's modulus, E/E_* against the volume fraction of sand, f_s as a function of E_s/E_* . Similarly, the normalized Poisson's ratio, ν/ν_* against f_s are shown in Figs. 7(a)-(c), as a function of E_s/E_* and ν_s/ν_* . The following observations are made from these figures:

1) The larger the E_s/E_* -ratio is, the greater the rate of increment of E/E_* with f_s -value becomes. As a hypothetical case, when $E_s/E_* = 1.0$, the Young's modulus of the sand particles equals that of the ice-matrix, and the Young's modulus of the ice-sand mixture will definitely

Fig. 5. Prediction of Poisson's ratio by three models

Fig. 6. E/E_* - f_s relationships predicted for ice-sand composite materials with various sand particle to ice modulus ratios

be the same as that of the ice-matrix, irrespective of the f_s -values. Further, if the sand particle Young's modulus is smaller than that of the ice-matrix, the Young's modu-

Fig. 7. Predicted Poisson's ratio of ice-sand mixture as function of ratio of sand-ice Young's moduli

lus of the ice-sand mixture will be smaller than that of the ice-matrix.

2) When the v_s/v_* and f_s -value remain constant, the normalized Poisson's ratio, v/v_* increases with increasing E_s/E_* -ratio. In addition, the smaller v_s/v_* is, the greater the sensitivity of v/v_* to E_s/E_* will be.

In view of these observations, it is important to establish the ranges of expected or realistic Young's modulus and Poisson's ratio of sand particles. Table 4 was prepared based on data reported by Lambe and Whitman (1969). The measured Poisson's ratio v_s and v_s/v_* , and the Young's modulus E_s and E_s/E_* for various original materials are summarized in this table. It is found that the realistic range of the Poisson's ratio and Young's modulus of natural sand particles are 0.12–0.30 and 66–123 GPa, respectively. Figures 8(a) and (b) show the possible values of the Young's modulus and Poisson's ratio of the ice-sand mixture in the range of f_s from 0 to 0.7, which are calculated by Eqs. (2) to (4) using data from Table 4, in which the two elastic parameters of the ice matrix were used as reference values. It is noted that the differences of both the Young's modulus and the Poisson's ratio of the ice-sand mixture for the geotechnical materials quoted are not so significant, considering the scatter of the experimental data mentioned above. The validity of this numerical observation is supported by measurements for different kinds of frozen sand reported by Baker and Kurfurst (1985) and Andersen et al. (1995), which indicated that even if the characteristics of sand particles as an inclusion were different, the Young's moduli of comparable magnitude could be observed in the case of same f_s value.

(b) Estimation of temperature dependency

Young's modulus, E_* and Poisson's ratio, v_* of granular ice are assumed in this paper to be isotropic, and may be related to temperature as presented by Sinha (1989) such that:

$$E_* = E_{*m} + c(T_m - T) \quad (5a)$$

$$v_* = v_{*m} + d(T_m - T) \quad (5b)$$

Table 4. Poisson's ratio and Young modulus for potential inclusion materials

Material	Poisson's ratio v_s	v_s/v_*	Young's modulus E_s (GPa)	E_s/E_*
Dolomite	0.3	0.97	112–123(118)	12.4–13.7(13.1)
Felspathic Gneiss	0.15–0.20(0.175)	0.48–0.65(0.56)	84–120(102)	9.3–13.4(11.3)
Granite	0.23–0.27(0.25)	0.74–0.87(0.81)	74–88 (81)	8.2–9.7 (9.0)
Limestone	0.27–0.30(0.285)	0.87–0.97(0.92)	88–109 (99)	9.8–12.1(11.0)
Mica Schist	0.15–0.20(0.175)	0.48–0.65(0.56)	81–103 (92)	8.9–11.4(10.2)
Obsidian	0.12–0.18(0.15)	0.39–0.58(0.48)	66–81 (74)	7.3–9.0 (8.2)
Quartzite	0.12–0.15(0.135)	0.39–0.48(0.44)	83–98 (90)	9.3–10.9(10.1)
Aluminium	0.34–0.36(0.35)	1.10–1.16(1.13)	56–77 (67)	6.2–8.6 (7.4)
Steel	0.28–0.29(0.285)	0.90–0.94(0.92)	203	22.6

* Poisson's ratio and Young modulus for ice: $v_*=0.31$, $E_*=9$ GPa (Shinha, 1989)

** Values in parentheses indicate mean values of modulus.

*** Based on Table 12.5 summarized by Lambe & Whitman (1969)

Fig. 8. Predicted elastic properties of ice-sand mixture with different sand particles as inclusions

where, E_{*m} and ν_{*m} are defined as the Young's modulus and Poisson's ratio at the melting point temperature, respectively, and 'c' and 'd' are experimental constants. Based on the data from Dantl (1969), E_{*m} , c, ν_{*m} and d are given by $E_{*m}=8.93$ (GPa), $c=0.012$ (GPa/K), $\nu_{*m}=0.308$ and $d=0.00007$ (1/K), respectively. Equation (5a) predicts the Young's modulus of granular ice to increase linearly by a small amount with decrease in temperature.

In order to evaluate the dependency of temperature on the Young's moduli of the ice-sand mixture, the following assumptions are introduced:

- 1) The temperature effect on the components influences only those of the ice-matrix, Eqs. (5a) and (5b).
- 2) The elastic moduli of the sand inclusions are simply represented by mean values shown for Quartzite in Table 4, whose values are $E_s=90$ GPa and $\nu_s=0.135$, respectively. This treatment seems to be realistic, when considering the results in Fig. 8.

Figures 9(a) and (b) show the changes of Young's modulus and the incremental percentage of this quantity with respect to the values at 0°C, E_0 , with decreases in temperature up to -50°C over the range of f_s -values

Fig. 9. Temperature dependency and sensitivity of Young's modulus of ice-sand mixture as composite material

from 0 to 0.7. Similar figures related to the Poisson's ratio are shown in Figs. 10(a) and (b). These assumptions lead to prediction of temperature dependency for both elastic parameters; however, the percentage of the differences with decrease in temperature is not significant from the practical and engineering point of view. The trend of the changes is somewhat different from the Young's modulus and Poisson's ratio. The Young's modulus increases monotonically with the decrease in temperature. As the amount of sand in the mixture is increased, the influence of the ice and hence the rate of increase of Young's modulus is reduced. The Poisson's ratio either increases or decreases with the decrease in temperature, although the difference is insignificant with no apparent change for a value of f_s from 0 to 0.7. Figure 11 shows the comparison of the predicted temperature effects on the elastic parameters over the range of f_s -values from 0.5 to 0.7 with experimental data reported by Baker and Kurfurst (1985) and Andersen et al. (1995). It is found that the predicted temperature dependency of the elastic parameters is relatively insignificant, compared with the range of the experimental data obtained.

Fig. 10. Temperature dependency and sensitivity of Poisson's ratio of ice-sand mixture as composite material

TANGENT MODULI OF ICE-SAND MIXTURE AT SMALL STRAIN

Basic Equation for Nonlinear Behaviour

It is assumed that the hyperbolic equation proposed by Kondner (1963), and Duncan and Chang (1970) may effectively represent the nonlinear stress-strain relationship of frozen sand with different volume fractions up to an upper yield stress, q_{uy} . This is defined as the first point on the stress-strain curve where the slope either becomes zero or reaches its minimum positive value before significant strain hardening (Andersen et al., 1995). The basic equation is given by Eq. (6a), and this equation is also rewritten as Eq. (6b):

$$q = \frac{\varepsilon_a}{A + B\varepsilon_a} \quad (6a)$$

$$\frac{\varepsilon_a}{q} = A + B\varepsilon_a \quad (6b)$$

where, q and ε_a are an applied deviator stress and axial strain under uniaxial or triaxial with σ_3 constant loading conditions, respectively, and 'A' and 'B' are experimen-

Fig. 11. Comparison of predicted temperature effect on elastic parameters of ice-sand mixture with experimental data

tal constants, which correspond to the inverse values of the Young's modulus E at extremely small strain level and the asymptotic value of the stress q_{ult} , respectively. In addition, the following relationship between q_{uy} and q_{ult} is assumed, introducing the failure ratio R_f defined by Duncan and Chang (1970) such that:

$$q_{uy} = R_f q_{ult} \quad (7)$$

When the Young's modulus at extremely small strain level is assumed to be equivalent to the Young's modulus E of the sand-ice mixture in Eqs. (2) to (4), and substituting Eq. (7) into Eq. (6a), it may be rewritten in the following general form:

$$q = \frac{\varepsilon_a}{\frac{1}{E} + \frac{\varepsilon_a R_f}{q_{uy}}} \quad (8)$$

A hyperbolic assumption for stress-strain behaviour of ice and frozen sands at small strain seems to be appropriate in Fig. 12(a), based on typical $q - \varepsilon_a$ relationships of ice and frozen sand with different volume fractions, temperatures and strain rates from Hawkes and Mellor (1972) and Andersen et al. (1995). The $\varepsilon_a/q - \varepsilon_a$ relation-

Fig. 12. Hyperbolic assumptions for stress-strain behaviour at small strain of ice or ice-sand mixtures as composite material

ships are depicted in Fig. 12(b), showing more or less the expected straight lines. It is also important to note that when the Young's moduli of ice-matrix and sand inclusion are represented by Eq. (5a) and the mean value of quartzite shown in Table 4 respectively, the Young's modulus E in Eq. (8) can be automatically estimated as a function of the f_s -value.

Figures 13(a) and (b) show the characteristics of the inverse value of E computed by Eqs. (2) to (4), and the failure ratio R_f in the realistic range of f_s values from 0 to 0.7, respectively together with the estimated values of R_f from Fig. 12(b). The experimental upper yield stresses of ice and frozen sand for determining the R_f values are also shown in Fig. 12(a), which are obtained as the first point on the stress-strain curve where the slope reaches its minimum positive value before significant strain hardening as defined by Andersen et al. (1995). It is found that the computed values of the Young's moduli may be used as the Young's modulus E in Eq. (8), and also that parameter R_f is about 0.9, irrespective of f_s values. The Young's modulus obtained from Eqs. (2)–(4) and 0.9 are therefore used as a Young's modulus E and R_f value, respectively, in order to evaluate the stress-strain behaviour

Fig. 13. Characteristics of parameter related to Young's modulus and parameter R_f for ice and frozen sand

of ice-sand mixtures at small strains up to an upper yield stress obtained from Eq. (8).

Tangent Modulus for Ice-Sand Mixture

As mentioned by Duncan and Chang (1970), the tangent modulus is needed to conduct the incremental stress analysis. In general, the tangent Young's modulus is defined as:

$$E_t = \frac{\partial q}{\partial \epsilon_a} \quad (9)$$

Then, substituting Eq. (8) into Eq. (9), E_t is given by:

$$E_t = \left(1 - R_f \frac{q}{q_{uy}} \right)^2 E \quad (10)$$

Further, performing substitution of Eqs. (2) to (4) into Eq. (10), and normalizing the tangent Young's modulus, E_t by the Young's modulus of ice-matrix, E_* , the following expression is obtained:

$$\frac{E_t}{E_*} = \left(1 - R_f \frac{q}{q_{uy}} \right)^2 \frac{E}{E_*}$$

Fig. 14. Relationship between E_t/E_* and f_s with various relative shear levels of $q/q_{uy,0}$

$$= \left(1 - R_f \frac{q}{q_{uy}}\right)^2 \frac{\left(\left(\frac{E_s}{E_*}\right)^{0.5} - 1\right) f_s + 1}{f_s \left(\frac{E_*}{E_s}\right)^{0.5} + (1 - f_s)}. \quad (11)$$

This expression shows that the tangent Young's modulus can also be represented as a function of the parameters which describe the ice-sand mixture. Figure 14 shows the predicted results of E_t/E_* with the increasing normalized applied stress q/q_{uy} , in the range of f_s from 0 to 0.7. The trend in the changes in E_t/E_* with applied normalized stress is found in connection with f_s values. In the following section, the characteristics of an ice-sand mixture in the upper yield stress range will be discussed with respect to Eq. (11).

Effect of Sand Content on Upper Yield Stress

The upper yield stress under a certain confining pressure may be represented as a function of the volume fraction of sands, strain rate and temperature such that:

$$q_{uy} = g(f_s, \dot{\epsilon}_a, |T|) \quad (12)$$

where, $\dot{\epsilon}_a$ is the axial strain rate, and $|T|$ is defined as the absolute value of the temperature. Based on the analytical representation of creep and strength data (Andersland and Ladanyi, 1994), Eq. (12) is normally expressed as an analogy to:

$$q_{uy} = q_{uy,0}(|T_0|) \left(\frac{\dot{\epsilon}_a}{\dot{\epsilon}_{a,0}}\right)^{1/n(|T|)} \quad (13)$$

where, $q_{uy,0}$ is a standard upper yield stress, which is defined as the value of q_{uy} when $|T| = |T_0|$, $\dot{\epsilon}_a = \dot{\epsilon}_{a,0}$ and $f_s = f_{s,0}$ and 'n' is a parameter depending on the temperature. Typical values of 'n' for ice and frozen sands against temperature are shown in Fig. 15, which are depicted based on data from the literature (Andersen et al., 1995; Hawkes and Mellor, 1972; Jones, 1982). As the characteristics of the second term in Eq. (13) including

Fig. 15. Temperature-dependency of parameter n

parameter 'n' have already been discussed elsewhere, in the following, the first term is mainly discussed from the view of an ice-sand composite geomaterial.

When considering the characteristics of $q_{uy,0}$ of an ice-sand mixture related to the upper yield stress $q_{uy^*,0}$ of an ice-matrix in the same way as the Young's modulus mentioned before, the following relationship is proposed:

$$\frac{q_{uy,0}}{q_{uy^*,0}} = \alpha \frac{E}{E_*} \rightarrow q_{uy,0} = q_{uy^*,0} \left(\alpha \frac{E}{E_*} \right). \quad (14)$$

This means that when the normalized Young's modulus, E/E_* , is determined together with a parameter α , the normalized upper yield stress, $q_{uy,0}/q_{uy^*,0}$, can easily be computed. Figures 16(a) and (b) are introduced to make the characteristics of $q_{uy,0}/q_{uy^*,0}$ clear, where the relationship between $q_{uy,0}$ and f_s is shown in Fig. 16(a) as a function of strain rates, and also the values of $q_{uy,0}/q_{uy^*,0}$ against f_s are shown for three different strain rates in Fig. 16(b). The experimental data reported by Andersen et al. (1995) is plotted on this figure. Figure 16(a) demonstrates that $q_{uy,0}$ decreases with the slower strain rates, and also further that $q_{uy,0}$ decreases gradually with decreasing f_s -values. Figure 16(b) appears to confirm that the normalized upper yield stress is uniquely determined as a function of f_s , irrespective of the strain rate according to the Reuss equation given in Table 3. Introducing the Reuss model with the key assumption that the mixtures will be subjected to a uniform stress to estimate the ratio $\alpha(E/E_*)$ in Eq. (14), $q_{uy,0}$ is expressed as:

$$\begin{aligned} q_{uy,0} &= q_{uy^*,0} \left(\alpha \frac{E}{E_*} \right) = q_{uy^*,0} \left(\frac{E}{E_*} \right)_{\text{Reuss}} \\ &= \frac{q_{uy^*,0}}{1 + f_s \left(\frac{E_*}{E_s} - 1 \right)}. \end{aligned} \quad (15)$$

The background of this assumption is that the mechanism governing the response of the ice-sand mixture changes gradually due to the shear deformation, namely, from a mechanism based on Omine's model, which as-

Fig. 16. Characteristics of upper yield and normalized upper yield stress related to volume fraction of sand

sumes equivalent strain energy increments in the inclusion and matrix at extremely small strain, to that of the Reuss model in the range of deformation at the upper yield stress. Calculating data from Eq. (15) and plotting these in Fig. 16(b) strongly supports this assumption. It is important to point out that, based on the above discussion, the referred tangent Young's modulus, $E_{t,0}$, obtained by $q_{uy*,0}$ can be expressed as a combination of Omine's model and the Reuss model, such that:

$$E_{t,0} = \left(1 - R_f \frac{q}{q_{uy,0}}\right)^2 E \quad (16a)$$

where

$$q_{uy,0} = \frac{q_{uy*,0}}{1 + f_s \left(\frac{E_*}{E_s} - 1\right)} \quad (16b)$$

$$E = E_{Omine} = \frac{E_* \left[\left(\left(\frac{E_s}{E_*} \right)^{0.5} - 1 \right) f_s + 1 \right]}{f_s \left(\frac{E_s}{E_*} \right)^{-0.5} + (1 - f_s)} \quad (16c)$$

Fig. 17. Strain rate dependency of ice samples at two different temperatures

Figure 17 shows how the upper yield stress in the ice-matrix $q_{uy*,0}$ is strongly dependent on the strain rate at two different temperatures, and may also be imaginably dependent on the temperature. This figure is helpful for estimating the value of $q_{uy*,0}$ in Eqs. (15) and (16b). It is important to emphasize that, when E_s , E_* and R_f are practically assumed as 90 GPa, 9 GPa and 0.9 mentioned above, the Young's modulus is easily computed as a function of $q_{uy*,0}$ and f_s values alone.

Some Applications of the Model

The secant Young's moduli E_{sec} during shear are often used to evaluate the stress-strain behaviour of geomaterials at small strain, and this is normally defined by:

$$E_{sec} = \frac{q}{\epsilon_a} \quad (17)$$

Substituting Eq. (8) into Eq. (17), Eq. (17) can be rewritten as:

$$E_{sec} = \frac{1}{\frac{1}{E} + R_f \frac{\epsilon_a}{q_{uy}}} \quad (18)$$

Further, introducing Eq. (15), based on the Reuss model, to determine $q_{uy,0}$ from $q_{uy*,0}$, and normalizing $E_{sec,0}$ by the composite value of E from Eqs. (2)–(4), the following relationship is derived:

$$\frac{E_{sec,0}}{E} = \frac{1}{1 + R_f E \epsilon_a \left(\frac{1 + f_s \left(\frac{E_*}{E_s} - 1 \right)}{q_{uy*,0}} \right)} \quad (19)$$

This relationship demonstrates that the secant Young's modulus of an ice-sand mixture may also be determined at a specific axial strain based on the characteristics of the ice-matrix and sand inclusion alone. Figures 18(a) and (b) show the normalized secant moduli against axial strains, computed parametrically from Eq. (19). Figures

Fig. 18. Changes of normalized Young's modulus with axial strain: a) f_s -dependency, b) dependency on upper yield stress of ice

18(a) and (b) indicate the dependency on f_s and q_{uy*0} respectively. At a constant axial strain, the normalized secant moduli, $E_{sec,0}/E$, decrease with higher f_s -values and a lower upper yield stress of the ice matrix, q_{uy*0} . Figure 19 shows the comparison of the predicted $E_{sec,0}$ at $\epsilon_a = 1.5\%$ in frozen sand with the experimental data obtained from the uniaxial compression tests at constant temperature by Kuribayashi et al. (1985). Note that all the deviator stresses at $\epsilon_a = 1.5\%$ are approximately in the upper yield stress region. The predicted and experimental $E_{sec,0}$ are shown against the upper yield stress. Since f_s -values for the experimental data were not found in the literature, values of 0.5 to 0.65 were used for the prediction, and these correspond to $D_r = 0\% - 100\%$. Although the comparable data is limited, the predicted results tend to somewhat overestimate the experimental data. This is to be expected due to bedding errors and the measurement accuracy obtained using an uniaxial apparatus.

The predicted results also imply that the effect of f_s changing from 0.5 to 0.65 has practically no influence on E_{sec} at $\epsilon_a = 1.5\%$ in the frozen sand.

CONCLUSIONS

1) Three different types of models based on the theory of mixtures were introduced to evaluate the elastic proper-

Fig. 19. Applicability of model to estimation of secant Young's modulus at $\epsilon_a = 1.5\%$ in frozen sand

ties for fairly homogeneous saturated frozen sands under uniaxial or triaxial loading at constant temperature. It is found that although the experimental data is quite scattered, most of the data points plot in-between the results predicted by the Omine and Reuss models. It is also noted that, based on the model used, the Young's modulus increases monotonously but nonlinearly with the increasing volume fraction of sand.

2) In basing the ice-sand mixture model on Eqs. (2) to (4), the Young's modulus of the ice-sand mixture depends only on the ratio of ice-matrix Young's modulus E_* to sand particle Young's modulus E_s and the volume fraction of the sand. The corresponding Poisson's ratio is dependent on not only the values of E_s/E_* but also the ratio of ice-matrix Poisson's ratio ν_* to sand particle Poisson's ratio ν_s .

3) The ranges of expected or realistic Young's modulus and Poisson's ratio of the ice-sand mixtures were investigated based on a range of geotechnical materials. It is noted that the differences in both the Young's modulus and the Poisson's ratio of the ice-sand mixture for the range of geotechnical materials quoted are not so significant, considering the scatter of experimental data.

4) Based on the model, the Young's modulus increases with the decrease in temperature, and the percentages of the increments decrease slightly with greater volume fraction of sand. The Poisson's ratio either increases or decreases with increasing volume fraction of sand and with the decrease in temperature, although the differences are not very significant.

5) Stress-strain elastic parameters were combined with a hyperbolic stress-strain model to represent the entire stress-strain range up to upper yield stress, and these were derived based on the theory of mixtures. The key assumption is that the mechanism governing the response of the ice-sand mixture is thought to change gradually due to the shear deformation: namely, from a mechanism based on the Omine model which assumes equivalent strain energy increments of inclusion and matrix at ex-

tremely small strain, to that of the Reuss model, which assumes uniform stress increments in the range of deformation at the upper yield stress. It is important to point out that this model can be expressed as a function of the volume fraction of sand and parameters relating to the sand inclusion and the ice matrix alone. Although the comparable data is limited, the predicted results of secant Young's modulus at the deviator stresses with 1.5% axial strain tend to somewhat overestimate the experimental data.

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