ELASTODYNAMIC MODEL FOR LARGE-DIAMETER END-BEARING SHAFTS

GEORGE MYLONAKIS

ABSTRACT

The dynamic response of large-diameter end-bearing cylindrical shafts is studied. First, the popular plane-strain model of Novak is reviewed and its limitations are discussed. An improved model is then developed which, while retaining the simplicity of the original model, accounts for the third dimension by considering the normal and shear stresses acting on the upper and lower faces of a horizontal soil slice. These stresses are incorporated in the analysis by implementing a dynamic Vlasov-Leontiev approximation based on integrating the governing equations over the thickness of the soil layer. It is shown that this operation leads to a set of elastodynamic equations which are similar to those in the plane-strain model, yet properly incorporate the salient 3-D effects. Explicit closed-form solutions are obtained for: (i) the dynamic soil reaction along the shaft; (ii) the dynamic impedance of the shaft; (iii) the displacement field in the soil; and (iv) the dynamic interaction factors between neighboring shafts. Both vertical and lateral oscillations are analyzed for single and grouped shafts. Results are presented in terms of dimensionless graphs which highlight the importance of soil-foundation interaction on the response. It is shown that the proposed model avoids the limitations of the plane strain model.

Key words: cutoff frequency, dynamics, group, impedance, interaction factor, pile, shaft (ICG: E8/E13)

INTRODUCTION

Most of the foregoing research on the dynamics of deep foundations has been devoted to slender foundations such as piles and long drilled shafts (see pertinent review articles by Novak, 1991; Pender, 1993; Martin and Lam, 1995; Gazetas and Mylonakis, 1998). Less flexible foundations such as large-diameter shafts and caissons have received less research attention (Harada et al., 1981; Gazetas and Dakoulas, 1998; Mok et al., 1999). This type of foundation is often warranted when a shallow soft soil layer overlains a firm stratum. Large-diameter shafts founded directly on the firm stratum can then be used to support a heavy superstructure. As compared to the more common pile foundations, large-diameter shafts have some distinct advantages (Bowles, 1996): (i) they eliminate the need for pile caps (superstructure elements can be supported directly by the shafts); (ii) they use fewer foundation elements (thereby group effects are not as complex and dominant as in pile groups); (iii) almost any foundation diameter can be produced. Accordingly, it is of no surprise that large-diameter deep foundations are receiving increasing attention by engineers. Yet, their dynamic behavior remains to a large extent unexplored.

Based on lateral flexibility, deep foundations can be roughly classified into three groups: (I) flexible pile-type structures having a depth to width ("slenderness") ratio which is typically higher than 10; (II) rigid caisson-type structures with slenderness ratio which is typically less than 4; (III) foundations of intermediate slenderness. For foundations in Group I, flexibility is such that the response due to a lateral load applied at the top is confined within a certain depth (named active length), with no response carried over to the tip. For foundations in Group II, bending is very small in comparison to the deformation of the surrounding soil so the foundation moves essentially as a rigid body with a significant portion of the external head load carried by the base. The "intermediate" Group III represents a perhaps more complicated case since bending develops along the foundation while the external load has a carry over to the base.

Most of the current knowledge on the dynamics of deep foundations is restricted to piles (Group I). Some limited information on the behavior of rigid caissons (Group II) comes indirectly from studies of the dynamic response of embedded footings (Gazetas, 1991). In contrast, our knowledge on foundations of intermediate slenderness (Group III) is very limited, and will be the focus of the present study.

The scope of this paper is fourfold: (1) to critically review Novak's plane-strain model for embedded cylindrical foundations; (2) to derive an improved model with emphasis on large-diameter end-bearing shafts; (3) to apply the new model to analyze the vertical and lateral oscillations of such shafts; (4) to study dynamic group effects between neighboring shafts.

Assistant Professor, City University of New York, New York 10031, USA; mylonakis@ccny.cuny.edu
Manuscript was received for review on May 15, 2000.
Written discussions on this paper should be submitted before January 1, 2002 to the Japanese Geotechnical Society, Sugayama Bldg. 4F, Kanda Awaji-cho 2-23, Chiyoda-ku, Tokyo 101-0063, Japan. Upon request the closing date may be extended one month.
PROBLEM DEFINITION

The problem studied in this paper is shown in Fig 1: a large-diameter cylindrical shaft embedded in a homogeneous soil stratum resting on a rigid base, subjected to dynamic head loading. The soil is assumed to be a linear viscoelastic material of thickness $L$, Young’s modulus $E_s$, Poisson’s ratio $\nu$, mass density $\rho_s$, and linear hysteretic damping $\beta$, [expressed through the complex shear modulus $G^* = G_s(1 + 2i\beta_s)$]. The shaft is a solid elastic cylinder of Young’s modulus $E$, and diameter $d$. Perfect bonding (i.e., no gap or slippage) is assumed between the shaft and the soil. The system is subjected to a harmonic vertical or lateral displacement $U(t) = U_0 \exp [i\omega t]$, or a rotation $\Theta(t) = \Theta_0 \exp [i\omega t]$ at the shaft head. It is noted that the shaft is assumed to be long enough (say, $L > 4d$) so that flexural deformations are dominant during lateral oscillations. Also, the vertical shear stresses that develop along the soil-shaft interface during lateral-rocking oscillations (which can be important for rigid caissons in Group III) are considered negligibly small. Positive notation for stresses and displacements is provided in Fig. 1.

THE PLANE-STRAIN MODEL

Probably the simplest model for analyzing the dynamic response of an embedded cylindrical foundation is the one proposed by Novak (1974). In this model, the soil is divided into an infinite number of thin horizontal “slices”, with each slice being subjected to dynamic plane-strain deformation. This is tantamount to replacing the soil with a bed of distributed springs and dashpots, with the springs representing stiffness while the dashpots representing damping due to radiation and hysteretic energy dissipation. This representation leads to the so-called Dynamic Winkler Foundation Model (DWFM) (Novak, 1974; Roesset, 1980; Dobry et al., 1982) which is the dynamic counterpart of the familiar static models (Randolph and Wroth, 1978; Scott, 1981; Reese, 1986).

The plane strain model has been used extensively in pile dynamics (Novak, 1974; Takemiki and Yamada, 1981; Konagai and Nogami, 1987; Trochanis, 1988; El-Naggar and Novak, 1992), and variations of it (Gazetas and Dobry, 1984; Veletsos and Dotson, 1986; Matsubara and Hoshiya, 1999) have been applied to dynamic analyses of various types of embedded structures.

In the plane-strain model and cylindrical coordinates, the equilibrium equations of the harmonically oscillating medium are (Novak et al., 1978)

\[ \eta^* \frac{\partial}{\partial r} \left( \frac{1}{r^2} \frac{\partial}{\partial \theta} \left( ru + \frac{\partial v}{\partial \theta} \right) \right) + \left( \frac{\omega}{V_s^*} \right)^2 u = 0 \]

\[ \eta^* \frac{1}{r^2} \frac{\partial}{\partial \theta} \left( \frac{\partial}{\partial r} (ru) + \frac{\partial v}{\partial \theta} \right) + \left( \frac{\omega}{V_s^*} \right)^2 v = 0 \]

\[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial w}{\partial r} \right) + \left( \frac{\omega}{V_s^*} \right)^2 w = 0 \]

which pertain to lateral (Eqs. (1), (2)) and vertical axisymmetric oscillations (Eq. (3)). $u$, $v$, and $w$ denote the associated soil displacements in the radial, tangential, and vertical directions, respectively (Fig. 1); $\omega$ stands for the cyclic vibrational frequency and $V_s^*$ for the complex shear wave velocity $V_s^* = V_s \sqrt{1 + 2i:\beta}$. Finally, $\eta$ expresses the ratio of the dilatational and shear wave velocities in the medium

\[ \eta = \frac{2(1 - \nu)}{1 - 2\nu} \]

The restraining action of the soil can be expressed through a complex dynamic impedance, $\zeta$, which is defined as the force required to produce a harmonic displacement of unit amplitude at the soil-foundation interface. In the case of horizontal and vertical oscillations, the plane-strain model yields, respectively, the following impedance functions (Novak et al., 1978)

\[ \zeta = -\pi G_z s^2 \]

\[ \times A K_s(q) K_s(s) + s K_s(q) K_s(s) + q K_s(q) K_s(s) \]

\[ \frac{q K_s(q) K_s(s) + s K_s(q) K_s(s) + q s K_s(q) K_s(s)}{K_s(s)} \]

\[ \zeta = 2\pi G_z s \]

\[ \times \frac{K_s(s)}{K_s(s)} \]

\[ \text{Fig. 1. System and positive notation considered} \]

1 According to Timoshenko’s beam theory the effect of shearing deformations on the natural frequencies of cylindrical cantilevered beams with $L/d > 4$ does not exceed 4% for the first six vibrational modes (Jacobsen and Ayre, 1958, p. 502).

2 The model is often referred to as “Novak-Baranov” model after the work of Baranov (1967) on the subject.
where \( K_0(\cdot) \) and \( K_1(\cdot) \) denote the modified Bessel functions of the second kind and order zero and one, respectively; \( s \) and \( q \) stand for the dimensionless complex parameters
\[
S = \frac{\alpha_0}{2\sqrt{1 + 2i\beta_s}} \quad (7a)
\]
\[
q = \frac{s}{\eta} \quad (7b)
\]
which pertain to shear and dilatational waves in the medium, respectively. Finally, \( \alpha_0 \) denotes the well-known dimensionless frequency factor
\[
\alpha_0 = \frac{\omega d}{V_s} \quad (8)
\]
Results obtained using Eqs. (5) and (6) are shown Figs. 2 and 3. They are cast in the form:
\[
\zeta = \delta G_s (1 + 2i\beta_s) \quad (9)
\]
where \( \delta = \delta(\omega) \) and \( \beta = \beta(\omega) \) are real coefficients representing soil stiffness and damping, respectively. Note the higher values of the lateral stiffness coefficient \( \delta_s \) as compared to the vertical coefficient \( \delta_v \) (recall that the material is stiffer in compression-extension than in shearing), and the approximately equal damping coefficients.

Limitations of the Plane-Strain Model

With reference to Figs. 2 and 3, the following shortcomings can be identified:
1. In the low-frequency range \( (\omega_0 < 0.05) \), \( \delta_v \) and \( \delta_w \) decrease rapidly with decreasing frequency and become zero at \( \omega_0 = 0 \). Accordingly, the model cannot predict static stiffness. This deficiency has been identified in studies of the static settlement of piles (Randolph and Wroth, 1978).
2. It is well known that a soil layer resting on a stiffer base exhibits a characteristic frequency, termed cutoff frequency, below which no propagating waves exist in the medium and, thereby, no radiation damping is generated. The cutoff frequency is associated with the fundamental natural frequency of the layer in compression-extension or shearing (depending on the mode of vibration) and a sudden increase in damping. Evidently, since both stiffness and damping coefficients in Figs. 2 and 3 are smooth functions of frequency, cutoff-frequency effects cannot be captured by the plane-strain model. This deficiency can be particularly pronounced in relatively shallow layers (which are of particular importance for the type of foundations studied herein), since their fundamental natural frequency is typically quite high.
3. The complex dynamic impedance \( \zeta \) in Eqs. (5) and (6) is independent of foundation-soil stiffness contrast \( (E_s/E_f) \) and slenderness ratio \( (L/d) \). In contrast, numerous analytical studies have shown that \( \zeta \) does depend on these parameters (Vesic, 1961; Blaney et al., 1976; Nogami and Novak, 1976; Roesset, 1980; Dobry et al., 1982).

DEVELOPMENT OF AN IMPROVED MODEL

The sources of the shortcomings of the plane-strain model can be identified with reference to: (i) the lack of continuity in the medium in the vertical direction; (ii) the inability of the model to account for the thickness of the profile (i.e., by considering just a thin horizontal layer). The lack of continuity fails to provide the means for transferring forces vertically in the medium (i.e., by either shearing of compression-extension) and is responsible for the lack of stiffness at low frequencies. On the other hand, the inability of the model to "see" the actual thickness of the profile does not allow for cutoff frequency to be incorporated.

Basis of the proposed improved model is that a horizontal layer in the soil medium is acted upon along its upper and lower faces by horizontal shearing stresses (in lateral vibrations), and vertical normal stresses (in vertical vibrations), as shown in Fig. 4. The variation of these stresses with depth, \( \partial \tau / \partial z \) and \( \partial \sigma / \partial z \), provides for the capacity of the medium to transfer forces vertically, by shearing and compression-extension. The significance of the former type of stresses on the lateral dynamic
response of retaining walls has been demonstrated in earlier studies by Arias et al. (1981) and Veletsos and Younan (1994b; 1995). In contrast, the significance of the variation of normal stresses $\sigma_r/\sigma_z$ on vertical response has received less attention (Nogami and Novak, 1976).

To account for these stresses, the governing Eqs. (1)–(3) should be revised as follows:

\[
\eta_1 \frac{\partial}{\partial r} \left[ \frac{\partial}{\partial r} (ru) + \frac{\partial v}{\partial \theta} \right] - \frac{1}{r^2} \frac{\partial}{\partial \theta} \left[ \frac{\partial (rv)}{\partial \theta} - \frac{\partial u}{\partial \theta} \right] + \frac{\partial^2 u}{\partial z^2} + \left( \frac{\omega}{V_s^2} \right)^2 u = 0
\]

(10)

\[
\eta_2 \frac{1}{r^2} \frac{\partial}{\partial \theta} \left[ \frac{\partial}{\partial r} (ru) + \frac{\partial v}{\partial \theta} \right] + \frac{\partial}{\partial r} \left( \frac{\partial w}{\partial r} \right) + \frac{\partial^2 v}{\partial z^2} + \left( \frac{\omega}{V_s^2} \right)^2 v = 0
\]

(11)

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\partial^2 w}{\partial r^2} \right) + \eta_3 \frac{\partial^2 w}{\partial z^2} + \left( \frac{\omega}{V_s^2} \right)^2 w = 0
\]

(12)

where $\eta_1$ and $\eta_2$ are dimensionless coefficients depending solely on Poisson’s ratio. Their numerical values will be discussed later on. The above equations differ from the corresponding plane-strain expressions in the presence of terms $\partial^2 u/\partial z^2$, $\partial^2 v/\partial z^2$ and $\eta_3 \partial^2 w/\partial z^2$ which make the above formulation essentially three dimensional. It is noted that the horizontal and vertical vibrations are uncoupled, as in the plane strain model. In that sense, the above equations can be viewed as an intermediate case between the plane-strain and the three-dimensional elastodynamic theory (Achenbach, 1973).

**Proposed Solution: Vlasov-Leontiev Approximation**

Analytical solutions to Eqs. (10)–(12) have already been developed for the dynamic analysis of embedded structures (Tajimi, 1969; Nogami and Novak, 1976; Veletsos and Younan, 1994b). All these solutions utilize eigenvalue expansions over the vertical coordinate to obtain the response in terms of infinite trigonometric series. Because of their complicated nature, application of these solutions to engineering practice has been limited. Another drawback is that soil impedance is obtained as a function of depth which complicates its routine engineer-

ing use. In this paper, a more convenient approach will be adopted: Instead of expanding the solution in trigonometric series, the vertical coordinate will be suppressed by integrating the governing equations over the thickness of the soil layer. In this way, the three-dimensional problem will be reduced to a two dimensional problem analogous to that in the plane-strain model. It will be shown that the proposed solution will be free of the drawbacks of the earlier model and provide satisfactory predictions of the dynamic response.

Basis of the proposed solution is the assumption that the displacement field in the soil can be decomposed into a function of the horizontal coordinates ($r$ and $\theta$) and a dimensionless function of the vertical coordinate ($z$) i.e.,

\[
U(r, \theta, z) = U(r, \theta) \chi(z)
\]

(13)

where $U(r, \theta, z)$ stands for any of the displacement components $u, v, w$.

Introducing Eq. (13) into Eqs. (10)–(12), multiplying each of the equations by $\chi(z)$ and integrating over the thickness of soil layer, the terms involving double differentiation with respect to $z$ are replaced as follows (see Appendix II)

\[
\frac{\partial^2 U}{\partial z^2} \to b^2 U
\]

(14)

where $b$ is a constant (dimensions $1$/Length) given by

\[
b^2 = \int_0^z \left| \frac{d\chi(z)}{dz} \right|^2 dz \int_0^z \chi(z)^2 dz
\]

(15)

The above technique was apparently first employed by Vlasov and Leontiev (1966) for the static analysis of surface footings, and was later extended to piles by Georgiadis and Butterfield (1983). The above authors showed that this operation leads to a two-parameter foundation model (Pasternak, 1954; Nogami et al., 1992). In this study the technique is used in a somewhat different way. While in the foregoing studies the integration was performed in the direction perpendicular to the soil-foundation interface (i.e., the vertical direction for a footing or the horizontal plane for a pile), in this work the integration is performed parallel to the soil-foundation interface (see Eq. (15)). It will be shown this modification will lead to an one-parameter model analogous to that of Novak.

Using the substitution in Eq. (15), Eqs. (10)–(12) take the simpler form

\[
\eta_1 \frac{\partial}{\partial r} \left[ \frac{\partial}{\partial r} (ru) + \frac{\partial v}{\partial \theta} \right] - \frac{1}{r^2} \frac{\partial}{\partial \theta} \left[ \frac{\partial (rv)}{\partial \theta} - \frac{\partial u}{\partial \theta} \right] + \left( \frac{\omega}{V_s^2} \right)^2 u = 0
\]

(16)

\[
\eta_2 \frac{1}{r^2} \frac{\partial}{\partial \theta} \left[ \frac{\partial}{\partial r} (ru) + \frac{\partial v}{\partial \theta} \right] + \frac{\partial}{\partial r} \left( \frac{\partial w}{\partial r} \right) + \left( \frac{\omega}{V_s^2} \right)^2 v = 0
\]

(17)

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\partial^2 w}{\partial r^2} \right) + \left( \frac{\omega}{V_s^2} \right)^2 w = 0
\]

(18)
where \( b_z \) and \( b_w \) are obtained from Eq. (15) using two pertinent functions \( \chi_z(z) \) and \( \chi_w(z) \) (i.e., for the horizontal and vertical modes, respectively). More information on these functions will be given later on. Many interesting features are worthy of note. First, the independent variable \( z \) has been suppressed. Second, Eqs. (16)–(18) are identical to those in the original plane-strain model Eqs. (1)–(3) except for the presence of terms \( (b_z) \) and \( (\eta_w b_w) \). Accordingly, Novak’s solution Eqs. (5), (6) is still applicable, with the dimensionless coefficients \( s \) and \( q \) being replaced by

\[
\begin{align*}
\frac{s_a}{\eta_a} &= \frac{1}{2} \left[ a_{\alpha} - \frac{d_{\alpha}}{1 + 2\beta_s} \right] \quad (19a) \\
\frac{q_a}{\eta_a} &= \frac{1}{\eta_a} s_a \\
\frac{s_w}{\eta_w} &= \frac{1}{2} \left[ a_{\alpha} - \frac{d_{\alpha}}{1 + 2\beta_s} \right] \\
\end{align*}
\]

where \( s_a \) and \( q_a \) correspond to lateral vibrations while \( s_w \) to vertical vibrations.

With reference to the dimensionless constants \( \eta_a \), \( \eta_w \), if the vertical displacement \( w \) is neglected during lateral vibrations, \( \eta_w \) would be equal to \( \eta \) in Eq. (4) (Veletsos and Younan, 1994b). Similarly, if the horizontal displacements \( u \) and \( v \) are neglected during vertical vibrations, \( \eta_w \) will also be equal to \( \eta \).

The main drawback stemming from the use of \( \eta \) is that the solution will exhibit a high sensitivity to Poisson’s ratio (recall that \( \eta \) tends to infinity as \( \nu \rightarrow 0.5 \)) which is not observed in rigorous numerical solutions of such problems (Gazetas and Dobry, 1984; Veletsos and Younan, 1994a). In this paper a more realistic assumption is adopted: instead of assuming vanishing displacements, vanishing stresses will be considered. For example, if the vertical normal stress in the medium, \( \sigma_z \), is assumed to be zero during lateral vibrations, \( \eta_a \) should be replaced by (Veletsos and Younan, 1994b; Mylonakis, 2000)

\[
\eta_a = \frac{2 - \nu}{1 - \nu} \quad (21)
\]

which, clearly, is not singular at \( \nu = 0.5 \). Veletsos and Younan (1994a) showed that this assumption is superior over that of vanishing displacements for the dynamic response of shallow retaining walls. The same is also expected to be true for the system investigated herein.

For the vertical mode, assuming that the horizontal stresses \( \sigma_z \) and \( \sigma_y \) are zero yields

\[
\eta_w = \sqrt{2(1 + \nu)} \quad (22)
\]

which corresponds to the ratio of P and S wave velocities in a rod. A perhaps better choice for the problem at hand is to consider \( \sigma_z = 0 \) and \( \varepsilon_y = 0 \) which would take into account, approximately, for the partial lateral restraint due to the zero tangential displacement \( (\nu = 0) \) in axisymmetric vibrations. In this case \( \eta_w \)

\[
\eta_w = \frac{2}{1 - \nu} \quad (23)
\]

With reference to coefficient \( \eta_w \), Eqs. (4), (22) and (23) are plotted in Fig. 5. It is seen that for low values of Poisson’s ratio all three expressions attain similar values (at \( \nu = 0 \) they are all equal to 1.41). As \( \nu \) approaches 0.5, however, \( \eta \) in the constrained medium tends to infinity, while Eqs. (22) and (23) yield the similar values 1.73 and 2, respectively. Either one of the latter expressions can provide acceptable engineering estimates for the vertical soil response. In the sequel, Eq. (23) will be used.

The improvement offered by the new solution becomes apparent by examining Eq. (19) for the case of zero material damping \( (\beta_s = 0) \). At low frequencies \( (a_z < a_{\alpha} \) and \( a_w < a_{\alpha} \)), the factors \( s_a \), \( q_a \) and \( s_w \) are real numbers as opposed to imaginary in the original model. Accordingly, the impedances \( k_a \) and \( k_w \) in Eqs. (5) and (6) are real valued which implies that no waves develop in the medium. In the high-frequency range \( (a_z > a_{\alpha} \) and \( a_w > a_{\alpha} \)), \( s_a \), \( q_a \) and \( s_w \) will turn complex and an imaginary part will appear in Eqs. (5) and (6). The two frequencies at which the coefficients \( s_a \), \( q_a \) and \( s_w \) become zero (i.e., \( a_z = a_{\alpha} \) and \( a_w = a_{\alpha} \)) correspond to the cutoff frequencies of the system in lateral and vertical oscillations, respectively.

The variation with frequency of soil impedances obtained from the proposed formulation are plotted in Fig. 6 for a medium with zero material damping. The stiffness parameter \( k \) (which corresponds to either \( o_{\alpha} \) or \( o_{\alpha} \)) is normalized with respect to its static value, while frequency is normalized with respect to the cutoff frequency \( o_{\text{cutoff}} \) (i.e., \( a_{\alpha} \) or \( a_{\alpha} \)). Some interesting trends are observed: for frequencies below cutoff, the real part of the impedance decreases monotonically with increasing frequency while the imaginary part is zero. At cutoff frequency, the stiffness of the undamped medium becomes zero while the imaginary part exhibits a sudden increase. Beyond cutoff, both real and imaginary parts increase...
monotonically with frequency (as in Figs. 2 and 3). Interestingly, with this normalization the behavior in the horizontal and vertical modes is almost identical which is different than what observed in Figs. 2 and 3. It is noted that this representation is not possible with the original model since neither static stiffness nor cutoff frequency exist.

The effect of material damping on dynamic soil impedance is presented in Fig. 7 for a pre-specified cutoff frequency $a_{cut}=0.1$. With non-zero material damping, stiffness tends to decrease as compared to the undamped medium while damping tends to increase. At the cutoff frequency the drop in stiffness is understandably not as dramatic as in the undamped medium. For frequencies below cutoff, damping is practically equal to the soil material damping (shaft is assumed to be undamped). The behavior in the horizontal and vertical modes is, again, very similar.

**Selection of Shape Functions**

To compute the impedance of the soil medium using the proposed method, a pair of pertinent functions $\chi_{v}(z)$ and $\chi_{h}(z)$ is needed. For instance, for a stiff shaft vibrating vertically a linear shape function can be employed

$$\chi_{v}(z) = 1 - \frac{z}{L}$$  \hspace{1cm} (24)

which corresponds to the deformed shape of an axially-loaded cantilever. In the case of a more compressible shaft or stiffer soil, a perhaps more appropriate choice is the exponential function

$$\chi_{v}(z) = \cosh \lambda z \left( 1 - \frac{\tanh \lambda z}{\tanh \lambda L} \right)$$  \hspace{1cm} (25a)

which is obtained using Winkler theory (Scott 1981); $\lambda$ is given by

$$\lambda = \sqrt{\frac{E_s}{2E_p A_p}}$$  \hspace{1cm} (25b)

where $A_p$ denotes the cross-sectional area of the shaft.

With the above shape functions, the cutoff frequencies of system are Eqs. ((15) and (20b))

$$a_{sw} = 3 \eta w \left( \frac{L}{d} \right)^{-1}$$  \hspace{1cm} (26)

$$a_{sw} = \lambda d \eta w \frac{\sinh 2\lambda L + 2\lambda L}{\sinh 2\lambda L - 2\lambda L}$$  \hspace{1cm} (27)

for the linear and the exponential shapes, respectively. It is noted that Eq. (26) is independent of pile-soil stiffness contrast, while the more elaborate expression (27) depends on both $E_p/E_s$ and $L/d$. It will be shown that Eq.
(26) will suffice for engineering estimates of vertical response.

In the lateral mode, a sinusoidal shape function can be employed

\[ \chi_s(z) = \cos \left( \frac{\pi z}{2L} \right) \]  
(28)

The cutoff frequency obtained with this shape is (Eqs. (15) and (20a))

\[ a_{cd} = \frac{\pi}{2} \left( \frac{L}{d} \right)^{-1} \]  
(29)

which coincides with the fundamental natural frequency in shearing oscillations of a homogeneous layer.

An alternative shape function can be obtained using Winkler theory,

\[ \chi_s(z) = e^{-\mu z} \left( -e^{2\mu L} \left( \cos \mu L - \sin \mu L \right) + e^{4\mu L} \left( \cos \mu L + \sin \mu L \right) - e^{2\mu L} \left( 1 - e^{2\mu L} \right) \cos (2\mu L - \mu z) + (1 + e^{2\mu L}) \sin (2\mu L - \mu z) \right) \]  
(30)

which corresponds to the deformed shape of a fixed-head hinged-base cylinder. The corresponding cutoff frequency of the system is given from the complicated expression

\[ a_{cd} = \mu \frac{1 + E_s - 2E_s \left( 1 + 4 \mu L + E_s \left( 1 - 4 \mu L \right) \right) C_2}{2E_s \left( 1 + 4 \mu L \right) S_2 + 2E_s \left( 8 \mu L + S_2 \right)} \]  
\[ - 3 \left( 1 + E_s \right) + 2E_s \left[ 3 \left( 1 + E_s \right) C_2 - (3 - 4 \mu L + E_s (3 + 4 \mu L) + 4E_s C_2) S_2 \right] \]  
(31a)

where \( E_{ij} \), \( S_{ij} \) and \( C_{ij} \) denote, respectively, the functions \( \exp (j \times \mu L) \), \( \sin (j \times \mu L) \), \( \cos (j \times \mu L) \). \( \mu \) in the above equations is a shape parameter that can be approximated by (Poulos and Davis, 1980; Scott, 1981; Dobry et al., 1982)

\[ \mu = \left( \frac{E_s}{4E_s I_p} \right)^{1/4} \]  
(31b)

Static stiffness coefficients obtained from the above expressions are presented in Fig. 8. Ranges of values pertaining to pile foundations (Novak et al., 1978; Roesset, 1980; Dobry et al., 1982) are also shown. It is seen that for small slenderness ratios soil stiffness is higher than that for piles and tends to increase with decreasing \( L/d \) and \( E_p/E_s \). The trend is more pronounced in the vertical mode. Interestingly, the coefficients in the two modes are approximately equal which is in contrast to the predictions of the plane strain model (recall that \( \delta_v = 2\delta_s \) in Figs. 2 and 3). The results obtained with the different shape functions agree quite well especially for large soilfoundation stiffness contrasts (\( E_p/E_s = 1000 \)). The linear and sinusoidal shape functions (Eqs. (26) and (29)) will be used in the remainder of the paper.

The sensitivity of \( \delta_v \) and \( \delta_s \) to Poisson’s ratio is examined in Fig. 9. Both coefficients tend to increase with increasing \( v \), with the trend being more pronounced in the vertical coefficient. Nevertheless, recalling that the response of the shaft is proportional to \( (\delta_v)^{1/4} \) and \( (\delta_s)^{1/2} \) (see Eqs. (34b) and (35b) below), the effect of Poisson’s ratio on the dynamic response of the system is of secondary importance.

Incidentally, it is noted that \( \delta_v \) and \( \delta_s \) can be approximated by the expressions

\[ \delta_v = \frac{2\pi}{\ln \left( \frac{2}{s_w} \right) - \gamma} \]  
\[ \frac{5/4 + 2\left( -1 + 2\gamma + \ln \frac{s_w}{4} \right)}{2\left( \ln \left( \frac{2}{s_w} \right) - \gamma \right)^2} \]  
(32)

\[ \delta_s \approx \frac{4\pi \eta^2}{\left( 1 + \frac{1}{\eta^2} \right)^{1/2}} \left( \ln \frac{4\eta - \gamma}{s_w} \right) \]  
(33)

which were obtained from Eqs. (5) and (6) using series expansions around \( s = 0 \). In the above equations, \( \gamma \) denotes Euler’s number (\( \approx 0.577 \)).

**Shaft Impedance**

Considering the soil springs to be uniformly distributed along the shaft, the lateral dynamic impedance, \( X_{ij} = k_{ij}(\omega) + i\omega c_{ij}(\omega) \) at the head of a shaft carrying zero moment at the toe is (Mylonakis, 1995)
Table 1. Static pile stiffness obtained from the proposed model:

<table>
<thead>
<tr>
<th>$L/d$</th>
<th>$E_s/E_n$</th>
<th>$K_v/E_n d$</th>
<th>$K_{hv}/E_n d$</th>
<th>$K_{hv}/E_n d^2$</th>
<th>$K_v/E_n d^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>100</td>
<td>28.23</td>
<td>3.14</td>
<td>2.97</td>
<td>5.79</td>
</tr>
<tr>
<td></td>
<td>300</td>
<td>80.61</td>
<td>4.35</td>
<td>6.33</td>
<td>15.67</td>
</tr>
<tr>
<td></td>
<td>1000</td>
<td>263.90</td>
<td>8.21</td>
<td>17.82</td>
<td>50.06</td>
</tr>
<tr>
<td>5</td>
<td>100</td>
<td>15.78</td>
<td>2.85</td>
<td>2.67</td>
<td>5.10</td>
</tr>
<tr>
<td></td>
<td>300</td>
<td>42.03</td>
<td>3.58</td>
<td>4.46</td>
<td>11.74</td>
</tr>
<tr>
<td></td>
<td>1000</td>
<td>133.70</td>
<td>4.69</td>
<td>8.93</td>
<td>32.80</td>
</tr>
<tr>
<td>7</td>
<td>100</td>
<td>14.09</td>
<td>2.59</td>
<td>2.55</td>
<td>5.00</td>
</tr>
<tr>
<td></td>
<td>300</td>
<td>36.63</td>
<td>3.40</td>
<td>4.34</td>
<td>11.25</td>
</tr>
<tr>
<td></td>
<td>1000</td>
<td>115.20</td>
<td>4.38</td>
<td>7.61</td>
<td>28.00</td>
</tr>
<tr>
<td>10</td>
<td>100</td>
<td>11.22</td>
<td>2.35</td>
<td>2.38</td>
<td>4.84</td>
</tr>
<tr>
<td></td>
<td>300</td>
<td>27.12</td>
<td>3.10</td>
<td>4.14</td>
<td>11.04</td>
</tr>
<tr>
<td></td>
<td>1000</td>
<td>82.17</td>
<td>4.18</td>
<td>7.46</td>
<td>26.91</td>
</tr>
</tbody>
</table>

Note that in the extreme case of an infinitely-long shaft, the ratios in the right-hand side of Eqs. (34a) and (35a) tend to unity and Eqs. (34a) and (35a) converge to the well-known expressions (Pender, 1993): $K_{hv}=4E_pI_d\mu^2$; $K_v=2E_pI_d\mu^2$; $K_{hv}=2E_pI_d\mu$; $K_v=E_pA_p\lambda$.

Table 1 presents results static shaft stiffnesses obtained with the proposed model. The effect of slenderness ratio on the shaft stiffness is quite substantial which highlights the participation of the base on the response. Comparison with results obtained by the author using the
rigorous method of Kaynia and Kausal (1982) showed good agreement. For example, for $L/d=10$ and $E_p/E_s=1000$, the rigorous solution yielded the values 81.6, 47.2, 7.68, and 28.9, which are in agreement with those in Table 1 (last row). Similar good agreement was obtained for other $E_p/E_s$ and $L/d$ ratios (not shown).

Results for dynamic shaft impedance are presented in Figs. 10 to 12. Figure 10 refers to a stiff shaft ($E_p/E_s=1000$) and 5% soil material damping. In the vertical mode, shaft stiffness is almost independent of frequency ($K/K_{\text{static}}$ varies between 1 and 0.98) and damping is very small (less than 1%). This can be explained by recalling that the response is governed mainly by the compliance of the shaft rather than the soil. On the other hand, the horizontal impedance is more sensitive to frequency: at cutoff frequency the stiffness may be less than 65% of its static value. Damping is more pronounced than in the vertical mode and may exceed 40% beyond cutoff. The cutoff frequencies shown in the graph can be easily verified using Eqs. (26) and (29).

A soft shaft ($E_p/E_s=100$) is examined in Fig. 11. The variation of stiffness with frequency is stronger than for the stiff shaft of the previous graph. Also, damping is much higher—which is expected since the compliance of the system is controlled to a larger extent by the dissipative soil medium. An intermediate behavior is observed with the shaft of medial stiffness in Fig. 12.

A comparison of the proposed approximate model against results from the rigorous solution of Kaynia and Kausal (1982) and of Novak’s plane-strain model are presented in Fig. 13 for the shaft-soil system of Fig. 10. The accord between the presented approximate model and the rigorous solution is very good over the whole range of frequencies examined. In contrast, the plane-strain solution is less accurate, especially in the lateral mode and at low frequencies.

INTERACTION BETWEEN ADJACENT SHAFTS

It is well known that in addition to loading transmitted to the foundation from the superstructure, grouped foundation members experience additional loading imposed to them by waves emitted from the neighboring members. Accordingly, the dynamic impedance of a group may be quite different from the superimposed impedances of the individual elements. For piles, group effects can be modeled approximately with the superposition method which makes use of interaction factors (Poulos and Davis, 1980; Kaynia and Kausal, 1982; Dobry and Gazetas, 1988; El-Marsafawi et al., 1992; Mylonakis and Gazetas, 1999). The interaction factor is defined as the head response (translation or rotation) of a pile carrying no load at its head ("receiver" pile), subjected to the vibrations of a neighboring head-loaded pile.
Fig. 13. Comparison of dynamic shaft stiffness and damping obtained with the proposed analytical model versus results from the rigorous numerical solution of Kaynia and Kausel (1982) and the simplified plane-strain model of Novak (1974): \( L/d = 10, \ E_1/E_2 = 1000, \nu = 0.4, \rho_1/\rho_2 = 1.25, \beta_1 = 0.05 \). [In the plane-strain solution, stiffness was normalized with the static stiffness of the proposed model.]

Fig. 14. Attenuation of soil displacement with radial distance from the shaft, unded static loading: (top) vertical mode, (bottom) lateral mode, \( E_1/E_2 = 1000, \nu = 0.4, \rho_1/\rho_2 = 1.25, \beta_1 = 0.05 \).

("source" pile). In this section, the superposition method will be extended to large-diameter shaft foundations.

In recent studies on piles, Mylonakis and Gazetas (1998, 1999) showed that the dynamic interaction factor between two adjacent foundations, \( \alpha \), can be written as a product of two complex-valued functions

\[
\alpha = \psi \times \zeta
\]

(36)

Of these functions \( \psi \) describes the attenuation of soil displacement around a head-loaded foundation ("source" foundation), while \( \zeta \) is diffraction factor accounting for the diffraction of the induced wave field at the location of a neighboring "receiver" foundation.

The proposed model yields an explicit solution for factor \( \psi \). For example, in the vertical mode \( \psi \) is given by

\[
\psi_w(r, \omega) = \frac{w(r)}{w(d/2)} = \frac{K_0(2r s_w/d)}{K_0(s_w)}
\]

(37)

Equation (37) is identical to obtained from the plane-strain model except for the coefficient \( s_w \) which is given by Eq. (19c) instead of Eq. (7a). The corresponding expression for the horizontal mode is more complicated and is provided in Appendix II.

The attenuation of soil displacement around a statically-loaded shaft obtained from the proposed model is presented in Fig. 14. Also plotted in the graph is an approximate expression derived by Dobby and Gazetas (1988) using plane-strain theory. [Note that the plane-strain equations of Novak are not applicable to static conditions as they diverge as \( \omega \to 0 \).] It is seen that soil displacement attenuates quickly and practically vanishes beyond about 6 to 8 shaft diameters from the foundation centerline. The plane-strain formula predicts a significantly slower attenuation which is anticipated since it does not account for the presence of the rigid base.

The effect of frequency on the attenuation functions is illustrated in Fig. 15, in which the amplitude of \( \psi \) is plotted as function of frequency for seven shaft-to-shaft separations. Corresponding results obtained from the plane-strain model are also shown for comparison. In the frequency range \( 0 < \alpha_0 < 1 \), \( |\psi| \) is always higher than its static value which implies that shaft-to-shaft interaction is more pronounced with dynamic loads. At the cutoff frequencies \( \alpha_0 = 0.32, \alpha_{cu} = 0.18 \) a significant increase in \( \psi \) is observed which may exceed 100% of the static value particularly at large distances from the shaft. This implies that special care should be taken in computing group effects if the predominant periods of the excitation
Fig. 15. Attenuation of soil displacement around a shaft: (top) vertical mode, (bottom) lateral mode. $L/d = 10$, $E_s/E_i = 1000$, $v = 0.4$, $\rho_s/\rho_i = 1.25$, $\beta = 0.05$

are near the cutoff frequency of the system. Similar findings have been reported for pile foundations by El-Mar- safawi et al. (1992) and Gazetas et al. (1998). The predictions of the plane-strain model are in agreement with the present model beyond cutoff, but they diverge (significantly overestimate $\psi$) at smaller frequencies.

Incidentally, it is noted that the attenuation functions (37) and (II-1) can be approximated by the expressions

$$\psi_u(r, 0) = -\frac{d}{2\tau} \exp \left[-\left(-\frac{r}{d} - \frac{1}{2}\right) a_0^2 - \frac{a_0^2}{1 + 2i\beta} \right]$$

and

$$\psi_u(r, \pi/2) = \frac{d}{2\tau} \exp \left[-\left(-\frac{r}{d} - \frac{1}{2}\right) a_0^2 - \frac{a_0^2}{1 + 2i\beta} \right]$$

(39a)

Equation (38) is asymptotic to Eq. (37) at large arguments. In contrast, Eqs. (39) are approximate expressions based on the model of Dobry and Gazetas (1988).

The diffraction factor $\zeta$ can be determined analytically by applying the wave interference model of Mylonakis and Gazetas (1998, 1999). Details on that method are given in the above publications and won't be repeated here. In the case of a shaft hinged at the base

$$\zeta_{SP} = \frac{\kappa_u}{\kappa_u - m\omega^2} \frac{8\mu\lambda - 2T_3(4\mu\lambda C_3 + 3S_3) + 3S_4 - 6C_2 H_2 + 3H_4}{8(C_4 - T_2)(S_2 - H_2)}$$

(40a)

$$\zeta_{SM} = \zeta_{SP}$$

(40b)

$$\zeta_{SM} = \frac{\kappa_u}{\kappa_u - m\omega^2} \frac{2T_2 S_2 + S_2 - 2(C_2 + 4\mu\lambda S_2) H_2 - H_4}{8(C_4 - T_2)(S_2 - H_2)}$$

(40c)

where $\zeta_{SP}$, $\zeta_{SM}$, $\zeta_{SM}$ denote the diffraction functions for swaying, rocking, and cross-swaying rocking, respectively; $S_0$, $C_0$, $T_0$ and $H_0$ stand for the functions $\sin (j \times \mu L)$, $\cos (j \times \mu L)$, $\cosh (j \times \mu L)$, and $\sinh (j \times \mu L)$, respectively.

In the vertical mode, the corresponding function is (Mylonakis and Gazetas, 1998)

$$\zeta_v = \frac{\kappa_u}{\kappa_u - m\omega^2} \frac{1}{2} \left[ 1 + \frac{2\lambda L}{\sinh (\lambda L)} \right]$$

(41)

The variation with shaft length of the above functions is illustrated in Fig. 16. For zero foundation length all lateral diffraction functions tend to unity—an anticipated behavior since a hinged-base shaft of zero length has no flexural resistance and follows exactly the motion imposed by a neighboring shaft. The functions tend to decrease with increasing foundation length reaching the asymptotic values $\zeta_{SP} = 3/4$; $\zeta_{SM} = 1/2$; $\zeta_{SM} = 1/4$ at around $\mu L = 3$. These values are identical to those obtained by Mylonakis and Gazetas (1999) for long floating piles. In the vertical mode, the diffraction function is zero at $\lambda L = 0$ (an end-bearing shaft of zero length does not settle) and increases monotonically with increasing shaft length approaching the limiting value $\zeta_v = 1/2$. 

Fig. 16. Effect of shaft length on diffraction functions $\zeta$ for an end-bearing shaft in a homogeneous soil stratum, for hinged conditions at the base
Application: 2 × 1 Shaft Group

With the superposition method the dynamic impedance of a grouped foundation can be calculated by superimposing the interaction factors between the individual pairs of members. For instance, for a group of \(N\) identical shafts oscillating vertically, the dynamic impedance, \(\mathcal{K}_w\), is obtained as (Poulos and Davis, 1980; Kaynia and Kausel, 1982)

\[
\mathcal{K}_w = \mathcal{K}_w \{1\}^T \alpha_u^{-1} \{1\}
\]

where \(\mathcal{K}_w\) is the vertical impedance of a single shaft, \([\alpha_u]\) is a \(N\) by \(N\) complex matrix containing the interaction factor between the shaft pairs, and \(\{1\}\) is the \(N\) by 1 unit vector.

With only two shafts, Eq. (42) reduces to the simple expression (Dobry and Gazetas, 1988)

\[
\mathcal{K}_w^{2 	imes 2} = 2 \mathcal{K}_w (1 + \alpha_u)^{-1}
\]

in which \(\alpha_u\) is given by Eq. (36). In the case of lateral vibrations, the stiffness of a 2 by 1 group is

\[
\mathcal{K}_w^{2 	imes 1} = 2 \mathcal{F}_n (1 + \alpha_{sl}) / [\mathcal{F}_{uu} \mathcal{F}_n (1 + \alpha_{sp})(1 + \alpha_{sl}) - \mathcal{F}_{nn} (1 + \alpha_{sl}})
\]

were \(\mathcal{F}_{nn}\) denote the flexibility coefficients of a single shaft.

In Fig. 17, the dynamic impedance of a 2×1 shaft group is plotted in terms of the well-known group efficient factor (which is defined as the dynamic impedance of the group divided by the sum of the static stiffnesses of the individual shafts). At low "static" frequencies, the group factor reduces to the familiar static efficiency factor which, for elastic conditions, is always smaller than 1. At the fundamental natural frequency of the layer, the lateral impedance decreases substantially while the vertical impedance remains practically constant. At higher frequencies, wave interference phenomena become apparent leading to efficiency factors that far exceed unity. This behavior is analogous to that observed in pile foundations (Kaynia and Kausel, 1982; El-Marasfawi et al., 1992; Dobry and Gazetas, 1988; Mylonakis and Gazetas, 1998; 1999).

A different behavior is observed in Fig. 18 for a shorter \(L/d = 5\) shaft group. Lateral interaction effects are much stronger than in the longer shafts of the previous figure. Beyond the cutoff frequency, in particular, group efficiency becomes very small which suggests a significant drop in stiffness. More research is needed to quantify these effects for design purposes.
CONCLUSIONS

An approximate analytical model was presented for determining the dynamic response of large-diameter end-bearing shafts subjected to vertical and lateral harmonic head loads. The method is essentially an extension to three dimensions of Novak’s plane-strain model, formulated in conjunction with a modified Vlasov-Leontiev approximation based on integrating the governing equations over the vertical coordinate. The enhanced model is free of the shortcomings of the earlier model reproducing the static stiffness and cutoff frequency of the shaft-soil system. Shaft impedances and interaction factors between adjacent shafts are determined analytically and valuable insight is gained on the physics of the problem. Predictions from the method were found to be in good accord with more rigorous solutions.

It is emphasized, however, that the proposed model is limited by the assumptions of linearity in the soil and shaft materials, and perfect bonding at the shaft-soil interface. Therefore the method must be modified before being applied to situations involving nonlinear effects.

NOTATIONS

<table>
<thead>
<tr>
<th>English Symbols</th>
<th>Greek Symbols</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_o, a_w, a_v$</td>
<td>$\alpha$</td>
</tr>
<tr>
<td>$A_s$</td>
<td>$\beta, \beta'$</td>
</tr>
<tr>
<td>$b, b_o, b_w$</td>
<td>$\delta, \delta_o, \delta_v$</td>
</tr>
<tr>
<td>$d$</td>
<td>$\zeta$</td>
</tr>
<tr>
<td>$E_s, G_s, G_i$</td>
<td>$\eta_s, \eta_v, \eta_w$</td>
</tr>
<tr>
<td>$m$</td>
<td>$\theta$</td>
</tr>
<tr>
<td>$E_y$</td>
<td>$\lambda, \mu$</td>
</tr>
<tr>
<td>$E_p$</td>
<td>$\sigma, \tau$</td>
</tr>
<tr>
<td>$I_s, I_w$</td>
<td>$v$</td>
</tr>
<tr>
<td>$I_y$</td>
<td>$\nu$</td>
</tr>
<tr>
<td>$k, k_{aux}$</td>
<td>$X, X_{os}, X_o$</td>
</tr>
<tr>
<td>$l, l_o, l_w$</td>
<td>$\omega$</td>
</tr>
<tr>
<td>$s, s_o, s_v$</td>
<td>$\omega$</td>
</tr>
<tr>
<td>$u, v, w, U$</td>
<td>$\omega$</td>
</tr>
<tr>
<td>$v_s, V_s$</td>
<td>$\omega$</td>
</tr>
<tr>
<td>$z$</td>
<td>$\omega$</td>
</tr>
</tbody>
</table>

REFERENCES

23) Mylonakis, G. and Gazetas, G. (1998): Vertical vibrations and additional distress of grouped piles in layered soil, Soils and Founda-
APPENDIX I: DERIVATION OF EQUATIONS (16)–(18)

Multiplying Eq. (12) by $\chi(z)$, integrating over the shaft length and introducing Eq. (13) leads to

$$\int_0^l \chi(z)dz \left[ \frac{d^2w}{dr^2} + \frac{1}{r} \frac{dw}{dr} + \left( \frac{\omega}{V_s^*} \right)^2 w \right] + \int_0^l \frac{d^2\chi(z)}{dz^2} \chi(z)dz \eta^2 w = 0$$

(I-1)

Using integration by parts and enforcing the boundary conditions $w(L, r) = 0, \sigma_0(0, r) = 0$, it can be shown that

$$\int_0^l \frac{d^2\chi}{dz^2} \chi dz = -\int_0^l \frac{d\chi}{dz} \frac{dz}{dz}$$

(I-2)

Substituting Eq. (I-2) into Eq. (I-1) and dividing by $\frac{d\chi(z)}{dz}$, Eq. (18) is obtained, with $b_\alpha$ given by Eq. (15). An analogous procedure can be applied to derive Eqs. (16) and (17) from Eqs. (10) and (11).

APPENDIX II: LATERAL ATTENUATION FACTOR

The induced free-field displacement in lateral vibrations $\psi_\alpha$ is given by

$$\psi_\alpha(r, \theta) = \psi_\alpha(r, 0) \cos^2 \theta + \psi_\alpha(r, \pi/2) \sin^2 \theta$$

(II-1a)

where

$$\psi_\alpha(r, 0) = \frac{d}{2r} \left[ -K_s(q2r/d) + q2r/dK_s(q2r/d) \right] \left[ 2K_s(q) + sK_s(q) \right] + K_l(s2r/d) \left[ 2K_l(q) + qK_l(q) \right]$$

(II-1b)

$$\psi_\alpha(r, \pi/2) = \frac{d}{2r} \left[ K_l(s2r/d) + s2r/dK_l(s2r/d) \right] \left[ 2K_l(q) + qK_l(q) \right] - K_s(q2r/d) \left[ 2K_s(q) + sK_s(q) \right]$$

(II-1c)