EFFECTS OF FOOTING SIZE AND ASPECT RATIO ON THE BEARING CAPACITY OF SAND SUBJECTED TO ECCENTRIC LOADING

Mitsu Okamura(i), Atsunori Mihara(ii), Jiro Takemura(iii) and Jiro Kuwano(iii)

ABSTRACT

The current practice of estimating bearing capacity usually employs the conventional bearing capacity formula originally developed for strip footings under vertical central loading. In order to account for the effect of footing shape and eccentricity and inclination of loads, correction factors are introduced in the formula, which are derived based on a number of small-scale model test observations.

This paper describes research on the bearing capacity of rectangular footings on sand subjected to vertical eccentric loading. Two aspects, namely the effects of footing size and of footing shape on the bearing capacity and deformation characteristics, are focused on. A series of loading tests was conducted in a centrifuge on rectangular footings with aspect ratios from 1 to 5, at two different centrifugal accelerations. In addition, finite element analyses were performed in which factors influencing the angle of shear resistance including stress level dependency, anisotropy and coefficient of intermediate principal stress, were taken into account.

It was found that the shape factor of footing apparently increased with increasing footing width. This indicates that the shape factor used in the current practice underestimates bearing capacity of footings. This was also the case for failure locus in the $M/B-V$ (moment-vertical) load plane. Normalized failure locus for wider footings with a smaller aspect ratio is considerably larger than that reported in the literature. The stress level dependency of the angle of shear resistance appeared to be responsible for the scale effects of footings on the failure locus.

Key words: bearing capacity, centrifuge test, deformation, eccentric loading, finite element analysis, sand, shallow rectangular footing (IGC: E3/E14)

INTRODUCTION

In current practice in estimating bearing capacities of shallow footings on sand, it is widely recognized that at least the following three factors should be properly taken into account, since they generally contribute considerably to reduce the bearing capacity factors: i) loading conditions including eccentricity and inclination of applied loads, ii) scale effect of footing on the bearing capacity factors, and iii) effect of footing shape.

As for footings subjected to central vertical loading, the scale effect of footing on the bearing capacity factors has been extensively explored by a large number of experiments conducted at 1G as well as in geotechnical centrifuges. Observed variations in the bearing capacity factors with footing width were compared with the analytical results from the extended stress characteristic method and the stress level dependency of the angle of shear resistance, $\phi'$, was confirmed to be a key aspect to estimate the scale effect on the bearing capacity factors (Oda and Koishikawa, 1979; Hettler and Gudehus, 1988; Kusakabe et al., 1991; Okamura et al., 1993). Existence of significant anisotropy in the angle $\phi'$ (Oda, 1972; Fukushima and Tatsuoka, 1984; Tatsuoka et al., 1986) was also found to be another cause of discrepancy for bearing capacity between calculated value from the stress characteristic method and experimental observations (Kimura et al., 1982, 1985; Kusakabe et al., 1991; Okamura et al., 1993). A practical method of determining $\phi'$ with due consideration of footing size was proposed by Kutter et al. (1988).

With regard to the effect of footing shape, the shape factors were introduced in the conventional bearing capacity formula originally developed for strip footings under vertical central loading conditions (e.g. Terzaghi, 1943; Meyerhof, 1951). The shape factors were derived empirically based mainly on a number of small-scale model tests (De Beer, 1970; Vesic, 1973). Thereafter, it was revealed by Kusakabe et al. (1991) and Okamura et al. (1993) that the shape factor did not take a constant value but increased or decreased with footing width.

For footings subjected to eccentric and/or inclined loading, Meyerhof (1953, 1963) introduced the inclination factors in the bearing capacity formula and the con-
cept of 'effective foundation width' to account for the load inclination and eccentricity, respectively. Validity of this method was examined by comparing small-scale footing test observations.

More recently, an alternative approach for estimating bearing capacities of footings under combined vertical ($V$), horizontal ($H$) and moment ($M$) loads has been developed (Butterfield and Tico, 1979; Georgiadis and Butterfield, 1988; Butterfield and Gottardi, 1994) in which a macroscopic failure envelope of an entire soil-footing system was considered in the $V$-$H$-$M$/$B$ load space. Note that $M$ is divided by footing width, $B$, to preserve dimensional homogeneity.

The plastic potential of such a soil-footing system, which prescribes incremental footing displacement components, has also been investigated. The apparent advantage of this approach is that the failure envelope in conjunction with the plastic potential is capable of providing methods to directly describe footing response under general combined loads. In fact, the idea has been successfully applied to footing and retaining wall problems (Nova and Montransio, 1991; Okamura and Matsuo, 2002).

A possible criticism on the research work concerning footings under combined loadings, however, is that all the model tests that stemmed and verified the inclination factors, the failure envelope and the plastic potential were carried out at gravitational field (i.e. 1G) using small footing width—mostly smaller than 100 mm in breadth—with exceptions for tests conducted in a centrifuge on particular spudcan footings (e.g. Dean et al., 1997) and on foundations on high gravel mounds (Terashi and Kitazume, 1987). It can also be pointed out that, in most of these test cases, strip footings or rectangular footings with an aspect ratio higher than five, which can be regarded essentially as strip footings, were used. To date, available data to examine the effects of either footing width or footing shape on the bearing capacity of footings under combined loadings is very limited in the literature.

The main objective of this paper is to provide data about variation of size and shape of failure envelope and plastic potential of a soil-footing system with footing size and aspect ratio, which can be used to refine practical bearing capacity formulae and the methods to directly describe footing responses. In this study, a series of centrifuge tests was conducted to investigate the effects of size and shape of footings on the bearing capacity and the deformation characteristics of eccentrically loaded surface footings on sand. Information from the load-displacement records are presented and discussed to provide the basis for modeling the footing response at failure conditions. Finite element analyses on the bearing capacity were also performed, in which influential factors on the angle $\phi'$ including stress level, anisotropy and coefficient of intermediate principal stress were taken into account. This makes it possible to examine the influence of these factors on the bearing capacity in relation to the effects of scale and shape of footings.

![Centerline along long axis](centerline along short axis)

(a) Perspective view of model footing

![Container and footing](container and footing)

(b) Arrangement of model footing; footing was set at the center of sand surface with its long axis perpendicular to the long axis of container

Fig. 1. Schematic illustration of model footing and its arrangement in container: (a) Perspective view of model footing, (b) Arrangement of model footing set at center of sand surface with long axis perpendicular to long axis of container.

**CENTRIFUGE TESTS PROCEDURES AND CONDITIONS**

Toyoura sand with a relative density $D_r$ ranging between 88 and 90% was used in all centrifuge tests reported in this paper. Dry Toyoura sand ($D_{so} = 0.19$, $G_s = 2.640$, $e_{max} = 0.973$ and $e_{min} = 0.609$) was rained to a depth of about 200 mm from a single hole hopper into a rigid model container with internal dimensions of 490 mm long, 300 mm wide and 400 mm deep. The hopper was manually moved back and forth and the free falling height was kept constant to provide uniform sand deposits with the desired relative density. The sand surface was prepared flat, the sample height was measured at several points and averaged to give the initial model height. A 40 mm wide rectangular steel footing with a rough base was placed at the center of the sand surface with its long axis perpendicular to the long axis of the container, as shown in Fig. 1. The footing was specifically designed to be rigid and thin so that the change in load eccentricity as the footing rotated during loading was minimal. In order to make sure that the footing was free to rotate at the loading point and the loading ram did not transmit significant horizontal load to the footing, the vertical load from the loading jack was applied to the footing through a ball bearing. The ball bearing was set on the short axis of footings, at a distance $e$, from the center as shown in Fig. 1. In the cases of central loading (i.e. $e_s = 0$), however, a footing with much higher flexural rigidity (40 mm in thickness) was connected rigidly to the load cell. This was largely due to the fact that deforma-
Table 1. Summary of test conditions

<table>
<thead>
<tr>
<th>Centrifugal acceleration, (N(G))</th>
<th>Footing width, (B) (m)</th>
<th>Prototype footing width, (NB) (m)</th>
<th>Aspect ratio (L/B)</th>
<th>Initial eccentricity ratio (e/B)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.04</td>
<td>0.04</td>
<td>1</td>
<td>0, 0.05, 0.1, 0.15, 0.2, 0.25, 0.3</td>
</tr>
<tr>
<td>3</td>
<td>0.04</td>
<td>0.04</td>
<td>3</td>
<td>0, 0.05, 0.1, 0.15, 0.2, 0.25, 0.3</td>
</tr>
<tr>
<td>5</td>
<td>3.0</td>
<td>3.0</td>
<td>5</td>
<td>0, 0.05, 0.1, 0.15, 0.2, 0.25, 0.3</td>
</tr>
<tr>
<td>1</td>
<td>0.04</td>
<td>0.04</td>
<td>1</td>
<td>0, 0.05, 0.1, 0.15, 0.2, 0.25, 0.3</td>
</tr>
<tr>
<td>3</td>
<td>0.04</td>
<td>0.04</td>
<td>3</td>
<td>0, 0.05, 0.1, 0.15, 0.2, 0.25, 0.3</td>
</tr>
</tbody>
</table>

Fig. 2. Schematic illustration of test setup

Fig. 3. Model footing with built-in load cells used for contact pressure measurement

tion of the footing itself and the loading ram became marked for some central loading tests at high centrifugal acceleration with large load application. Several preliminary tests, however, confirmed that the fixity condition did not affect the load settlement behavior of the footing under central-vertical loading before the peak load, as reported by Kutter et al. (1988).

After a loading jack and three LVDTs used for vertical settlement measurement of the footing were set, the test setup shown in Fig. 2 was brought to the Mark II centrifuge at Tokyo Institute of Technology. Loading tests were then carried out in either a 75 G centrifugal acceleration environment or gravitational field (1G). Three footing lengths, \(L = 40\) mm, 120 mm and 200 mm were selected to provide the aspect ratio of footings, \(L/B\), from 1 to 5, which approximately covers two extreme conditions of axial symmetry and plane strain condition (Kusakabe et al., 1991), where \(B\) is the footing width. The initial load eccentricity \(e\), that is, the distance from the ball bearing to the center of footing ranged from 0 (central loading) to 12 mm at an interval of 2 mm along with the short axis of the footings (in the direction of footing width), giving rise to the eccentricity ratio \(e/B\) ranging from 0 to 0.30. A total of 41 tests were conducted, and test conditions are summarised in Table 1. Width of footings and depth of sand deposits were carefully determined through preliminary tests at 1G so that side and bottom boundaries imposed by the rigid soil container did not influence the bearing capacity.

In addition to these experiments, a test of a rectangular footing with \(L/B = 3\) was conducted at 75G on a similar model ground. This additional test aimed to investigate the base contact pressure distribution of rectangular footings. The model footing shown in Fig. 3 was used for this purpose; it was equipped with a total of 20 small load cells for measuring vertical contact pressures.

The centrifuge tests conducted in this study were aimed at studying the effects of size and aspect ratio of footings on the bearing capacity characteristics. Therefore, the centrifuge experiments presented in this paper simulated both 40 mm wide and 3.0 m wide rectangular footings with different aspect ratios resting on a uniform sand bed, which were subjected to vertical eccentric loadings. It should be mentioned here that the scale effect of footing on the bearing capacity factors (De Beer, 1970) is considered to arise mainly from two causes (Tatsuoka et al., 1991). One is the stress level dependency of the mechanical property of sand (De Beer, 1965; Row, 1969; Yamaguchi et al., 1976), which is referred to as the “stress level effect”. The increase in the stress level in sand due to the increase in the footing width makes sand more contractive with its peak angle of shear resistance lowered, resulting in the decrease of the bearing capacity.
TEST RESULTS AND DISCUSSIONS

Bearing Capacity for Central Loading

Load-settlement curves of the centrally loaded footings ($e_l = 0$) are shown in Fig. 4 in the form of normalized load intensities $2q/N_{r}B$ versus normalized settlement of footings, $s/B$, where $q$, $s$, and $N_{r}$ denote load intensity, footing settlement and unit weight of sand at 1G, respectively, and $N$ represents a prototype/model scaling factor. The normalized load intensities clearly decreased with increasing prototype footing width $NB$ and with decreasing aspect ratio $L/B$. Figure 5 gives the variation with $NB$ of the bearing capacity factors $N_{r} = 2q_{l}/N_{r}B$, in which $q_{l}$ is the peak load intensities of the curves in Fig. 4. The broken lines in the figure correspond to $N_{r}$ obtained from the tests of both strip and circular footings conducted by Okamura et al. (1993). Their models were constructed in much the same way as that in this study, but at a slightly lower relative density ($D_r = 86\%$). It can be seen that $N_{r}$ observed in this study is quite consistent with Okamura et al. (1993), and the bearing capacities of the square footings and rectangular footings with $L/B = 5$ are essentially the same as those for circular and strip footings, respectively. This observation is supported by a large number of 1G model tests (De Beer, 1970; Ingra and Baecher, 1983). Figure 6 indicates the relationship between shape factor $s_r$ and $NB$. The shape factor $s_r$ is defined as the ratio of the bearing capacity $q_{l}$ of a rectangular surface footing to $q_{l}$ of a surface strip footing. But, since tests with strip footings were not conducted in this study, the factor $s_r$ as represented in Eq. (1), is plotted in Fig. 6 instead.

$$s_r = \frac{q_{l}}{q_{l}(L/B=5)}$$

It is observed that the shape factor increased with $L/B$ and reached approximately unity for $L/B$ higher than three. Results of centrifuge tests reported by Kusakabe et al. (1991) are consistent with this observation. It also appeared in Fig. 6 that the shape factor is larger for wider footings. This indicates that the shape factors widely used in practice which were derived mainly from small scale model tests may yield conservative bearing capacity predictions for the practical footing width.
The vertical loads at failure, \( V_{\text{max}} = q/L \), of these central loading tests will be used in subsequent sections to normalize the vertical and moment load components.

**Failure Locus in \( M/B-V \) Plane**

Normalized load-settlement curves of both centrally and eccentrically loaded footings for cases of \( L/B = 1 \) and \( 5 \) are depicted in Fig. 7. In the cases of eccentric loading, the center of the footing base was taken as the reference point of the footing settlement. As the eccentricity ratio \( e/B \) increased, the load intensities became significantly smaller and peak loads appeared at smaller footing settlements.

The relationship between vertical and moment load \((V, M/B)\) at failure normalized with respect to the maximum vertical load \( V_{\text{max}} \) is plotted in Fig. 8, together with least-square-fit parabolas given by Eq. (2).

\[
\frac{M}{BV_{\text{max}}} = \varphi \frac{V}{V_{\text{max}}} \left(1 - \frac{V}{V_{\text{max}}} \right)
\]  

(2)

Note again that moment \( M \) is divided by \( B \) to preserve dimensional homogeneity. The moment at the peak load is calculated by

\[
M = V(e, \cos \theta + H \sin \theta)
\]  

(3)

where \( \theta \) and \( H \) are footing rotation angle at failure and height of loading point from footing base to the top of the ball bearing, respectively. It can be observed that the maximum moment occurred at about half the maximum vertical load and the parabolas can provide a reasonably good fit to the data points, and thus the failure locus. These observations are quite consistent with those reported by Georgiadas and Batterfield (1988) and Gottardi and Batterfield (1993).

The constant \( \varphi \) in Eq. (2) represents the initial slope of the failure locus at the origin on the \( M/B-V \) plane. Since the maximum possible eccentricity of any vertical load is 0.5\( B \), the initial slope must be 0.5. However, \( \varphi \) for the best-fitted parabolas did not take a constant value but apparently depended on both the aspect ratio and the footing width. Figure 9 gives the variation of \( \varphi \) with \( L/B \) obtained from tests in this study, as well as that available in the literature. It is observed in this plot that \( \varphi \) decreases as \( L/B \) increases, with \( \varphi \) for \( NB = 3.0 \) m being greater than that for \( NB = 40 \) mm. It can also be seen that the footings of \( NB = 40 \) mm with \( L/B = 5 \) exhibit the lowest value of \( \varphi = 0.325 \). This value of \( \varphi \) is quite consistent with small-scale 1G model test results of strip footings and rectangular footings with \( L/B = 5 \) on sand beds conducted by Georgiadas and Batterfield (1988), Gottardi and Batterfield (1993) and Nova and Montransio (1991). These facts demonstrate that the constant \( \varphi \) for larger footings with aspect ratios close to unity is considerably higher than that reported in the literature, approximately 0.35.

In Fig. 8 the failure locus derived from the concept of effective width (Meyerhof, 1953) is shown by dashed lines. It can be seen that the concept of effective width provides conservative predictions of the bearing capacity, as mentioned by Meyerhof (1953) and Georgiadas and Batterfield (1988).

Another observation that can be made in Figs. 8 and 9...
investigate the effect of aspect ratio on the bearing capacity, an attempt was made to measure footing base contact pressures. The model footing used for this purpose had dimensions of 40 mm wide by 120 mm long \((L/B=3)\), and was composed of aluminum blocks and four load cell blocks, with each load cell block equipped with five load cells from ‘a’ to ‘e’. The arrangement of load-cell blocks and aluminum blocks is shown in Fig. 3. The four load-cell blocks were set in the 1st, 2nd, 4th and 6th rows. The footing was vertically loaded with the eccentricity ratio \(e_i/B = 0.2\) at 75G. Averaged contact pressure of each load cell block at failure normalized with respect to the bearing capacity \(q_i\) is plotted against the distance of the block from the footing center in Fig. 10. It is found in this figure that the measured average contact pressure distribution along the long axis of the footing is more or less uniform except for the very end of the footing (1st row). This observation implies that, even for eccentric loading conditions, the idealization to estimate bearing capacity of rectangular footings proposed by De Beer (1970) could be valid. By the idealization, the bearing capacity can be given by simply superimposing the axisymmetric (circular footing) bearing capacity and the plane strain (strip footing) bearing capacity, as illustrated in Eq. (4) and Fig. 11.

\[
q_{f(rectangular \, footing)} = \frac{L - B}{L} q_{f(strip \, footing)} + \frac{B}{L} q_{f(circular \, footing)}
\]  

(4)

Bearing capacities of the rectangular footing calculated using Eq. (4) are depicted by solid lines in Fig. 6, in which \(q_i\) for footings with \(L/B = 1\) and 5 were used in the equation as a substitute for circular and strip footings' bearing capacity, respectively. The data points of the observed bearing capacity for \(L/B = 3\) lay on the lines irrespective of the footing width, showing the validity of De Beer’s idealization.
Deformation Characteristics

Displacement of footings during loading is depicted in Fig. 12 in the form of rotation $\theta$ versus normalized settlement, $s/B$. Open circles in the figure indicate footing deformation when the load intensity reached the peak. It can be observed that the slope of the curves, that is, the ratio of rotation angle increment to the settlement increment, $\dot{\theta}B/s$, increased as the footing penetration proceeded. This has also been reported by Georgiadis and Butterfield (1988) and Gottardi and Butterfield (1995). The curves then became essentially constant (straight line) after reaching inflection points. Appearance of the inflection point coincided with the time when the applied load reached a minimum, leveling off after the peak load. The change in the slope of the curves in Fig. 12 is more apparent for footings with smaller widths and with higher aspect ratios, while the curves tended to be more or less straight lines through the course of loading sequence for wider footings with lower aspect ratios. These observations seem to be closely associated with the base contact pressure distribution. Test results of the footing with built-in load cells previously described in Fig. 3 are presented in Fig. 13 in the form of contact pressures versus normalized footing settlement at the center of the base, $s/B$. Observations at load cell block 1 near the end of the footing and at block 6 near the footing center are depicted in this figure as a typical result. Figure 14 also shows the contact pressure distribution along with the short axis of the footing at three different footing settlements, $s_i/3$, $2s_i/3$ and $s_i$, where $s_i$ denotes footing settlement at the peak load. In this figure, the contact pressures measured by each load cell were normalized with respect to the average contact pressure of each load cell block. It is apparent in Fig. 13 that the footing settlement at peak contact pressure near the footing toe (load cell ‘a’) was smallest and the footing settlement at the peak contact pressure increased with distance from the toe, implying the propagation of ‘yielding of contact pressure’ from the toe toward the heel of the footing. This is because the settlement of the footing base was the largest at the toe and smallest at the heel due to the footing rotation. The drop in the contact pressure after the peak near the toe, and

![Fig. 11. Superimposition of contact pressure for estimation of bearing capacity of rectangular footing](image)  
![Fig. 12. Relationship between footing rotation and settlement during loading test](image)
thus the change in contact pressure distribution along with short axis of footing shown in Fig. 14, causes a significant degradation of sustainable moment. This change of the contact pressure distribution is considered to be responsible for the increase of the deformation ratio $\delta B/s$ mentioned in Fig. 12. Furthermore, the change in the contact pressure distribution measured at the 1st row of load cell block appears to be less marked than that at the 6th row. This observation is consistent with the fact that the deformation ratio $\delta B/s$ was more or less constant during the course of the loading test for a smaller aspect ratio.

The relationship between the deformation ratio $\delta B/s$ and the load eccentricity ratio at failure $e_f/B$ is shown in Fig. 15. The ratio $\delta B/s$ increased approximately linearly in the range of $e_f/B$ less than about 0.3. The ratio $\delta B/s$ was generally higher for footings with a larger aspect ratio. In order to prescribe the displacement characteristics of foundations under combined loading, the flow rule has

Fig. 13. Base contact pressure-settlement curves ($L/B = 3$, $NB = 3.0$ m, $e_f/B = 0.2$)

Fig. 14. Contact pressure distribution at load cell block 1 and 6 along short axis of footing ($L/B = 3$, $NB = 3.0$ m, $e_f/B = 0.2$)

Fig. 15. Variation of displacement ratio with eccentricity at failure
been examined, in which a macroscopic constitutive law for the entire soil-footing system was considered. Applicability of the associated flow rule has been investigated by several researchers (Nova and Montrasio, 1991; Gottardi and Butterfield, 1995; Martin and Houlshby, 2000), where the failure envelope also describes the plastic potential defining the relative magnitudes of the incremental displacement during failure. Figure 16 illustrates the normalized failure loads together with the incremental displacement vectors \( \theta B, s \). The \( \theta B-s \) incremental displacement plane is superimposed on the \( M/B-V \) load plane in this figure. It can be seen that the incremental displacement vectors are generally not normal to the failure locus and this is more apparent as \( V/V_{\text{max}} \) decreases. Since the safety factor for the bearing capacity of most shallow foundations is more than two or three, the associated flow rule may not be applicable on the \( M/B-V \) plane for practical purposes. This observation is consistent with those reported by Gottardi and Butterfield (1995), and suggests the clear need for a separate plastic potential to describe foundation behavior. In order to refine a plastic potential function in the general load space (Vertical-horizontal-moment load space) a greater accumulation of test results is needed. A refined plastic potential in conjunction with the failure locus can provide the basis of an effective means for predicting the response of footings under general load application (Gottardi and Butterfield, 1995).

FINITE ELEMENT ANALYSIS OF BEARING CAPACITY

In this section, two-dimensional and three-dimensional finite element analyses of both centrally loaded and eccentrically loaded rectangular footings on uniform sand beds are described. In the analysis, three factors affecting \( \phi' \) values, that is, anisotropy, stress level and coefficient of intermediate stress, were taken into account. The bearing capacity is focused on, but the deformation of sand is beyond the scope of the calculation. Effects of these factors on the bearing capacity are studied.

Procedures of Analysis

The elasto-visco-plastic formulation (Zienkiewicz and Cormeau, 1974) was adopted in the analysis for its ability to compute collapse loads accurately (e.g. Kobayashi, 1984; Iizuka et al., 1987). In this method, the time aspect was purely fictitious; the material is allowed to violate the failure criterion for finite periods and visco-plastic flow. The rate is related to the amount by which yield is violated and is allowed to continue until the rate of straining becomes negligible. Computational procedures are demonstrated in Fig. 17. The visco-plastic calculation is repeated in the fictitious time domain and the angle \( \phi' \) was updated according to the stresses of each element in every time step. Figure 18 presents 2-D and 3-D finite element meshes utilized for the analysis of square and strip footings for a half domain considering symmetry, which were constructed using a total of 222 and 1914 elements, respectively. The analysis was conducted using the meshes of eight-node brick elements for 3-D analysis and four-node quadrilateral elements for 2-D analysis with reduced integration to avoid the stiff response and to improve the quality of solutions (Zienkiewicz et al., 1971).

The foundation soil was assumed to be an ideal linear elastic-perfectly plastic body with the use of the Mohr-Coulomb failure envelope and the associated flow rule. Although the associated behavior apparently contradicts real sand observations and provides excessive dilation, it was found that the variation of the bearing capacity with the angle of dilation was much smaller than that with \( \phi' \) (Zienkiewicz and Humpheson, 1975; De Borst and Ver-
Soil Strength Parameters Used in the Analysis

The angle \( \phi' \) of each element was determined based on the stress conditions of each element, that is, the mean principal stress, \( \sigma_m \), the angle of major principal stress direction relative to the horizontal, \( \delta \), and the coefficient of intermediate principal stress, \( b = (\sigma_2 - \sigma_1)/\sigma_1 - \sigma_3 \).

Figure 19 shows the relationship between \( \phi' \) and \( \sigma_m \) determined based on triaxial and plane strain tests on Toyoura sand samples prepared at almost the same relative density as that of the centrifuge models (Tatsuoka et al., 1986; Fukushima and Tatsuoka, 1984; Fujii, 1976). The angle \( \phi' \) was assumed to be constant in the range of mean principal stress \( \sigma_m \) lower than 49 kPa, and to decrease linearly with an increase in the logarithm of \( \sigma_m \) in the range of higher stress levels. Existence of the plateau in the range of low stress levels was confirmed by Tatsuoka et al. (1986) and Fukushima and Tatsuoka (1984). Tatsuoka et al. (1986) and Fukushima and Tatsuoka (1984) also investigated the relationship between \( \phi' \) and the angle of the major principal stress direction relative to the bedding plane of the sand sample. In order to represent this anisotropy of \( \phi' \), an ellipse with its foci located on the x-axis was introduced as shown in Fig. 20, which is the same method of expression used to represent the undrained strength anisotropy of London clay (Davis and Christian, 1971). The length of a segment connecting the origin to a point on the ellipse at an angle of 20° from the x-axis represents the ratio of \( \phi' \) at \( \delta \) to that at \( \delta = 90^\circ \). The variation of \( \phi' \) with the coefficient of intermediate principal stress \( b \) used in the analysis is illustrated in Fig. 21, which was determined from triaxial,
plane strain and torsional simple shear test results (Tatsuoka et al., 1986; Fukushima and Tatsuoka, 1984; Lam and Tatsuoka, 1988). The angle $\phi'$ was assumed to increase linearly from $b=0$ (triaxial compression) to $b=0.3$ (plane strain compression), and then level off for $b$ values higher than 0.3.

Bearing capacity analyses were carried out for two footing shapes, namely, square footing ($L/B = 1$) and strip footing ($L/B = \infty$), and for two footing widths, $NB = 0.9$ m and 3.0 m, as summarized in Table 2, although a smaller footing width $NB = 40$ mm was tested in the centrifuge experiments, instead of 0.9 m in the analysis. This is because of the limitation of calculation time; in order to obtain the ultimate load accurately, calculation time generally becomes much longer as $\phi'$ increases and the angle $\phi'$ is higher for cases of smaller footing due to the stress level dependency of $\phi'$.

### Table 2. Summary of conditions of FE analysis

<table>
<thead>
<tr>
<th>Prototype footing width, NB (m)</th>
<th>Footing shape</th>
<th>Eccentricity ratio $e/B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.9</td>
<td>Square ($L/B = 1$)</td>
<td>0, 0.125, 0.25, 0.375</td>
</tr>
<tr>
<td>1.0</td>
<td>Strip ($L/B = \infty$)</td>
<td></td>
</tr>
</tbody>
</table>

Figure 22 compares normalized load settlement curves from the analyses for an eccentrically loaded footing ($e/B = 0.25$) with $NB = 3.0$ m and $L/B = 1$. There are five curves in the figure corresponding to the following cases:

(a) $\phi' = \phi_0 = 51.4^\circ$ was used irrespective of stresses of elements. This angle corresponds to $\phi'$ for plane strain compression with $\delta = 90^\circ$ and $\sigma_m \leq 49$ kPa.

(b) $\phi' = \phi_1 = 44.1^\circ$ was used irrespective of stresses of elements. This angle corresponds to $\phi'$ for triaxial compression with $\delta = 90^\circ$ and $\sigma_m \leq 49$ kPa.

(c) Only the dependency of $\phi'$ on the $b$-value was considered. In this case, therefore, the angle $\phi'$ took a value between $\phi_0 (= 51.4^\circ)$ and $\phi_1 (= 44.1^\circ)$ depending on the $b$-value of each element.

### Results and Discussions

The dependency of $\phi'$ on the $b$-value and stress level was considered. In addition to the factors in case (c), the effect of the stress level of each element was taken into account in determining $\phi'$. All the influential factors of $\phi'$ were considered. Thus, the effect of anisotropy was taken into account in addition to the factors in case (d).

It can be seen in Fig. 22 that the bearing capacity for case (c) is close to case (b), and is considerably lower than case (a). In fact, the difference in the bearing capacity factor between case (b) and case (c) corresponds to a difference in $\phi'$ of less than one degree according to the solution from the theory of plasticity (e.g., Hansen, 1970). This implies that, in the failure zone under square footings, the coefficient of intermediate principal stress is typically close to zero. This may also indicate the adequacy in estimating bearing capacities of square or circular footings to make use of $\phi'$ obtained from triaxial tests under appropriate stress level incorporation with bearing capacity solutions for axisymmetric condition. It can also be observed in Fig. 22 that both effects of the stress level and the anisotropy on $\phi'$ have marked influences on the calculated bearing capacity, showing the importance of these factors being taken into account in the bearing capacity estimation. These factors degraded bearing capacities by 57% and 39%, respectively, in this particular analysis.

The bearing capacity factor $N_c$ from the finite element analysis (FEA) is plotted against prototype footing width in Fig. 5 for cases of centrally loaded footings. For cases of square footings ($L/B = 1$) the bearing capacity factor from centrifuge tests and FEA are in good agreement. However for footings with a larger aspect ratio, $N_c$ from FEA shows a similar trend of decreasing with increasing $NB$ as observed in the model tests, but FEA overestimated $N_c$ by approximately 45%, corresponding to a difference in $\phi'$ of approximately $2^\circ$. A possible reason for the discrepancy of $N_c$ between FEA and test observations has been hypothesized to be the progressive nature of the failure surface developed in the sand. The peak
value of $\phi'$ is assumed to be mobilized in the failure zone at the ultimate condition in FEA; however, failure surfaces in real sand beds develop more or less progressively. The peak value of $\phi'$ can not be mobilized simultaneously along the failure surface and this is more marked for sand under footings with larger aspect ratios, since the stress-strain behavior of sand in the plane strain condition shows a clearer peak than that of triaxial specimens under the same confining pressure. The shape factor $s_r$ obtained from FEA is plotted in Fig. 6. The factor $s_r$ from FEA increased with $NB$ as observed in the centrifuge tests.

Figure 23 indicates the relationship between $V/V_{max}$ and $M/BV_{max}$ at the ultimate condition obtained from FEA, together with least-square-fit parabolas. The parabolas given by Eq. (2) also provide a reasonably good fit to the data points from FEA. The constant $\phi'$'s of these parabolas are plotted in Fig. 24 against $NB$, together with those from centrifuge tests. The constant $\phi'$'s from FEA increased with an increase in $NB$, with $\phi$ for $L/B=1$ being higher than those for $L/B=\infty$ (strip footings), which is quite consistent with the test results. It was confirmed that the finite element analysis incorporated with the dependencies of $\phi'$ on $\sigma_m$, $\delta$ and the $b$-value is capable of representing the scale and shape effect of footings on the bearing capacity of sand under both central and eccentric loading conditions.

In order to study how the three influential factors of $\phi'$ contribute to the scale and shape effects of footings, additional analyses were performed. The bearing capacity of the square footing with both $NB=0.9$ m and 3.0 m was calculated for the following four cases; 

(i) All the influential factors of $\phi'$, i.e. effects of $\sigma_m$, $b$-value and anisotropy, were considered in determining the angle $\phi'$. This is identical to the case (e) mentioned above.

(ii) All the influential factors except for the anisotropy were considered, where the angle $\phi'$ corresponding to $\delta=90^\circ$ was used. This is identical to the above mentioned case (d).

(iii) All the influential factors except for the $b$-value were considered, where the angle $\phi'$ corresponding to $b=0$ (triaxial compression) was used.

(iv) All the influential factors except for the stress level dependency were considered, where the mean principal stress $\sigma_m$ was set as 49 kPa in determining $\phi'$.

The constant $\phi'$'s of best-fit parabolas for each case are shown in Fig. 25. It can be seen in this figure that $\phi'$ for cases (i) through (iii) are essentially the same although maximum vertical load capacity $V_{max}$ and thus the value of $\phi'$ differ considerably for different cases. This implies that $\phi$ does not change with the angle $\phi'$, as long as the variation of $\phi'$ with stress level are taken into account. This is quite consistent with the observations of 1G small-scale tests conducted by Nova and Montrasio (1991) that $\phi$ was substantially independent of relative density of sand beds. In contrast, for case (iv), $\phi$ is apparently lower and no scale effect can be observed. The constant $\phi=0.43$ in this case is close to that obtained from the test with $NB=40$ mm ($\phi=0.45$). In case (iv), $\phi'$ did not change with stress level as much as the similar situation of small scale footing tests at 1G where $\phi'$ is essentially independent of
stress level since the stress level in the failure zone is mostly lower than a certain value, approximately 50 kPa for dense Toyoura sand. This clearly suggests that the stress level dependency of $\phi'$ raises $\varphi$, and small scale model tests conducted at 1G significantly underestimate $\varphi$.

**ESTIMATION OF PRACTICAL BEARING CAPACITY**

It is recommended that the bearing capacity of a surface rectangular footing on sand under eccentric loading be calculated as follows. Firstly, the bearing capacity of the strip footing with the same width under central vertical loading condition is evaluated using the angle $\phi'$ from a plane strain compression test taking the stress level effect and anisotropy into account. The use of the mean value of $\phi'$ for $\delta=90^\circ$ and for $\delta=0$, which is typically several per cent smaller than $\phi'$ for $\delta=90^\circ$, may provide a good approximation of the effect of anisotropy (Okamura et al., 1993). The angle $\phi'$ for plane strain compression may be assumed to be 10% larger than triaxial compression. Secondly, the bearing capacity of rectangular footing is estimated by multiplying the shape factor by the bearing capacity of the strip footing, in which the scale effect of the shape factor should be considered. Lastly, the bearing capacity of the rectangular footing under eccentric loading with arbitrary eccentricity ratio can be obtained using the failure locus. Taking the effect of width and shape of footings into account, the constant $\varphi$ may be determined using Fig. 9.

**CONCLUSION**

This paper describes research on the bearing capacity of rectangular footings on sand subjected to vertical eccentric loading conducted at Tokyo Institute of Technology. Two aspects, namely, the effects of footing size and of footing shape on the bearing capacity and deformation characteristics, were mainly focused on. A series of loading tests was conducted in the centrifuge on the rectangular footings having aspect ratios from 1 to 5, at two different centrifugal accelerations to simulate different prototype footing widths. In addition, finite element analyses were performed in which influences of factors affecting the angle of shear resistance $\phi'$ including the stress level, the anisotropy and the coefficient of intermediate principal stress were taken into account.

The following main conclusions are obtained from these results:

1. For footings subjected to central vertical loadings, the bearing capacity factor $N_y$ was observed in the tests to decrease with increasing prototype footing width $NB$, irrespective of footing shape. The prototype width $NB$ also appeared to have a substantial effect on the shape factor $s_y$, which increased with $NB$. This indicates that the shape factor widely used in practice, derived mainly from small scale model tests, may yield conservative bearing capacity predictions for the practical footing width. The same trend of variations in the bearing capacity factor $N_y$ and the shape factor $s_y$ with $NB$ can be obtained from FEA. The bearing capacity factor $N_y$ for square footings from FEA agreed well with that observed from the tests; however, FEA overestimated $N_y$ to some extent for rectangular footings with higher aspect ratios. The discrepancy of $N_y$ between FEA and test observations may be partly due to the progressive nature of the failure surface developed in the sand.

2. In the failure zone under square footings, it was found that the coefficient of intermediate principal stress was very close to zero. This suggests the adequacy in estimating bearing capacities of square or circular footings to make use of $\phi'$ obtained from triaxial tests under appropriate stress level incorporation with bearing capacity solutions for axisymmetric condition.

3. For footings under eccentric loadings, the moment divided by footing width, $M/B$, was plotted against the vertical load $V$ at failure. Maximum moment occurred at about half the maximum vertical load and the parabolas can provide a reasonably good fit to the data points in $M/B-V$ load plane, and thus failure locus. The normalized failure locus with respect to the maximum vertical load capacity was larger for wider footings and for smaller aspect ratios. The constant $\varphi$, which represents the size of normalized failure locus, for footings of $B=40$ mm with $L/B=5$ exhibits the lowest value of $\varphi = 0.325$ in this study. This value of $\varphi$ is quite consistent with small-scale 1G model test results of rectangular footings with the higher aspect ratios reported in the literature. The results of FEA as well as test observations confirmed the existence of the apparent scale effect and the shape effect of footings on $\varphi$. The constant $\varphi$ for larger footings with aspect ratio close to unity is considerably higher than that reported in the literature. The scale effect of footings on $\varphi$ was closely estimated by the analysis taking into account the stress level dependency of $\phi'$ only.

4. In order to consider a macroscopic constitutive law for the entire soil-footing system, incremental displacement vectors at failure were superimposed on the $M/B-V$ load plane. The incremental displacement vectors were generally normal to the parabola, indicating that an associated flow rule was not applicable on the $M/B-V$ plane. This observation suggests the clear need for separate plastic potential to describe foundation behavior.

**REFERENCES**


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