THE AXISYMMETRIC LOADING OF A FLEXIBLE DISC INCLUSION EMBEDDED IN CROSS-ANISOTROPIC LAYER

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ABSTRACT

The paper examines the axi-symmetric problem related to a loaded thin flexible disc inclusion embedded in bonded contact with a cross-anisotropic layer. The solution representations governing the cross-anisotropic layer are given in a convenient matrix form. The associated mixed boundary value problem is reduced to two coupled Fredholm integral equations of the second kind, in which the deflection of the disc is expressed by the stresses applied on the disc in an analytical way. The equations are evaluated numerically. Numerical results presented in this paper illustrate the influence of the anisotropy, the relative thickness of the layer and the relative rigidity of the disc on the load-deflection behavior of the disc for a wide variety of material and geometric conditions.

Key words: cross-anisotropy, deflection, disc inclusion, Fredholm integral equations, Hankel transform (IGC: E12)

INTRODUCTION

Investigations concerning the rigid and flexible disc inclusions contained in an elastic medium is of considerable importance in geotechnical engineering such as soil-foundation interaction and disc-shaped anchors etc. Topics in this area have been of great interest to many researchers, and have been analysed by a variety of methods. Selvadurai (1979b, 1993) and Gladwell (1980) presented a detailed review of various analyses applicable to the present class of problems. In previous investigations into the above study, the surrounding medium has been considered as an infinite isotropic (Selvadurai, 1976, 1979a and 1980a; Yue et al., 1995) or infinite cross-anisotropic (Zureick, 1988) medium, an isotropic half space (Selvadurai, 1990, 1993 and 2001) or cross-anisotropic half space (Selvadurai, 1980b; Rajapakse, 1988), a non-homogeneous isotropic half space (Rajapakse et al., 1991) and an isotropic layer with finite thickness (Rowe et al., 1979; Selvadurai et al., 1994).

In this paper the surrounding medium of the disc will be modelled as a cross-anisotropic layer resting on rigid and rough base. Many soils have been created through a sedimentation process which creates a typical soil fabric in which horizontal stratification planes can be distinguished. This process gives the soil mass a marked degree of cross-anisotropy in that the stress-strain relationships are different in horizontal and vertical directions. In this case, it is very necessary to consider the anisotropic behavior. Furthermore, in practical engineering, soils resting on rigid rock are often encountered, the settlement of the foundation embedded in such soil is mainly con-

trolled by the settlement of the soil layer. Therefore, if the surrounding medium is modelled as a finite cross-anisotropic layer, better estimation of the settlement of the foundation than that modelled as an isotropic or infinite medium can be made.

A variational scheme and associated discretization procedures have been used by Rajapakse (1988 and 1991) and Selvadurai (1979a and 1980b) to study the interaction between flexible plates and surrounding soil. The key step of this method is the selection of a proper deflection function; a power series of radial coordinate is employed to closely model the deflection of the circular plate, and the coefficients of the power series are determined through the principle of minimum potential energy. The accuracy of the solution depends on not only the number of the ring elements used to discretize the circular disc, but also the number of terms considered in the assumed power series displacement representation. The method of the Fredholm integral equations of the second kind has been used to analyze the interaction between a rigid disc (Selvadurai, 1980a, 1990, 1993, 1994, 2001; Yue et al., 1995) or a disc with rigid core (Iguchi et al., 1982) and its surrounding elastic medium. In the present paper, this method will be extended to examine the axi-symmetric problem related to a complete elastic disc inclusion embedded in bonded contact with a cross-anisotropic layer, where the deflection of the disc is directly expressed by the pressure and net stress applied on the disc in an analytical way; no assumption is made for the disc deflection function. The accuracy of the solution depends on only the discretization of the integral equations, so higher accuracy can be obtained compared with the variational

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scheme. Analysis of the interaction between a thin flexible disc and its surrounding cross-anisotropic medium can be fully analysed by making use of numerical methods, such as finite element method (FEM); however, the solution can be much more time consuming and requires a large amount of data preparation. Moreover, the FEM lacks the ability to simulate the infinite boundary in the horizontal extent, so some assumptions and approximations have to be made.

In this paper, the basic equations governing the cross-anisotropic elastic layer are solved by expressing them in the form of matrix and employing Hankel transform technique together with Cayley-Hamilton theorem. The net contact pressure acting on the disc in the Hankel transform domain is represented by a finite Fourier cosine transformation. By employing the Abel transform we reduce the associated mixed boundary value problem to two coupled Fedholm integral equations of the second kind, in which the deflection of the disc is considered in an analytical way. The coupled integral equations are solved in a numerical fashion. A group of numerical results are given to illustrate the influence of the cross-anisotropy, the relative thickness of the layer and the relative rigidity of the disc on the load-deflection behavior of the disc for a wide variety of material and geometric conditions.

GOVERNING EQUATIONS

Consider a cross-anisotropic homogeneous layer with thickness H , and assume that the layer is infinite in the horizontal extent. A flexible disc with radius R is embedded at depth h below the surface of the layer, and an axisymmetric load $q(r)$ is applied on the disc. The layer extends a further distance D below the disc. The layer rests on rigid and rough base.

Static Equilibrium Equations

In the absence of an increase in body forces, static equations of equilibrium with respect to a conventional cylindrical polar coordinate system take the form

$$\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = 0, \quad \frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{rz}}{\partial r} + \frac{\tau_{rz}}{r} = 0 \quad (1)$$

where $\sigma_r(r, z)$, $\sigma_\theta(r, z)$ are the stresses taken as positive in tension and $\tau_{rz}(r, z)$ is the shear stress.

Constitutive Equations

The constitutive equations governing the linear relations between stresses and strains expressed in terms of displacements in a cross-anisotropic solid take the form (Gerrard and Harrison, 1971)

$$\sigma_r = a \frac{\partial u}{\partial r} + b \frac{u}{r} + c \frac{\partial w}{\partial z}, \quad \sigma_\theta = b \frac{\partial u}{\partial r} + a \frac{u}{r} + c \frac{\partial w}{\partial z} \quad (2a)$$

$$\sigma_z = c \frac{\partial u}{\partial r} + c \frac{u}{r} + d \frac{\partial w}{\partial z}, \quad \tau_{rz} = \frac{f}{2} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right) \quad (2b)$$

where $u(r, z)$, $w(r, z)$ are the bulk displacement compo-

nents in the r and z directions respectively, and a, b, c, d, f are defined as follows

$$\begin{aligned} a &= \frac{E_h(1 - \nu_{hv}\nu_{vh})}{(1 + \nu_h)(1 - \nu_h - 2\nu_{hv}\nu_{vh})}, \\ b &= \frac{E_h(\nu_h + \nu_{hv}\nu_{vh})}{(1 + \nu_h)(1 - \nu_h - 2\nu_{hv}\nu_{vh})}, \\ c &= \frac{E_h\nu_{vh}}{(1 - \nu_h - 2\nu_{hv}\nu_{vh})}, \\ d &= \frac{E_v(1 - \nu_h)}{(1 - \nu_h - 2\nu_{hv}\nu_{vh})}, \quad f = F_v \end{aligned} \quad (3)$$

where E_h and E_v are the horizontal and vertical elastic moduli respectively, F_v is the vertical shear modulus, ν_{hv} is the Poisson's ratio for the effect of vertical strain on horizontal strain, ν_{vh} is the Poisson's ratio for the effect of horizontal strain on vertical strain, ν_h is the Poisson's ratio in the horizontal plane and there is the relationship $\nu_{vh}/E_v = \nu_{hv}/E_h$ among the parameters.

By utilizing the Hankel integral transform and inverse Hankel transform techniques, we define vectors which consist of the field variables of displacements and stresses in the layer of infinite lateral extent in the Hankel transform domain and physical domain as follows:

$$\begin{aligned} \tilde{X}(\xi, z) &= \int_0^\infty r J(\xi r) X(r, z) dr, \\ X(r, z) &= \int_0^\infty \xi J(\xi r) \tilde{X}(\xi, z) d\xi \end{aligned} \quad (4)$$

where

$$\begin{aligned} \tilde{X}(\xi, z) &= [\tilde{u}(\xi, z), \tilde{w}(\xi, z), \tilde{\sigma}_z(\xi, z), \tilde{\tau}_{rz}(\xi, z)]^T, \\ X(r, z) &= [u(r, z), w(r, z), \sigma_z(r, z), \tau_{rz}(r, z)]^T, \\ J(\xi r) &= \begin{pmatrix} J_1(\xi r) & 0 & 0 & 0 \\ 0 & J_0(\xi r) & 0 & 0 \\ 0 & 0 & J_0(\xi r) & 0 \\ 0 & 0 & 0 & J_1(\xi r) \end{pmatrix} \end{aligned} \quad (5)$$

where J_m is the Bessel function of order m ($m=0, 1$).

By suitably manipulating Eqs. (1) and (2), we can express these partial differential governing equations in the form of matrix with differential operators as its components (Chen, 2002)

$$\frac{\partial}{\partial z} X(r, z) = A(r) X(r, z) \quad (6a)$$

where the matrix $A(r)$ takes the form

$$A(r) = \begin{pmatrix} 0 & -\frac{\partial}{\partial r} & 0 & \frac{2}{f} \\ -\frac{c}{d} \left(\frac{\partial}{\partial r} + \frac{1}{r} \right) & 0 & \frac{1}{d} & 0 \\ 0 & 0 & 0 & -\left(\frac{\partial}{\partial r} + \frac{1}{r} \right) \\ \frac{c^2 - ad}{d} \left(\frac{\partial}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \right) & 0 & -\frac{c}{d} \frac{\partial}{\partial r} & 0 \end{pmatrix} \quad (6b)$$

By performing Hankel transform to Eq. (6a), and noting that there are the following relationships in Hankel's transform

$$\begin{aligned} \int_0^\infty r \frac{\partial}{\partial r} f(r) J_1(\xi r) dr &= -\xi \int_0^\infty r f(r) J_0(\xi r) dr \\ \int_0^\infty r \left[\frac{\partial}{\partial r} + \frac{1}{r} \right] f(r) J_0(\xi r) dr &= \xi \int_0^\infty r f(r) J_1(\xi r) dr \\ \int_0^\infty r \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \right] f(r) J_1(\xi r) dr \\ &= -\xi^2 \int_0^\infty r f(r) J_1(\xi r) dr \end{aligned} \quad (6c)$$

we get the following ordinary differential matrix equation in the Hankel transform domain

$$\frac{d}{dz} \tilde{X}(\xi, z) = \tilde{A}(\xi) \tilde{X}(\xi, z) \quad (7a)$$

where

$$\tilde{A}(\xi) = \begin{pmatrix} 0 & \xi & 0 & \frac{2}{f} \\ -\frac{c}{d} \xi & 0 & \frac{1}{d} & 0 \\ 0 & 0 & 0 & -\xi \\ \frac{ad-c^2}{d} \xi^2 & 0 & \frac{c}{d} \xi & 0 \end{pmatrix} \quad (7b)$$

The solution of the matrix ordinary differential Eq. (7a) in the Hankel transform domain takes the form

$$\tilde{X}(\xi, z) = T(\xi, z) \tilde{X}(\xi, 0) \quad (8)$$

where $\tilde{X}(\xi, 0)$ is the vector at the surface $z=0$ of the layer, $T(\xi, z) = \exp [z \tilde{A}(\xi)]$ is a matrix of dimensions 4×4 and named the transfer matrix because it can transfer the vector $\tilde{X}(\xi, z)$ from one depth to another depth. Note that in the following we will simplify $T(\xi, z)$, $\tilde{X}(\xi, z)$ into $T(z)$, $\tilde{X}(z)$. According to Cayley-Hamilton theorem (Braun, 1983), we obtain the matrix $T(z)$ as follows

$$T(z) = T_\rho^s ch(\rho \xi z) + T_\phi^c ch(\phi \xi z) + T_\rho^s sh(\rho \xi z) + T_\phi^s sh(\phi \xi z) \quad (9a)$$

ii) For the case of $\beta^2 = 0$, the matrix $T(z)$ is defined by

$$T(z) = \frac{1}{2} (e^{\alpha \xi z} T_1 + e^{-\alpha \xi z} T_2 + \alpha \xi z e^{\alpha \xi z} T_3 + \alpha \xi z e^{-\alpha \xi z} T_4) \quad (9b)$$

where $\rho = \alpha + \beta$, $\phi = \alpha - \beta$, $\alpha^2 = (ad - c^2 - cf + f\sqrt{ad})/2df$, $\beta^2 = (ad - c^2 - cf - f\sqrt{ad})/2df$, T_ρ^c , T_ϕ^c , T_ρ^s , T_ϕ^s and T_1 , T_2 , T_3 , T_4 are matrices defined in Appendix B.

THE MIXED BOUNDARY VALUE PROBLEMS

We consider the problem of a flexible disc inclusion which is embedded in a cross-anisotropic layer and sub-

jected to axisymmetric vertical load $q(r)$. There is no traction condition on the surface of the layer. Here we define two regions S and $\bar{S}$ which correspond to the regions on $z=h$ occupied by the inclusion and its exterior respectively. The boundary and continuity conditions are as follows:

$$\begin{Bmatrix} u(r, z) \\ w(r, z) \end{Bmatrix}^{(1)} = \begin{Bmatrix} u(r, z) \\ w(r, z) \end{Bmatrix}^{(2)} = \begin{Bmatrix} 0 \\ w(r) \end{Bmatrix}, \quad (r, z) \in S \quad (10a)$$

$$X^{(1)}(r, z) = X^{(2)}(r, z) \quad (r, z) \in \bar{S} \quad (10b)$$

$$\begin{Bmatrix} u(r, H) \\ w(r, H) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (10c)$$

$$\begin{Bmatrix} \sigma_z(r, 0) \\ \tau_{rz}(r, 0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (10d)$$

$$\int_0^R r \begin{Bmatrix} \sigma_z^{(2)}(r, z) - \sigma_z^{(1)}(r, z) \\ \tau_{rz}^{(2)}(r, z) - \tau_{rz}^{(1)}(r, z) \end{Bmatrix} dr = \int_0^R \begin{Bmatrix} q(r) \\ 0 \end{Bmatrix} r dr \quad (10e)$$

where $w(r)$ is the deflection of the disc, the superscript (1) denotes the layer region occupying the region $r \in [0, \infty)$, $z \in [0, h]$ and the superscript (2) denotes the layer region occupying the region $r \in [0, \infty)$, $z \in [h, H]$.

FREDHOLM INTEGRAL EQUATIONS GOVERNING THE FLEXIBLE DISC INCLUSION PROBLEM

By utilizing the solution in Eq. (8) and boundary and continuity conditions in Eqs. (10b)~(10d), we get

$$\begin{Bmatrix} \tilde{u}(0) \\ \tilde{w}(0) \end{Bmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{Bmatrix} \tilde{\sigma}_z^{(2)}(h) - \tilde{\sigma}_z^{(1)}(h) \\ \tilde{\tau}_{rz}^{(2)}(h) - \tilde{\tau}_{rz}^{(1)}(h) \end{Bmatrix} \quad (11)$$

where

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = \begin{pmatrix} T_{11}(H) & T_{12}(H) \\ T_{21}(H) & T_{22}(H) \end{pmatrix}^{-1} \begin{pmatrix} T_{13}(D) & T_{14}(D) \\ T_{23}(D) & T_{24}(D) \end{pmatrix}$$

and $T_{ij}(z)$ is the element of matrix $T(z)$ at the i th row and the j th column. From the condition (10a) we have

$$\int_0^\infty \xi J'(\xi r) \begin{pmatrix} T_{11}(h) & T_{12}(h) \\ T_{21}(h) & T_{22}(h) \end{pmatrix} \begin{Bmatrix} \tilde{u}(0) \\ \tilde{w}(0) \end{Bmatrix} d\xi = \begin{Bmatrix} 0 \\ w(r) \end{Bmatrix} \quad (12)$$

$$\text{where } J'(\xi r) = \begin{pmatrix} J_1(\xi r) & 0 \\ 0 & J_0(\xi r) \end{pmatrix}.$$

Introducing the finite Fourier cosine transformation such that

$$\tilde{\sigma}_z^{(2)}(h) - \tilde{\sigma}_z^{(1)}(h) = \int_0^R \phi_\sigma(t) \cos(\xi t) dt \quad (13a)$$

$$\tilde{\tau}_{rz}^{(2)}(h) - \tilde{\tau}_{rz}^{(1)}(h) = \int_0^R \phi_\tau(t) \sin(\xi t) dt \quad (13b)$$

The transformations in Eqs. (13a) and (13b) satisfy the stress continuity conditions in (10b).

Substituting Eqs. (11), (13a) and (13b) into Eq. (12) results in

$$\int_0^R \int_0^\infty \xi \mathbf{J}'(\xi r) \mathbf{K} \begin{Bmatrix} \phi_\sigma(t) \cos(\xi t) \\ \phi_\tau(t) \sin(\xi t) \end{Bmatrix} d\xi dt = \begin{Bmatrix} 0 \\ w(r) \end{Bmatrix} \quad (14)$$

$$\text{where } \mathbf{K} = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} = \begin{pmatrix} T_{11}(h) & T_{12}(h) \\ T_{21}(h) & T_{22}(h) \end{pmatrix} \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix};$$

the elements of the matrix $\mathbf{K}$ are defined in Appendix B.

In Eq. (14) the thin disc deflection $w(r)$ satisfies the following equation

$$\frac{1}{r} \frac{d}{dr} \left\{ r \frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr} \left(r \frac{dw}{dr} \right) \right] \right\} = \frac{q(r) - p(r)}{D_d} \quad (15)$$

where $D_d = E_d h_d^3 / 12(1 - \nu_d^2)$, E_d , ν_d and h_d are the Young's modulus, Poisson's ratio and the thickness of the disc. $p(r)$ is the net contact stress acting on the disc, so we have

$$p(r) = \int_0^\infty \xi (\tilde{\sigma}_z^{(2)}(h) - \tilde{\sigma}_z^{(1)}(h)) J_0(\xi r) d\xi \quad (16a)$$

$$\tilde{p}(\xi) = \tilde{\sigma}_z^{(2)}(h) - \tilde{\sigma}_z^{(1)}(h) \quad (16b)$$

where $\tilde{p}(\xi)$ is the Hankel transform of $p(r)$. For the case of a uniformly loaded disc, $q(r) = q_0$, the solution of Eq. (15) takes the form (Iguchi et al., 1982)

$$w(r) = c_0 + c_1 r^2 + c_2 \ln r + c_3 r^2 \ln r + \frac{q_0 r^4}{64 D_d} - \int_0^\infty \frac{\tilde{p}(\xi) [r^2 \xi^2 + 4(J_0(\xi r) - 1)]}{4 D_d \xi^3} d\xi \quad (17)$$

If there is no concentrated force applied on the disc, we have $c_2 = c_3 = 0$. The disc inclusion we are considering is a thin plate with $h_d \ll R$, the load applied along the disc edge by the surrounding medium is very limited, so we assume the present buried disc is subjected to Kirchhoff boundary conditions which take the form

$$M_r(R) = Q(R) = 0 \quad (18)$$

where $M_r(R)$ and $Q(R)$ denote the bending moment and the shear force per unit length along the disc circumferential edge. Such boundary conditions have been examined to be applicable for an uniformly loaded disc (Rajapakse, 1988). We can obtain the unknown constant c_1 in Eq. (17) by using boundary condition (18); substituting c_1 into Eq. (17) results in

$$w(r) = c_0 - G(r) - \int_0^\infty \tilde{p}(\xi) \tilde{H}(\xi, r) d\xi \quad (19)$$

where

$$G(r) = \frac{(3 + \nu_d) q_0 R^2 r^2}{32 D_d (1 + \nu_d)} - \frac{q_0 r^4}{64 D_d},$$

$$\tilde{H}(\xi, r) = \frac{J_0(\xi r) - 1}{D_d \xi^3} + \frac{J_0(\xi R) r^2}{2 D_d (1 + \nu_d) \xi} - \frac{(1 - \nu_d) J_1(\xi R) r^2}{2 D_d (1 + \nu_d) R \xi^2}.$$

It should be mentioned that for many load functions $q(r)$, the deflection function $w(r)$ of the disc can be solved in an analytical way as that of in Eq. (19).

From Eq. (19) it can be seen that for a rigid disc, $D_d \rightarrow \infty$, $w(r) = c_0$; for a flexible disc, when $r = 0$, $w(r) = c_0$. So the constant c_0 is the translation for the rigid disc or the central settlement for the flexible disc.

By applying Abel transform (Gladwell, 1980) to Eq. (14), we get

$$\int_0^R \int_0^\infty \xi \mathbf{F}(\xi u) \mathbf{K} \begin{Bmatrix} \phi_\sigma(t) \cos(\xi t) \\ \phi_\tau(t) \sin(\xi t) \end{Bmatrix} d\xi dt = \begin{Bmatrix} 0 \\ \bar{w}(u) \end{Bmatrix} \quad (20)$$

where $\mathbf{F}(\xi u) = \begin{pmatrix} \sin(\xi u) & 0 \\ 0 & \cos(\xi u) \end{pmatrix}$ and $\bar{w}(u)$ is the

Abel transformation of $w(r)$ which takes the form

$$\bar{w}(u) = c_0 - \bar{G}(u) - \int_0^\infty \tilde{p}(\xi) \tilde{H}(\xi, u) d\xi \quad (21)$$

where

$$\bar{G}(u) = \frac{(3 + \nu_d) q_0 R^2 u^2}{16 D_d (1 + \nu_d)} - \frac{q_0 u^4}{24 D_d},$$

$$\tilde{H}(\xi, u) = \frac{\cos(\xi u) - 1}{D_d \xi^3} + \frac{J_0(\xi R) u^2}{D_d (1 + \nu_d) \xi} - \frac{(1 - \nu_d) J_1(\xi R) u^2}{D_d (1 + \nu_d) R \xi^2}.$$

From the Eqs. (13a) and (13b) we have

$$\begin{Bmatrix} \phi_\tau(u) \\ \phi_\sigma(u) \end{Bmatrix} - \frac{2}{\pi} \int_0^R \int_0^\infty \begin{Bmatrix} \phi_\tau(t) \sin(\xi t) \sin(\xi u) \\ \phi_\sigma(t) \cos(\xi t) \cos(\xi u) \end{Bmatrix} d\xi dt = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (22)$$

Multiplying the two Eqs. in (20) by $2/(\pi \omega_1)$ and $2/(\pi \omega_2)$ respectively, and adding them into the two Eqs. in (22) yields

$$\begin{Bmatrix} \phi_\tau(u) \\ \phi_\sigma(u) \end{Bmatrix} + \frac{2}{\pi} \int_0^R \int_0^\infty \begin{pmatrix} \sin(\xi u) & 0 \\ 0 & \cos(\xi u) \end{pmatrix} \times \begin{pmatrix} \frac{\xi K_{11}}{\omega_1} & \frac{\xi K_{12}}{\omega_1} - 1 \\ \frac{\xi K_{21}}{\omega_2} - 1 & \frac{\xi K_{22}}{\omega_2} \end{pmatrix} \begin{Bmatrix} \phi_\sigma(t) \cos(\xi t) \\ \phi_\tau(t) \sin(\xi t) \end{Bmatrix} \times d\xi dt = \begin{Bmatrix} 0 \\ 2\bar{w}(u) \end{Bmatrix} \quad (23)$$

where for the case of $\beta^2 \neq 0$:

$$\rho \phi \omega_1 = \omega_2 = \frac{1}{(\rho^2 - \phi^2) df} \left(\frac{2a - f\phi^2}{2\phi} - \frac{2a - f\rho^2}{2\rho} \right);$$

for the case of $\beta^2=0$:

$$\alpha^2 \omega_1 = \omega_2 = \frac{(1+\chi)\alpha}{2f}, \quad \chi = (\sqrt{ad} - c)/2\sqrt{ad}.$$

From the Eq. (13a) and the static equilibrium condition of the disc, we have

$$q_0 = 2 \int_0^R \phi_\sigma(t) dt / R^2 \quad (24)$$

Substituting Eqs. (13a), (16b), (21) and (24) into (23) results in

$$\begin{Bmatrix} \phi_\tau(u) \\ \phi_\sigma(u) \end{Bmatrix} + \frac{2}{\pi} \int_0^R \int_0^\infty \mathbf{K}' \begin{Bmatrix} \phi_\sigma(t) \\ \phi_\tau(t) \end{Bmatrix} d\xi dt = \begin{Bmatrix} 0 \\ c'_0 \end{Bmatrix} \quad (25)$$

where $c'_0 = 2(c_0 - \bar{G}(u))/(\pi\omega_2)$ and

$$\mathbf{K}'(u, t) = \begin{pmatrix} \frac{\xi K_{11}}{\omega_1} \sin(\xi u) \cos(\xi t) & \left(\frac{\xi K_{12}}{\omega_1} - 1 \right) \sin(\xi u) \sin(\xi t) \\ \left[\left(\frac{\xi K_{21}}{\omega_2} - 1 \right) \cos(\xi u) + \frac{\bar{H}(\xi, u)}{\omega_2} \right] \cos(\xi t) & \frac{\xi K_{22}}{\omega_2} \cos(\xi u) \sin(\xi t) \end{pmatrix} \quad (26)$$

Equation (25) is the Fredholm integral equation of the second kind governing the axi-symmetric problem related to the loading of a flexible disc inclusion embedded in a cross-anisotropic layer. This equation can only be solved in a numerical way. The elements of the matrix $\mathbf{K}'$ have the behavior of $\lim_{\xi \rightarrow \infty} K'_{ij} = 0$, where K'_{ij} is the element of matrix $\mathbf{K}'$ at the i th row and j th column. The elements K'_{11} , K'_{12} , K'_{22} and the term $((\xi K_{21}/\omega_2) - 1) \cos(\xi u) \cos(\xi t)$ of K'_{21} decay with ξ exponentially. As to the second term of K'_{21} , we find that when ξ tends to be zero $\bar{H}(\xi, u) \cos(\xi t)$ is limited, and the function $\bar{H}(\xi, u) \cos(\xi t)$ is a wave function that decays very fast.

A numerical method for evaluating $\int_0^\infty \bar{H}(\xi, u) \cos(\xi t) d\xi$ is presented as follows

$$\begin{aligned} \int_0^\infty \bar{H}(\xi, u) \cos(\xi t) d\xi &= \int_0^{\xi_1} \bar{H}(\xi, u) \cos(\xi t) d\xi \\ &+ \int_{\xi_1}^{\xi_2} \bar{H}(\xi, u) \cos(\xi t) d\xi + \cdots \\ &+ \int_{\xi_n}^{\xi_{n+1}} \bar{H}(\xi, u) \cos(\xi t) d\xi + \cdots \end{aligned} \quad (27)$$

where $\xi_1, \xi_2, \dots, \xi_n, \dots$ are the zero points of the function $\bar{H}(\xi, u) \cos(\xi t)$; each term on the right side of Eq. (27) can be integrated based on Simpson's quadrature technique. Because of the rapid decay of $\bar{H}(\xi, u) \cos(\xi t)$ with variable ξ , enough accuracy for the semi-infinite integral can be obtained by taking the initial four to five terms in Eq. (27) for practical purpose.

Therefore the four infinite integrals $\int_0^\infty K'_{ij} d\xi$ in the Eq. (25) are convergent.

For a rough understanding of the exponential decay of the elements in matrix $\mathbf{K}'$ with the increase of ξ , as an example, we select the term $(\xi K_{21}/\omega_2) - 1$ when $\beta^2 \neq 0$, to study its behavior in more detail. Dividing the numerator

and denominator of ξK_{21} by $e^{-(\rho+\phi)\xi H}$ respectively, we obtain

$$\xi K_{21} = \frac{N_0 + \sum_{i=1}^{19} N_i e^{-\xi p_i}}{D_0 + \sum_{j=1}^4 D_j e^{-\xi q_j}} = \frac{N_0 + N_k e^{-\xi p_k} + o(e^{-\xi p_k})}{D_0 + D_l e^{-\xi q_l} + o(e^{-\xi q_l})} \quad (28)$$

where N_0, N_i, p_i with $i=1, \dots, 19$, D_0, D_j, q_j with $j=1, 2, 3, 4$ are constants related to ρ, ϕ, C_p, C_ϕ , and $p_k = \min(p_1, p_2, \dots, p_{19})$, $q_l = \min(q_1, q_2, q_3, q_4)$. Because $N_0 D_0 = \omega_2$, we have (N_0, D_0, p_i, q_j) are presented in Appendix C for reference

$$\lim_{\xi \rightarrow \infty} \left(\frac{\xi K_{21}}{\omega_2} - 1 \right) = \frac{1}{\omega_2} \lim_{\xi \rightarrow \infty} \frac{N_k e^{-\xi p_k} + o(e^{-\xi p_k})}{D_0 + D_l e^{-\xi q_l} + o(e^{-\xi q_l})} \quad (29)$$

Based on Eqs. (27) and (29), the upper range of the infinite integrals $\int_0^\infty K'_{ij} d\xi$ can then be determined based on the zero points of $\bar{H}(\xi, u) \cos(\xi t)$ and the fast decay of the value of $e^{-\xi p_k}$.

By taking the limits $D_d \rightarrow \infty$, $D = H - h \rightarrow \infty$ and taking $\alpha = 1$ in Eqs. (B4) and (B5), we can obtain the simplified elements of the matrix $\mathbf{K}'$ which are the same as those by Selvadurai (1993 and 2001) for an isotropic half space with rigid disc inclusion. By taking $h=0$ and the limits $D_d \rightarrow \infty$, $H \rightarrow \infty$, our problem reduces to that of a cross-anisotropic half space with a rigid disc resting on its surface under an axial load, for this case Selvadurai (1980) gave the exact closed-form solution for the rigid disc translation as follows:

$$w = \frac{\pi R q_0 (k_1 - k_2)}{2f(1+k_1)(1+k_2)(\gamma_1 - \gamma_2)} \quad (30)$$

where

$$\begin{aligned} k_1 &= C_\phi / C_p, \quad k_2 = C_p / C_\phi, \\ \gamma_1 &= \frac{2\rho}{f} \left(\frac{dk_1 - c/\rho}{(1+k_1)} \right), \quad \gamma_2 = \frac{2\phi}{f} \left(\frac{dk_2 - c/\phi}{(1+k_2)} \right), \end{aligned}$$

and C_ρ , C_ϕ are defined in Appendix A. Moreover, when taking the limits $D_d \rightarrow \infty$, $D = H - h \rightarrow \infty$ and $h \rightarrow \infty$ at the same time, our problem reduces to that of a cross-anisotropic medium of infinite extent with a rigid disc inclusion under an axial load, for this case Zureick (1988) gave the exact closed-form solution for the rigid disc translation w as follows:

$$w = \frac{\pi R q_0}{4f(k_2 - k_1)(\phi k_2 - \rho k_1)} \quad (31)$$

NUMERICAL SOLUTIONS FOR THE INTEGRAL EQUATIONS

To solve the Fredholm integration equation of the second kind, numerical treatment techniques have been presented (Atkinson, 1997; Selvadurai, 1993 and 2001), and these techniques mainly involve two computational steps. The first step is the numerical integration of the infinite integrals of $\int_0^\infty K' d\xi$ inside Eq. (25). These integrations have been discussed in the previous section. The second step involves the numerical discretization of the two Fredholm integration Eq. (25). The integration interval $[0, R]$ is divided into N segments such that the discrete values of u and t are defined by

$$u_i = \frac{(i-0.5)R}{N}, \quad t_j = \frac{(j-0.5)R}{N} \quad \text{with } i, j = 1, 2, \dots, N \quad (32)$$

and the integral Eq. (25) and the static equilibrium Eq. (24) can be discretized into a matrix equation as follows

$$\begin{pmatrix} \Delta_{11} & \Delta_{12} & \cdots & \Delta_{1j} & \cdots & \Delta_{1N} & -E' \\ \Delta_{21} & \Delta_{22} & \cdots & \Delta_{2j} & \cdots & \Delta_{2N} & -E' \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ \Delta_{j1} & \Delta_{j2} & \cdots & \Delta_{jj} & \cdots & \Delta_{jN} & -E' \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ \Delta_{N1} & \Delta_{N2} & \cdots & \Delta_{Nj} & \cdots & \Delta_{NN} & -E' \\ E & E & \cdots & E & \cdots & E & 0 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_j \\ \vdots \\ \phi_N \\ c'_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ q_0 R N / 2 \end{pmatrix} \quad (33)$$

where

$$\Delta_{ij} = \frac{2}{\pi} \int_0^\infty \begin{pmatrix} K'_{11}(u_i, t_j) & K'_{12}(u_i, t_j) \\ K'_{21}(u_i, t_j) & K'_{22}(u_i, t_j) \end{pmatrix} d\xi \cdot \frac{R}{N}$$

obtained from step 1, $\Delta'_{kk} = \Delta_{kk} + I$, $k = 1, \dots, N$,

$$\phi_j = \begin{pmatrix} \phi_\sigma(t_j) \\ \phi_\tau(t_j) \end{pmatrix}, \quad E = \{1, 0\},$$

$$E' = \{0, 1\}^T, \quad \mathbf{0} = \{0, 0\}^T, \quad I = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The above numerical solution has been well investigated by many researchers, and it has been shown that with the increase in the segment number N , it is possible to obtain more accurate and readily convergent solutions

Fig. 1. Comparison of calculations the central settlement of the flexible disc between the present study and existing solutions

for the integral equations. Solutions of Eq. (33) generate the numerical values of c_0 , $\phi_\sigma(t_j)$ and $\phi_\tau(t_j)$. Then the disc deflection $w(r_0)$ at any radial coordinate r_0 can be obtained from Eqs. (13a), (16b) and (19).

NUMERICAL RESULTS

For the cross-anisotropic material, there are 6 mechanical parameters which are $(E_v, E_h, F_v, \nu_h, \nu_{vh}, \nu_{hv})$. By introducing the moduli ratios $n = E_h/E_v$, $m = F_h/F_v$ (where $F_h = E_h/(1 + \nu_h)$) (Conte, 1998), and considering the relationship $E_v \nu_{hv} = E_h \nu_{vh}$, we reduce the six parameters to four independent nondimensional parameters (n, m, ν_h, ν_{vh}). In the following, the nondimensional central settlement $w^*(0) = E_v w(0)/(a q_0)$ and nondimensional deflection $w_d^* = E_v [w(0) - w(R)]/(a q_0)$ will be investigated. Besides, the relative rigidity parameter $K_r = E_d h_d^3/(E_v R^3)$ is used to study its influence on the load-deflection behavior of the flexible disc inclusion. A review of the stiffness parameters for several cross-anisotropic clay soils, as assessed from both in situ and laboratory tests, can be found in a recent paper by Jamiolkowski et al. (Jamiolkowski et al., 1994). In particular, it was reported that values of the ratio n range between 0.55–4 and m range between 0.59–3. These values are here used for the numerical study.

Comparison with Existing Solutions

1) Comparison with the results by variational scheme (Rajapakse, 1988): numerical results by the present method are compared with those obtained by a variational scheme. The medium is modelled as an isotropic half space, the flexible disc is uniformly loaded, $h/R = 0.5, 2.0$ are selected, K_r ranges from 0.5 to 100 and Poisson's ratio of the disc is taken to be $\nu_d = 0.3$. The numerical results are shown in Fig. 1. It can be observed that the agreement between the results is excellent. Such agreement is obtained with $N = 15$.

2) Comparison with the exact solutions: numerical results by the present study are compared with those obtained from exact solution in Eq. (31) (Zureick, 1988), where the medium is modelled as a cross-anisotropic in-

Table 1. Comparison of $w^* = E_v w / (aq_0)$ of a rigid disc in cross-anisotropic medium between the present study and the exact solutions

n	0.55			1.0			4.0		
m	0.59	1.0	3.0	0.59	1.0	3.0	0.59	1.0	3.0
$h/R = D \quad R = 50$	0.6945	0.8675	1.4465	0.5306	0.6531	1.0700	0.2466	0.3003	0.4837
$h/R = D \quad R = 100$	0.6975	0.8710	1.4505	0.5314	0.6541	1.0707	0.2472	0.3011	0.4848
Exact solutions	0.6980	0.8716	1.4513	0.5317	0.6545	1.0712	0.2474	0.3013	0.4851

Fig. 2. Influence of layer cross-anisotropy, D/R and h/R on the translation of rigid disc inclusion

finite medium by taking $h/R = D/R = 50$ and $h/R = D/R = 100$ respectively, and considering the rigid disc. The numerical results are shown in Table 1, the close agreement of the results is again observed between the present study and the exact solutions. All the numerical solutions by the present study are obtained with $N = 15$.

By varying the segment number N in a large number of calculation cases under a wide variety of material and

geometric conditions, we found that when $N \geq 15$, stable and convergent solutions of displacement can be obtained.

Parametric Study

In this section, parametric studies are carried out to investigate the load-deflection behavior of the disc inclusion in the cross-anisotropic layer, and the segment num-

Fig. 3. Influence of layer cross-anisotropy, K_r , and D/R on the deflection of flexible disc ($v_d = 0.3$)

bers $N \geq 15$ are adopted in all the following numerical calculations. The nondimensional geometric variables (h/R , D/R) are adopted to examine the influences of both the embedment depth of the disc and the depth of the layer below the disc on the load-deflection behavior of disc inclusion.

The numerical results for parametric study are plotted in Figs. 2–4. In Fig. 2, only a rigid disc is considered in order to focus on the investigation into the influences of the embedment ratio h/R , especially the layer cross-anisotropy, and the ratio of the layer thickness below the disc D/R on the translation of the disc. The range for the D/R is selected to be from 2 to 1000. The influence of the embedment ratio h/R has been studied in other papers (Selvadurai, 1993 and 2001), so here only $h/R = 0.5$, 20 are selected to consider its influence. In Fig. 2(a) $n = 0.55$, 1, 4 with $m = 1$, $v_{vh} = v_h = 0.25$ are selected; in Fig. 2(b), $m = 0.59$, 1, 3 with $n = 1$, $v_{vh} = v_h = 0.25$ are selected; in Fig. 2(c), $v_h = 0.0$, 0.25, 0.5 with $n = m = 1$, $v_{vh} = 0.25$ are selected; and in Fig. 2(d), $v_{vh} = 0.0$, 0.25, 0.5 with $n = m = 1$, $v_h = 0.25$ are selected respectively.

From Fig. 2, it can be seen that an increase in the translation of the rigid disc is noted with D/R for $D/R < 100$, whereas for $D/R > 100$ the solution converges toward the value corresponding to a cross-anisotropic half space. The influences of the degree of anisotropy $n = E_h/E_v$ and $m = F_h/F_v$ on the translation of the rigid disc are significant, while the effects of the two Poisson's ratios v_h and v_{vh} are of secondary importance on the translation of

the rigid disc. Also, the embedment ratio h/R has a marked influence upon the translation of the disc, which has been observed before.

In Fig. 3, the deflection of the flexible disc is investigated with different relative rigidity ranging from 0.5 to 100. Two values of $D/R = 5$, 50 are selected to investigate its influence on the deflection. Similarly, in Fig. 3(a) with $m = 1$, $v_{vh} = v_h = 0.25$ are selected; in Fig. 3(b), $m = 0.59$, 1, 3 with $n = 1$, $v_{vh} = v_h = 0.25$ are selected; in Fig. 3(c), $v_h = 0.0$, 0.25, 0.5, with $n = m = 1$, $v_{vh} = 0.25$ are selected; and in Fig. 3(d), $v_{vh} = 0.0$, 0.25, 0.5 with $n = m = 1$, $v_h = 0.25$ are selected respectively. The influence of the embedment ratio h/R has been studied by Rajapakse (1988), so here only one embedment ratio $h/R = 0.5$ is selected in order to focus on the investigation into the influences of the layer cross-anisotropy, the variation of the layer thickness below the disc D/R and the relative rigidity of the disc on the nondimensional deflection w_d^* of the disc.

From Fig. 3, it can be observed that the results between $D/R = 5$ and $D/R = 50$ are almost the same as each other, therefore the influence of the ratio of the layer thickness below the disc on the deflection of the disc can be ignored. The influences of the degrees of anisotropy $n = E_h/E_v$ and $m = F_h/F_v$ on the deflection of the flexible disc are appreciable, while the effects of the two Poisson's ratios v_h and v_{vh} are very small. Moreover for $K_r > 50$, the deflection of the disc is convergent towards zero which behaves like a rigid disc.

In Fig. 4, the results are obtained for the case of a disc

Fig. 4. Influence of layer K_r and D/R on the ratio of the central disc settlement between flexible disc and rigid disc ($\nu_d = 0.3$)

embedded in an isotropic layer with $\nu_{vh} = \nu_h = 0.25$. The influences of the relative rigidity of the disc and the geometric ratio D/R on $w_f(0)/w_r(0)$, which represents the ratio of the central disc settlement between the flexible disc and the corresponding rigid disc, are investigated.

Figure 4 shows that with the decrease of the relative rigidity of the disc, the central settlement of the flexible disc increases, and the ratio $w_f(0)/w_r(0)$ is always greater than 1.0. With an increase of D/R , the central settlements of both the flexible and rigid disc inclusions should increase. It can be observed that their ratio decreases to a convergent value; therefore, it can be concluded that the influence of D/R on the central settlement of the rigid disc inclusion is greater than that of the flexible disc inclusion.

CONCLUSIONS

The technique of Fredholm integral equations of the second kind is extended to solve the axisymmetric problem related to a loaded thin flexible disc inclusion embedded in bonded contact with a cross-anisotropic layer with high efficiency. The group of numerical results are presented to fully examine the roles of the anisotropic parameters of the layer, relative layer thickness and the relative disc rigidity on the load-deflection behavior of the disc inclusion. Based on these numerical results, it can be concluded that:

a) The agreements among the numerical results by the present method, by the variational scheme and the exact solutions are excellent.

b) The ratio of the layer thickness below the disc D/R has appreciable influence upon the central settlement of the disc while its influence on the deflection of the disc can even be ignored, besides its influence on the central settlement of the rigid disc inclusion is greater than that of the flexible disc inclusion. For $D/R > 100$, the response of the disc convergences toward the value corresponding to a cross-anisotropic half space.

c) The influences of the degree of anisotropy $n = E_h/E_v$ and $m = F_h/F_v$ on the translation of the rigid disc and the deflection of the flexible disc are significant, while

the effects of the two Poisson's ratios ν_h and ν_{vh} are of secondary importance on the translation of the rigid disc and are very small on the deflection of the flexible disc.

d) With the decrease of the relative rigidity of the disc, the central settlement of the disc increases. For $K_r > 50$, the disc behaves like a rigid disc.

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APPENDIX A

(1) For the case of $\beta^2 \neq 0$: $T(z) = T_\rho^c ch(\rho \xi z) + T_\phi^c ch(\phi \xi z) + T_\rho^s sh(\rho \xi z) + T_\phi^s sh(\phi \xi z)$

$$T_\rho^c = \frac{1}{C_\rho - C_\phi} \begin{pmatrix} C_\rho & 0 & \frac{\eta}{\xi} & 0 \\ 0 & -C_\phi & 0 & -\frac{\eta}{\xi} \\ -\frac{\xi C_\rho C_\phi}{\eta} & 0 & -C_\phi & 0 \\ 0 & \frac{\xi C_\rho C_\phi}{\eta} & 0 & C_\rho \end{pmatrix} \quad T_\rho^s = \frac{1}{(C_\rho - C_\phi)} \begin{pmatrix} 0 & \frac{C_\rho}{\rho} & 0 & \frac{\eta C_\rho}{\xi \rho C_\phi} \\ -\rho C_\phi & 0 & -\frac{\eta \rho C_\phi}{\xi C_\rho} & 0 \\ 0 & -\frac{\xi C_\rho C_\phi}{\eta \rho} & 0 & -\frac{C_\rho}{\rho} \\ \frac{\xi \rho C_\rho C_\phi}{\eta} & 0 & \rho C_\phi & 0 \end{pmatrix}$$

$$T_\phi^c = I - T_\rho^c, \quad \rho T_\rho^s + \phi T_\phi^s = \tilde{A}(\xi)/\xi \quad (A1)$$

where $C_\rho = c + d\rho^2$, $C_\phi = c + d\phi^2$, $\eta = (2c + f)/f$ and I is unit matrix of dimensions 4×4 .

(2) For the case of $\beta^2 = 0$: $T(z) = \frac{1}{2} (e^{\alpha \xi z} T_1 + e^{-\alpha \xi z} T_2 + \alpha \xi z e^{\alpha \xi z} T_3 + \alpha \xi z e^{-\alpha \xi z} T_4)$

$$T_1 = \begin{pmatrix} 1 & \frac{\chi}{\alpha} & 0 & \frac{1+\chi}{\alpha \xi f} \\ \alpha \chi & 1 & \frac{\alpha(1+\chi)}{\xi f} & 0 \\ 0 & \frac{(1-\chi)\xi f}{\alpha} & 1 & -\frac{\chi}{\alpha} \\ \alpha(1-\chi)\xi f & 0 & -\alpha \chi & 1 \end{pmatrix}, \quad T_1 + T_2 = 2I$$

$$T_3 = \begin{pmatrix} 1 & \frac{1}{\alpha} & \frac{1}{\xi f} & \frac{1}{\alpha \xi f} \\ -\alpha & -1 & -\frac{\alpha}{\xi f} & -\frac{1}{\xi f} \\ -\xi f & -\frac{\xi f}{\alpha} & -1 & -\frac{1}{\alpha} \\ \alpha f \xi & \xi f & \alpha & 1 \end{pmatrix} (1-\chi), \quad T_4 = \begin{pmatrix} -1 & \frac{1}{\alpha} & -\frac{1}{\xi f} & \frac{1}{\alpha \xi f} \\ -\alpha & 1 & -\frac{\alpha}{\xi f} & \frac{1}{\xi f} \\ \xi f & -\frac{\xi f}{\alpha} & 1 & -\frac{1}{\alpha} \\ \alpha f \xi & -\xi f & \alpha & -1 \end{pmatrix} (1-\chi) \quad (A2)$$

APPENDIX B

(1) For the case of $\beta^2 \neq 0$

$$K_{11} \cdot g_1(\xi) = \frac{1}{2} [C_\rho^2 - C_\phi^2] \{ ch(\rho \xi H) ch(\phi \xi (H-2h)) + ch(\phi \xi H) ch[\rho \xi (H-2h)] - 2ch(\phi \xi h) ch(\rho \xi h) \}$$

$$+ \frac{1}{2} \left[C_\rho^2 \frac{\phi}{\rho} - C_\phi^2 \frac{\rho}{\phi} \right] \{ sh(\rho \xi H) sh[\phi \xi (H-2h)] + sh(\phi \xi H) sh[\rho \xi (H-2h)] + 2sh(\phi \xi h) sh(\rho \xi h) \}$$

$$- \left[\frac{C_\rho^3}{C_\phi} \frac{\phi}{\rho} - \frac{C_\phi^3}{C_\rho} \frac{\rho}{\phi} \right] sh[\phi \xi (H-h)] sh[\rho \xi (H-h)] = K_{22} \cdot g_1(\xi) \quad (B1)$$

$$\rho \phi K_{12} \cdot g_1(\xi) = \frac{1}{2} \left[\frac{C_\rho^3}{C_\phi} \frac{\phi}{\rho} + \frac{C_\phi^3}{C_\rho} \frac{\rho}{\phi} \right] [\rho sh(\rho \xi H) ch(\phi \xi H) - \phi ch(\rho \xi H) sh(\phi \xi H)]$$

$$\begin{aligned}
& -C_\rho C_\phi [\rho ch(\rho \xi H) sh(\phi \xi H) - \phi sh(\rho \xi H) ch(\phi \xi H) - 2\rho ch(\rho \xi h) sh(\phi \xi h) + 2\phi sh(\rho \xi h) ch(\phi \xi h)] \\
& + \frac{1}{2} \left[\frac{C_\rho}{C_\phi} \phi^2 - \frac{C_\phi}{C_\rho} \rho^2 \right] \left\{ \frac{C_\rho^2}{\rho} sh(\phi \xi H) ch[\rho \xi (H-2h)] + \frac{C_\phi^2}{\phi} sh(\rho \xi H) ch[\phi \xi (H-2h)] \right\} \\
& + \frac{1}{2} [C_\rho^2 - C_\phi^2] \left\{ \frac{C_\rho}{C_\phi} \phi ch(\phi \xi H) sh[\rho \xi (H-2h)] + \frac{C_\phi}{C_\rho} \rho ch(\rho \xi H) sh[\phi \xi (H-2h)] \right\} \\
& + 2\rho C_\phi^2 ch[\rho \xi (H-h)] sh[\phi \xi (H-h)] - 2\phi C_\rho^2 sh[\rho \xi (H-h)] ch[\phi \xi (H-h)]
\end{aligned} \quad (B2)$$

$$\begin{aligned}
K_{21} \cdot g_1(\xi) = & \frac{1}{2} \left[\frac{C_\rho^3}{C_\phi} \frac{\phi}{\rho} + \frac{C_\phi^3}{C_\rho} \frac{\rho}{\phi} \right] [\rho ch(\rho \xi H) sh(\phi \xi H) - \phi sh(\rho \xi H) ch(\phi \xi H)] \\
& - C_\rho C_\phi [\rho sh(\rho \xi H) ch(\phi \xi H) - \phi ch(\rho \xi H) sh(\phi \xi H) - 2\rho sh(\rho \xi h) ch(\phi \xi h) + 2\phi sh(\rho \xi h) sh(\phi \xi h)] \\
& + \frac{1}{2} \left[\frac{C_\rho}{C_\phi} \phi^2 - \frac{C_\phi}{C_\rho} \rho^2 \right] \left\{ \frac{C_\rho^2}{\rho} sh(\rho \xi H) ch[\phi \xi (H-2h)] + \frac{C_\phi^2}{\phi} sh(\phi \xi H) ch[\rho \xi (H-2h)] \right\} \\
& + \frac{1}{2} [C_\rho^2 - C_\phi^2] \left\{ \frac{C_\rho}{C_\phi} \phi ch(\rho \xi H) sh[\phi \xi (H-2h)] + \frac{C_\phi}{C_\rho} \rho ch(\phi \xi H) sh[\rho \xi (H-2h)] \right\} \\
& + 2\rho C_\phi^2 sh[\rho \xi (H-h)] ch[\phi \xi (H-h)] - 2\phi C_\rho^2 ch[\rho \xi (H-h)] sh[\phi \xi (H-h)]
\end{aligned} \quad (B3)$$

$$\text{where } g_1(\xi) = \frac{\xi(C_\rho - C_\phi)}{\eta} \left[-2C_\rho C_\phi + (C_\rho^2 + C_\phi^2) ch(\rho \xi H) ch(\phi \xi H) - \left(\frac{\phi}{\rho} C_\rho^2 + \frac{\rho}{\phi} C_\phi^2 \right) sh(\rho \xi H) sh(\phi \xi H) \right]$$

(2) For the case of $\beta^2 = 0$

$$\xi fK_{11} \cdot g_2(\xi) = \xi fK_{22} \cdot g_2(\xi) = (1-\chi)^2 [\alpha \xi (H-h)]^2 [ch^2(\alpha \xi h) - \chi sh^2(\alpha \xi h)] + (1+\chi) \{ (1-\chi)^2 (\alpha \xi h)^2 - \chi \} sh^2[\alpha \xi (H-h)] \quad (B4)$$

$$K_{12} = \frac{\sum_{i=1}^4 Y_i(\xi) + \sum_{i=5}^7 Y_i(\xi)}{\alpha f g_2(\xi) \xi}, \quad K_{21} = \frac{\sum_{i=1}^4 Y_i(\xi) - \sum_{i=5}^7 Y_i(\xi)}{f g_2(\xi) \xi / \alpha} \quad (B5)$$

$$\text{where } g_2(\xi) = ch^2(\alpha \xi H) - \chi^2 sh^2(\alpha \xi H) + (1-\chi)^2 (\alpha \xi H)^2, \quad Y_1(\xi) = \frac{1}{2} (1+\chi) (1-\chi^2) ch(\alpha \xi H) sh(\alpha \xi H);$$

$$Y_2(\xi) = \frac{1}{2} (1+\chi) \{ ch(\alpha \xi H) sh[\alpha \xi (H-2h)] + \chi^2 sh(\alpha \xi H) ch[\alpha \xi (H-2h)] \};$$

$$Y_3(\xi) = (1-\chi)^2 (1+\chi) (\alpha \xi h)^2 sh[\alpha \xi (H-h)] ch[\alpha \xi (H-h)];$$

$$Y_4(\xi) = -(1-\chi)^3 [\alpha \xi (H-h)]^2 sh(\alpha \xi h) ch(\alpha \xi h); \quad Y_5(\xi) = (1-\chi) [\alpha \xi (H-h)] [ch^2(\alpha \xi h) - \chi^2 sh^2(\alpha \xi h)];$$

$$Y_6(\xi) = -(1+\chi)^2 (1-\chi) (\alpha \xi h) sh^2[\alpha \xi (H-h)]; \quad Y_7(\xi) = (1-\chi)^3 \alpha^3 \xi^3 Hh(H-h)$$

APPENDIX C

$$\begin{aligned}
N_0 = & \left[\frac{1}{2} \left(\frac{C_\rho^3}{C_\phi} \frac{\phi}{\rho} + \frac{C_\phi^3}{C_\rho} \frac{\rho}{\phi} \right) - C_\rho C_\phi \right] (\rho - \phi), \quad D_0 = \frac{C_\rho - C_\phi}{\eta} \left(C_\rho^2 + C_\phi^2 - \frac{\phi}{\rho} C_\rho^2 - \frac{\rho}{\phi} C_\phi^2 \right) \\
p_1 = & 2\rho H, \quad p_2 = 2\phi H, \quad p_3 = 2(\rho + \phi)H, \quad p_4 = (\rho + \phi)D, \quad p_5 = \rho D + \phi(H+h), \quad p_6 = \phi D + \rho(H+h), \\
p_7 = & (\rho + \phi)(H+h), \quad p_8 = (\rho + \phi)h, \quad p_9 = \rho h + \phi(H+D), \quad p_{10} = \phi h + \rho(H+D), \quad p_{11} = (\rho + \phi)(H+D), \\
p_{12} = & 2\rho h, \quad p_{13} = 2\phi h, \quad p_{14} = 2\rho D, \quad p_{15} = 2\phi D, \quad p_{16} = 2(\rho H + \phi h), \quad p_{17} = 2(\phi H + \rho h), \quad p_{18} = 2(\rho H + \phi D), \\
p_{19} = & 2(\phi H + \rho D) \\
q_i = & p_i, \quad i = 1, 2, 3, \quad q_4 = (\rho + \phi)H
\end{aligned}$$