DISPLACEMENTS AND STRESSES DUE TO VERTICAL SUBSURFACE LOADING FOR A CROSS-ANISOTROPIC HALF-SPACE

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ABSTRACT

This paper presents the analytical solutions for estimating the displacements and stresses in a cross-anisotropic half-space due to various loading types of an axially loaded pile. The loading types include an embedded point load for an end-bearing pile, uniform skin friction, linear variation of skin friction, and non-linear variation of skin friction for a friction pile, respectively. The planes of cross-anisotropy are assumed to be parallel to its horizontal surface. The derived solutions have not yet appeared in the literature, and can be obtained by integrating the point load solutions in a Cartesian co-ordinate system for the cross-anisotropic half-space. A part of the solutions are identical with the Mindlin’s and Geddes’s solutions when the medium is isotropic. The presentation of these proposed solutions is concise and systematized; also, they indicate that the displacements and stresses in a cross-anisotropic half-space are affected by the type and degree of material anisotropy, and the loading types. Furthermore, two illustrative examples are given to investigate the effect of the type and degree of soil anisotropy, and four different loading types on the vertical surface displacement and vertical normal stress, respectively. The results show that the displacement and stress accounted for soil anisotropy are quite different from those estimated by isotropic solutions.

Key words: analytical solutions, an embedded point load, axially loaded pile, cross-anisotropic half-space, displacements and stresses, linear variation of skin friction, non-linear variation of skin friction, uniform skin friction (IGC: E13/G13/H1)

INTRODUCTION

In general, theoretical analyses of soil behavior usually assumed that the properties of soil are isotropic. However, there are many natural soils which have deposited through a geologic process of sedimentation over a period of time, such as flocculated clays, varved silts or sands. Under such circumstances, it is reasonable to suppose that the elastic properties differ in horizontal and vertical directions. Hence, in order to obtain more desirable results, it is imperative to consider anisotropic deformability. In this work, an elastic problem for a cross-anisotropic half-space is relevant.

It is known that piles transmit axial loads to the ground by the mechanism of end-bearing or skin friction. For an isotropic soil mass, the displacements and stresses are obtained in most cases by using Mindlin’s solutions (1936) subjected to a vertical point load acting in the interior of the half-space. Grillo (1948) presented the influence charts (Poisson’s ratio = 0.5) for vertical normal stress due to a pile point load, and uniform skin friction along the pile, respectively. However, his accompanying equations contained a number of inconsistencies and errors that have been pointed out by Geddes (1966a). An analytical method based on the Mindlin’s solutions (1936) for estimating the four stresses at any point, over the length of the pile in a cylindrical co-ordinate system due to a point load, uniform skin friction, and linear variation of skin friction was proposed by Geddes (1966a). He also prepared the most commonly used tables of stress coefficients for Poisson’s ratio equal to 0.1, 0.3, and 0.5, respectively (Geddes, 1966a, 1966b). Geddes (1969) later presented the Boussinesq’s solutions (1885) for a point load in a modified form related to the same subsurface loadings as mentioned above. However, these solutions are generally less accurate than those by using the Mindlin’s solutions (Bowels, 1996). Poulos and Davis (1968, 1980) also utilized the Mindlin’s solutions (1936) to predict the settlement behavior of a single axially loaded incompressible pile by considering it as a number of uniformly loaded cylindrical elements. Kaniraj and Ranganatham (1979) investigated the settlement of piles and pile groups in normally consolidated clay by using the equivalent foundation and uniform skin friction approaches. Apart from the analytical solutions, the numerical techniques by boundary element methods, (three-dimensional) finite element methods, simplified finite element methods, (three-dimensional) finite difference methods, also can deal with the related problems (Poulos, 2001).

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So far, to the best of this authors’ knowledge, no analytical solutions of displacements and stresses for a cross-anisotropic half-space resulting from an end-bearing and skin frictions have been proposed. The exact solutions for displacements and stresses in a cross-anisotropic medium due to an end-bearing (a point load), and various skin frictions (uniform, linearly varying, and non-linearly varying) of an axially loaded pile are derived in this paper, respectively. In deriving the proposed solutions, the medium is assumed as a homogeneous, linear elastic, and cross-anisotropic continuum. This cross-anisotropic half-space has planes of isotropy parallel to the boundary plane. In addition, the effect of pile diameter is not considered herein. Hence, in the present work, four cases of vertical loading are included as follows:

Case A. A point load $F$ (force), at depth $L$ (Fig. 1).

Case B. A total load of $Q$ (force per unit length), applied along the vertical axis $z$ in uniform distribution ($Q(z) = F/L$) from the surface to depth $L$ (Fig. 2).

Case C. A total load of $Q$ (force per unit length), applied along the vertical axis $z$ in increments varying linearly with depth ($Q(z) = (2F/L^2)z$), from zero at the surface to a maximum at depth $L$ (Fig. 3).

Case D. A total load of $Q$ (force per unit length), applied along the vertical axis $z$ in increments varying non-linearly with depth ($Q(z) = (3F/L^3)z^2$), from zero at the surface to a maximum at depth $L$ (Fig. 4).

The closed-form solutions proposed in this work can be directly obtained by integrating the point load solutions in a Cartesian co-ordinate system for a cross-anisotropic half-space (Wang and Liao, 1999). These solutions indicate that both of the displacements and stresses in a cross-anisotropic half-space are affected by the type and degree of material anisotropy, and different loading types. Two illustrative examples are given at the end of this paper to investigate the effect of the type and degree of soil anisotropy, and loading types on the vertical surface displacement and vertical normal stress in the isotropic/cross-anisotropic soils due to a point load, uniform skin friction, linear variation of skin friction, and non-linear variation of skin friction, respectively.

**CASE A: DISPLACEMENTS AND STRESSES DUE TO A POINT LOAD**

In this paper, the solutions for displacements and stresses in a cross-anisotropic half-space due to uniform
skin friction, linear variation of skin friction, and non-linear variation of skin friction of an axially loaded pile are directly integrated from the point load solutions (Wang and Liao, 1999). The planes of cross-anisotropy are assumed to be parallel to its horizontal surface. The analytical solutions for displacements and stresses in the Cartesian co-ordinate system subjected to a vertical point load $F$ acting at $z = h$ (from the surface) in the interior of a cross-anisotropic half-space are newly expressed as follows:

$$u_x^p = \frac{F}{4\pi} \left[ k(p_{311} - p_{312}) + m_1(T_1p_{311} - T_3p_{312}) \right] - m_2(T_2p_{311} - T_4p_{312}) \quad [L] \quad (1)$$

$$u_y^p = u_x^p \cdot \frac{y}{x} \quad [L] \quad (2)$$

$$u_z^p = \frac{F}{4\pi} \left[ m_1(T_1p_{321} - T_1p_{32a} - T_2p_{32b}) - m_2(T_2p_{322} + T_3p_{32a} - T_4p_{32b}) \right] \quad [L] \quad (3)$$

$$\sigma_{xx}^p = \frac{F}{4\pi} \left[ (A_{11} - u_1u_2)A_{13} - 2A_{66})(k(p_{311} + T_1m_{p_{31a}} - T_2m_{p_{31b}} - T_3m_{p_{31c}}) - T_3m_{p_{32a}} + T_4m_{p_{32b}}) \right] + 2A_{66}(k(p_{311} - p_{312}) + (p_{321} - p_{32a}) + m_1(T_1p_{321} - T_3p_{32a}) - T_4(p_{311} - p_{312})) \quad [FL^-2] \quad (4)$$

$$\sigma_{yy}^p = \frac{F}{4\pi} \left[ (A_{11} - u_1u_2)A_{13} - 2A_{66})(k(p_{311} + T_1m_{p_{31a}} - T_2m_{p_{31b}} - T_3m_{p_{31c}}) - T_3m_{p_{32a}} + T_4m_{p_{32b}}) \right] + 2A_{66}(k(p_{311} - p_{312}) + (p_{321} - p_{32a}) + m_1(T_1p_{321} - T_3p_{32a}) - T_4(p_{311} - p_{312})) \quad [FL^-2] \quad (5)$$

$$\sigma_{zz}^p = \frac{F}{4\pi} \left[ (A_{13} - u_1u_2)A_{33})(k(p_{311} + T_1m_{p_{31a}} - T_2m_{p_{31b}} - T_3m_{p_{31c}}) - T_3m_{p_{32a}} + T_4m_{p_{32b}}) \right] + 2A_{66}(k(p_{311} - p_{312}) + (p_{321} - p_{32a}) + m_1(T_1p_{321} - T_3p_{32a}) - T_4(p_{311} - p_{312})) \quad [FL^-2] \quad (6)$$

$$\tau_{xy}^p = \frac{F}{2\pi} \left[ k(p_{311} - p_{312}) + m_1(T_1p_{31a} + T_3p_{31c}) \right] - m_2(T_2p_{31a} - T_4p_{31c}) \quad [FL^-2] \quad (7)$$

$$\tau_{yz}^p = \frac{F}{4\pi} \left[ (u_1 + m_1)(k(p_{321} + T_1m_{p_{32a}} - T_2m_{p_{32b}} - T_4m_{p_{32c}}) - (u_2 + m_2)(k(p_{321} + T_1m_{p_{32a}} - T_2m_{p_{32b}}) - T_4m_{p_{32c}})) \right] \quad [FL^-2] \quad (8)$$

$$\tau_{xz}^p = \tau_{yz}^p \cdot \frac{x}{y} \quad [FL^-2] \quad (9)$$

where:

- The generalized Hooke's law for the cross-anisotropic medium in a Cartesian co-ordinate system can express the constitutive equations used in this paper as:

$$\sigma_{xx} = A_{11}e_{xx} + (A_{11} - 2A_{66})e_{yy} + A_{13}e_{zz} \quad [FL^-2] \quad (10)$$

$$\sigma_{yy} = (A_{11} - 2A_{66})e_{xx} + A_{11}e_{yy} + A_{13}e_{zz} \quad [FL^-2] \quad (11)$$

$$\sigma_{zz} = A_{13}(e_{xx} + e_{yy}) + A_{33}e_{zz} \quad [FL^-2] \quad (12)$$

$$\tau_{xy} = A_{66}y_{xy} \quad [FL^-2] \quad (13)$$

$$\tau_{yz} = A_{44}y_{yz} \quad [FL^-2] \quad (14)$$

$$\tau_{xz} = A_{44}y_{xz} \quad [FL^-2] \quad (15)$$

$A_{ij}$ ($i, j = 1 - 6$) are the elastic moduli or elasticity constants of the medium. For a cross-anisotropic material, the five engineering elastic constants, $E$, $E'$, $v$, $v'$, and $G'$ are defined as (Lee and Rowe, 1989):

1. $E$ is the Young's modulus in the horizontal direction.
2. $E'$ is the Young's modulus in the vertical direction.
3. $v$ is the Poisson's ratio for the effect of horizontal stress on complementary horizontal strain.
4. $v'$ is the Poisson's ratio for the effect of vertical stress on horizontal strain.
5. $G'$ is the shear modulus in the vertical plane.

Hence, $A_{ij}$ can be expressed in terms of these elastic constants as:

$$A_{11} = \frac{E(1 - v')}{(1 + v)(1 - v' - 2E/E'} \cdot \frac{E}{1 - v'}, \quad A_{13} = \frac{E}{1 - v'}, \quad A_{33} = \frac{E'(1 - v)}{1 - v'}, \quad A_{44} = G', \quad \frac{E}{2} (1 + v) \quad [FL^-2] \quad (16)$$

• Thermodynamic constraints require that the strain energy of an elastic material should always be positive, hence, the theoretical bounding values of the relevant elastic parameters are described as (Pickering, 1970; Sutcu, 1992):

$$E, E', G, G' > 0 \quad [FL^-2] \quad (17)$$

$$-1 < v, v' < \sqrt{\frac{E'}{E}} \quad [FL^-2] \quad (18)$$

• $u_1$ and $u_2$ are the roots of the following characteristic equation:

$$u^4 - su^3 + q = 0 \quad [1] \quad (19)$$

where:

$$s = \frac{A_{11}A_{33} - A_{13}(A_{13} + 2A_{66})}{A_{33}A_{44}}, \quad q = \frac{A_{11}}{A_{33}}$$

There are three categories of the characteristic roots, $u_1$ and $u_2$ as follows:

- case 1. $u_{1,2} = \pm \sqrt{\frac{1}{4} [s \pm \sqrt{(s^2 - 4q)}]}$ are two real distinct roots when $s^2 - 4q > 0$;
- case 2. $u_{1,2} = \pm s/2$, $-\pm s/2$ are double equal real roots when $s^2 - 4q = 0$;
- case 3. $u_1 = \pm \sqrt{s^2 + 4q - i\pm \sqrt{24q}} = y - i\delta$, $u_2 = y + i\delta$ are two complex conjugate roots (where $y$ cannot be equal to zero) when $s^2 - 4q < 0$. 


\[ \begin{align*}
\mathbf{m}_i &= (A_{13} + A_{44})u_0 = A_{11} - A_{44}u_i^2 \quad (i = 1, 2), \\
k &= \frac{(A_{11} + A_{44})}{A_{33}A_{44}(u_i^2 - u_0^2)}, \\
T_1 &= k \frac{u_1 + u_2}{m_1 - u_1}, \quad T_2 = k \frac{2u_2 + m_2}{m_1 - u_1 - u_2}(u_1 + m_1), \\
T_3 &= k \frac{2u_2 + (u_1 + m_1)}{m_1 - u_1 - u_2}, \quad T_4 = k \frac{u_1 + u_2}{m_2 - u_2 - u_1}, \\
p_{111} &= \frac{x}{R(R_i + z_i)}, \quad p_{211} = \frac{1}{R_i}, \\
p_{121} &= \frac{z_i}{R_i}, \quad p_{122} = \frac{1}{R_i(R_i + z_i)} - \frac{x^2(2R_i + z_i)}{R_i(R_i + z_i)^2}, \\
p_{221} &= \frac{y}{R_i(R_i + z_i)^2}, \quad p_{222} = \frac{y}{R_i^2}, \\
&\quad (i = 1, 2, a, b, c, d).
\end{align*} \]

The range of subscript \(i\) \((i = 1, 2, 3, a, b, c, d, e)\) utilized in Wang and Liao (1999) is different from that utilized in this paper \((i = 1, 2, a, b, c, d)\). The differences between them are mainly because the former solutions treat the three-dimensional subsurface point loads (one vertical and two horizontal directions); however, the latter ones only deal with a vertical subsurface load. Therefore, Wang and Liao’s solutions (1999) can be simplified as the form of Eqs. (1)–(9) for subjecting a vertical point load, and further defining \(p_{111} - p_{222}\) in Eqs. (1)–(3), and \(p_{111} - p_{222}\) in Eqs. (4)–(9) as the elementary functions for the displacements and stresses, respectively.

\[ \begin{align*}
R_i &= \sqrt{x^2 + y^2 + z_i^2} \quad (i = 1, 2, a, b, c, d). \end{align*} \]

Where \(z_i\) has three different forms, namely, (1) \(z_1 = u_i(z - h)\), (2) \(z_2 = u_1(z + h)\), (3) \(z_3 = u_1h\), (4) \(z_4 = u_1h + u_2h\), (5) \(z_5 = u_1h + u_2h - u_3z\).

If the vertical point load \(F\) is applied at depth \(L\) as depicted in Fig. 1, then, the displacements and stresses for an end-bearing pile in the cross-anisotropic half-space can be obtained by substituting \(h\) by \(L\) in the above-mentioned \(z_1\) \((i = 1, 2, a, b, c, d)\).

The presented formulations for displacements and stresses are identical with Mindlin’s (1936), and Geddes’s solutions in a Cartesian co-ordinate system (only for the stresses) (1966a) when the medium is isotropic.

**CASE B: DISPLACEMENTS AND STRESSES DUE TO A UNIFORM SKIN FRICTION**

The solutions for displacements and stresses in a cross-anisotropic half-space due to uniform, linearly varying, and non-linearly varying skin frictions can be directly integrated from the elementary functions of the point load solutions \(p_{111} - p_{222}\) in Eqs. (1)–(3), and \(p_{111} - p_{222}\) in Eqs. (4)–(9). The total load \(Q\) (force per unit length) distributed uniformly with depth from the surface to depth \(L\), as demonstrated in Fig. 2 is considered. Taking an infinitesimal element \(dh\) along the \(z\)-axis, the load can be divided into a finite number of elementary forces as follows:

\[ \int_{0}^{L} F \, dh \]

Table 1. The relationships of \(z\) \((i = 1, 2, a, b, c, d)\) and \(u_{n}z\) \((m = 1, 2)\)

<table>
<thead>
<tr>
<th>(z) ((i = 1, 2, a, b, c, d))</th>
<th>(u_{n}z) ((m = 1, 2))</th>
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<tbody>
<tr>
<td>(z_1 = u_1(z - h))</td>
<td>(u_1 = -u_1 = -u_1)</td>
</tr>
<tr>
<td>(z_2 = u_1(z + h))</td>
<td>(u_1 = -u_1 = -u_1)</td>
</tr>
<tr>
<td>(z_3 = u_1h)</td>
<td>(u_1 = u_1 = u_1)</td>
</tr>
<tr>
<td>(z_4 = u_1h + u_2h)</td>
<td>(u_0 = u_1 = u_0)</td>
</tr>
<tr>
<td>(z_5 = u_1h + u_2h - u_3z)</td>
<td>(u_0 = u_1 = u_0)</td>
</tr>
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</table>

\[ dQ = \left(\frac{F}{L}\right) dh. \]

Since the variable \(h\) associated with \(z\) has three different forms as described above, the complete expressions of integrations for all forms would be very lengthy. However, if we further set \(z\) \((i = 1, 2, a, b, c, d)\) to \(u_{n}z = u_{n}h = h(m = 1, 2)\), another word, it is more convenient to deal with the general form \((u_{n}z = u_{n}h)\) instead of three ones \((z)\). The relationships of \(z\) \((i = 1, 2, a, b, c, d)\) and \(u_{n}z = u_{n}h(m = 1, 2)\) are expressed in Table 1. Hence, \(h\) is integrated in the \(u_{n}z = u_{n}h\) of the elementary functions \(p_{111} - p_{222}\), and \(p_{111} - p_{222}\) between the limits 0 and \(L\) as:

\[ \begin{align*}
[U]^T &= \int_{0}^{L} [U]^T dQ \quad \text{[L]} \quad (21) \\
[\sigma]^T &= \int_{0}^{L} [\sigma]^T dQ \quad \text{[FL^{-2}]} \quad (22)
\end{align*} \]

where \([U] = [u_0, u_1, u_0]^T\), \([\sigma] = [\sigma_{x\tau}, \sigma_{y\tau}, \tau_{x\tau}, \tau_{y\tau}, \tau_{x\tau}]^T\) (superscript \(T\) denotes that the transpose matrix), and the superscripts \(r\) and \(P\) express the displacement and stress components are result from a uniform skin friction, and a point load, respectively. Upon integration, the explicit solutions for displacements and stresses in a half-space can be regrouped as the forms of Eqs. (1)–(9). This means that the analytical solutions for this case are the same as Eqs. (1)–(9) except that the displacement elementary functions \(p_{111} - p_{222}\), and stress elementary functions \(p_{111} - p_{222}\) are replaced by the displacement integral functions \(r_{111} - r_{222}\) for \(u_i^1, u_i^2, u_i^3, u_i^4\) and stress integral functions \(r_{111} - r_{222}\) for \(\sigma_{x\tau}, \sigma_{y\tau}, \tau_{x\tau}, \tau_{y\tau}, \tau_{x\tau}\) respectively. Similarly, the solutions for a linear, and a non-linear variation of skin frictions given below also can be expressed as the forms of Eqs. (1)–(9), except for the integral functions. Hence, only the displacement and stress integral functions will be presented in the following cases. For the case of uniform skin friction, the displacement and stress integral functions are given as follows.

**Integral Functions for a Uniform Skin Friction**

\[ \begin{align*}
r_{111} &= \frac{x}{u_0L(x^2 + y^2)} (u_0L + f_2 - f_1) \quad \text{[L^{-1}]} \quad (23) \\
r_{112} &= \frac{f_2}{u_0L} \quad \text{[L^{-1}]} \quad (24)
\end{align*} \]
\[ r_{sl} = \frac{1}{u_s L} \left( \frac{1}{f_1} - \frac{1}{f_2} \right) \]  \[ r_{sh} = \frac{x^2 - y^2}{(x^2 + y^2)^2} \left( 1 + \frac{1}{u_s L} \frac{(u_m z^2 - f_1)}{f_2} \right) \]  \[ - \frac{1}{u_s L} \frac{x^2 y^2}{f_1 f_2} \]  \[ r_{sh} = \frac{xy}{(x^2 + y^2)^2} \left( 2 + \frac{1}{u_s L} \left( x^2 + y^2 + 2u_m z^2 \right) \right) \]  \[ - \frac{x^2 + y^2 + 2f_1^2}{f_2} \]  \[ r_{sh} = - \frac{y}{u_s L} \frac{u_m z - f_1}{f_2} \]  \[ (i = 1, 2, a, b, c, d; m, n = 1, 2). \]

If the half-space is isotropic, the closed-form solutions for stresses are the same as Geddes's solutions in a Cartesian co-ordinate system (1966a) if the half-space is isotropic.

**CASE C: DISPLACEMENTS AND STRESSES DUE TO A LINEAR VARIATION OF SKIN FRICTION**

The linearly varying skin friction assumed for a total load of \( Q \) (force per unit length) is shown in Fig. 3. The load applied over a depth \( dh \) is given by:

\[ dQ = 2F \left( \frac{h}{L} \right)^2 dh. \]  \[ \text{[F]} \]  \[ (29) \]

Replacing Eq. (29) into the elementary functions \( p_{as} - p_{ah} \) (Eqs. (1)–(3)) and \( p_{as} - p_{ah} \) (Eqs. (4)–(9)), and integrating \( h \) from 0 to \( L \) as follows:

\[ [U]^n = \int_0^L [U]^p dQ \]  \[ \text{[L]} \]  \[ (30) \]

\[ [\sigma]^n = \int_0^L [\sigma]^p dQ \]  \[ \text{[FL}^{-2}] \]  \[ (31) \]

where the superscript \( l \) denotes the displacement and stress components that result from a linear variation of skin friction. The displacement integral functions \( l_{ai} = l_{ai} \) for \( u_{ai}, u_{ai}^*, u_{ai}^* \) and stress integral functions \( l_{ai} - l_{ai} \) for \( \sigma_{ai}, \sigma_{ai}^*, \tau_{ai}, \tau_{ai}^* \) are derived and listed in the following.

**Integral Functions for a Linear Variation of Skin Friction**

\[ l_{ai} = \frac{x}{(x^2 + y^2)^2} \left( 1 - \frac{1}{u_s L} \frac{(u_m z^2 - (u_m z - u_s L) f_1)}{f_2} \right) \]  \[ \text{[L}^{-1}] \]  \[ (32) \]

\[ l_{ai} = - \frac{2}{(u_s L)^2} \left( f_2 - f_1 + u_m z f_3 \right) \]  \[ \text{[L}^{-1}] \]  \[ (33) \]

\[ l_{ai} = \frac{2}{u_s L} \left( f_2 - f_1 \right) \]  \[ \text{[L}^{-2}] \]  \[ (34) \]

\[ l_{ai} = - \frac{x^2 - y^2}{(x^2 + y^2)^2} \left( 1 - \frac{1}{u_s L} \frac{(u_m z^2 - (f_2 - f_1) + f_3)}{f_2} \right) \]  \[ \text{[L}^{-2}] \]  \[ (35) \]

\[ l_{ai} = \frac{2x y}{(x^2 + y^2)^2} \left( 1 - \frac{1}{u_s L} \frac{u_m z f_3}{f_2} \right) \]  \[ \text{[L}^{-2}] \]  \[ (36) \]

\[ l_{ai} = \frac{2y}{(u_s L)^2} \left( f_2 - f_1 + u_m z u_s L \right) \]  \[ \text{[L}^{-2}] \]  \[ (37) \]

Also, the proposed solutions for stresses are equivalent to Geddes’s solutions in a Cartesian co-ordinate system (1966a) if the half-space is isotropic.

**CASE D: DISPLACEMENTS AND STRESSES DUE TO A NON-LINEAR VARIATION OF SKIN FRICTION**

A total load of \( Q \) (force per unit length), applied along the vertical axis \( z \) in increments varying non-linearly with depth, from zero at the surface to a maximum at depth \( L \) as seen in Fig. 4 is investigated. Similarly, by the same way as the presentations of Cases B and C, the increment load \( dQ \) over a depth \( dh \) can be expressed as:

\[ dQ = 3F \left( \frac{h^2}{L^3} \right) dh. \]  \[ \text{[F]} \]  \[ (38) \]

Substituting the foregoing equation into the elementary functions \( p_{as} - p_{ah} \) (Eqs. (1)–(3)) and \( p_{as} - p_{ah} \) (Eqs. (4)–(9)), and also integrating \( h \) from 0 to \( L \) gives the following expression:

\[ [U]^n = \int_0^L [U]^p dQ \]  \[ \text{[L]} \]  \[ (39) \]

\[ [\sigma]^n = \int_0^L [\sigma]^p dQ \]  \[ \text{[FL}^{-2}] \]  \[ (40) \]

where the superscript \( n \) expresses the displacement and stress components due to a non-linear variation of skin friction. The displacement integral functions \( n_{ai} - n_{ai} \) for \( u_{ai}, u_{ai}^*, u_{ai}^* \) and stress integral functions \( n_{ai} - n_{ai} \) for \( \sigma_{ai}, \sigma_{ai}^*, \tau_{ai}, \tau_{ai}^* \) are presented as.

**Integral Functions for a Non-linear Variation of Skin Friction**

\[ n_{ai} = \frac{x}{x^2 + y^2} \left( 1 - \frac{2(x^2 + y^2) - (u_m z)}{u_m z} \right) f_2 \]  \[ (39) \]

\[ + \frac{2(x^2 + y^2) - (u_m z - u_s L)^2}{(u_m z - u_s L) f_2} \]  \[ \text{[L}^{-2}] \]  \[ (40) \]

\[ - \frac{3x u_m z f_3}{(u_s L)^3} \]  \[ \text{[L}^{-1}] \]  \[ (41) \]

\[ n_{ai} = \frac{3}{2(u_s L)^3} \left( 3u_m z f_2 - (3u_m z - u_s L) f_3 \right) \]  \[ \text{[L}^{-2}] \]  \[ (42) \]
Fig. 5. Flow chart for computing the displacements and stresses for a cross-anisotropic half-space subjected to presented loading types

\[
\begin{align*}
  n_{h1} &= -\frac{3}{(u_L L)^3} \left[ 2f_2 - 2f_2^2 + u_L L (3u_m z + f_1) + 2u_m z f_4 \right] \quad [L^{-1}] \quad (42) \\
  n_{h2} &= -\frac{x^2 - y^2}{(x^2 + y^2)^2} \frac{1}{(u_L L)^3} \left[ (x^2 + y^2) \left( 2(2x^2 + y^2) f_2 - (x^2 + 2y^2) f_3 \right) \\
  &+ (x^2 - y^2) \left( u_m z \right)^2 (f_2 - f_3) + u_L L (u_m z - u_L z) f_3 \right] \\
  &+ \frac{3x^2 (f_2 + 2u_m z u_L L) - 3u_m z f_4}{(u_L L)^3 (x^2 + y^2)^3} \quad [L^{-2}] \quad (43) \\
  n_{h3} &= \frac{xy}{(x^2 + y^2)^2} \left[ 2 + 2 \left( \frac{f_3}{u_L L} \right)^3 \right. \\
  &- (x^2 + y^2) (f_2^2 + 2f_2 + u_m z^2) + 2f_1 (u_m z^2 + u_L^2 L^3) \left[ \frac{u_L L}{f_1} \right] \quad [L^{-2}] \quad (44) \\
  n_{h4} &= -\frac{3y}{(u_L L)^3} \left[ \frac{1}{x^2 + y^2} \left[ u_m z f_2 \right. \right. \\
  &- (x^2 + y^2) (u_m z - u_L z) + u_m z^2 f_1 \left. \right] \left. \right. \\
  &- f_4 \right] \quad [L^{-2}] \quad (45)
\end{align*}
\]

A flow chart that illustrates the derived solutions for computing the displacements and stresses due to the loading conditions of Cases A, B, C, and D in a cross-anisotropic half-space is presented in Fig. 5.

**ILLUSTRATIVE EXAMPLE**

A parametric study is conducted to verify the solutions derived and investigate the effect of the type and degree of material anisotropy, and loading types on the vertical surface displacement and vertical normal stress. For typical ranges of cross-anisotropic parameters, Gazetas (1982) summarized several experimental data regarding deformational cross-anisotropy of clays and sands. He concluded that the ratio \( E/E' \) for clays ranges from 0.6 to 4, and is as low as 0.2 for sands. However, for the heavily over-consolidated London clay, the range of the ratio \( E/E' \) is 1.35–2.37 (the average value is 1.84), and that of the ratio \( G'/E' \) is 0.23–0.44 (the average value is 0.38) (Ward et al., 1965; Gibson, 1974; Lee and Row, 1989; Tarn and Lu, 1991). Besides, a hypothetical ratio \( v/v' \) varying between 0.75 to 1.5 is chosen to take the possible range of Poisson’s ratios into account in this study. Hence, the degree of anisotropy of London clay including the ratios \( E/E' \), \( G'/E' \), and \( v/v' \) is accounted in order to investigate its effect on the displacement and stress. Several types of isotropic and cross-anisotropic soils are considered as the constituted foundation materials. The elastic properties of the materials are listed in Table 2. The values adopted in Table 2 of \( E \) and \( v \) are 50 MPa and 0.3, respectively.

<table>
<thead>
<tr>
<th>Soil type</th>
<th>( E/E' )</th>
<th>( G'/E' )</th>
<th>( v/v' )</th>
<th>Root type</th>
</tr>
</thead>
<tbody>
<tr>
<td>Soil 1. Isotropy</td>
<td>1.0</td>
<td>0.385</td>
<td>1.0</td>
<td>Equal</td>
</tr>
<tr>
<td>Soil 2. Cross-anisotropy</td>
<td>1.35</td>
<td>0.385</td>
<td>1.0</td>
<td>Distinct</td>
</tr>
<tr>
<td>Soil 3. Cross-anisotropy</td>
<td>2.37</td>
<td>0.385</td>
<td>1.0</td>
<td>Distinct</td>
</tr>
<tr>
<td>Soil 4. Cross-anisotropy</td>
<td>1.0</td>
<td>0.23</td>
<td>1.0</td>
<td>Distinct</td>
</tr>
<tr>
<td>Soil 5. Cross-anisotropy</td>
<td>1.0</td>
<td>0.44</td>
<td>1.0</td>
<td>Complex</td>
</tr>
<tr>
<td>Soil 6. Cross-anisotropy</td>
<td>1.0</td>
<td>0.385</td>
<td>0.75</td>
<td>Complex</td>
</tr>
<tr>
<td>Soil 7. Cross-anisotropy</td>
<td>1.0</td>
<td>0.385</td>
<td>1.5</td>
<td>Distinct</td>
</tr>
</tbody>
</table>

(Where \( E = 50 \) MPa, \( v = 0.3 \) are adopted)
point in the half-space can be computed. In this study, the vertical displacement at the surface \((z = 0)\), and vertical normal stress in the soil at the depth \(z\) below the pile tip \((z/L > 1)\) are presented. According to the results reported in Figs. 6–7, and Figs. 8–9, the effect of the type and degree of soil anisotropy, and loading types on the vertical surface displacement, and vertical normal stress is investigated below.

Firstly, Figs. 6(a)–6(d) present the normalized vertical displacement \(U_z/F\) for a point load case (Fig. 6(a)), \(U_z/F\) for a uniform skin friction case (Fig. 6(b)), \(U_z/F\) for a linear variation of skin friction case (Fig. 6(c)), and \(U_{z_0}/F\) for a non-linear variation of skin friction case (Fig. 6(d)), at the surface of the media \((y = z = 0)\) vs. the non-dimensional ratio \(x/L\), respectively. With knowledge of the type and magnitude of the loading, and soil types (Soils 1–7, as listed in Table 2), the vertical surface displacement can be estimated from these figures. Figures 6(a)–6(d) indicate that the vertical surface displacement [1] decreases with the increase of \(x/L\) from 0 to 2, [2] increases with the increase of \(E/E'\) from 1.0 to 2.37 (Soils 1, 2 and 3), [3] increases with the decrease of \(G'/E'\) from 0.385 to 0.23 (Soils 1 and 4), [4] increases with the increase of \(v/v'\) from 1.0 to 1.5 (Soils 1 and 7), [5] decreases with the increase of \(G'/E'\) from 0.385 to 0.44 (Soils 1 and 5), [6] decreases with the decrease of \(v/
Fig. 7. Effect of loading types on normalized vertical surface displacement for a) Soil 1, b) Soil 2, c) Soil 3, d) Soil 4, e) Soil 5, f) Soil 6 and g) Soil 7

\( \nu' \) from 1.0 to 0.75 (Soils 1 and 6), for most loading cases (except for the uniform skin friction case). Especially, the increase of the ratio of \( E/E' \), and decrease of the ratio of \( G'/E' \) do have a great influence on vertical displacement. It reflects the fact that the displacement increases with the increase of deformability in the direction parallel to the point or skin friction loads. Also, from Fig. 6(d), it is noted that with the increase of \( x/L \), the displacement for the non-linear variation of skin friction case might be swollen in Soil 6 (\( E/E' = 1.0, G'/E' = 0.385, \nu' = 0.75 \)). It means that the swelling phenomenon may appear at the surface of this medium. Figures 7(a)–7(g) clarify the effect as already mentioned of loading on the same displacement for Soils 1–7, respectively. The trend of Figs. 7(a), 7(b), 7(c), 7(d), and 7(g) is similar, and the order of induced displacement is as follows: the uniform skin friction case > the linear variation of skin friction case > the non-linear variation of skin friction case > the point load case. However, the trend of the induced displacement by the non-linear variation of skin friction case in Figs. 7(e) and 7(f) (for Soils 5 and 6) differs a little from that in Figs. 7(a)–7(d), and 7(g). According to Figs. 6 and 7, the magnitude of vertical surface displacement strongly depends on the type and degree of soil anisotropy, and the considered loading types.

Secondly, the non-dimensional vertical normal stress for Soils 1–7 along line of action (\( x = y = 0 \)) of a point load case (\( \sigma_{zz}^* \cdot L^2 / F \)), a uniform skin friction case (\( \sigma_{zz}^* \cdot L^2 / F \)), a linear variation of skin friction case (\( \sigma_{zz}^* \cdot L^2 / F \)), a non-linear variation of skin friction case (\( \sigma_{zz}^* \cdot L^2 / F \)) vs. the non-dimensional ratio \( z/L \) is given in Figs. 8(a)–8(d), respectively. In order to check the accuracy of the proposed solutions, comparisons with the isotropic solutions of Geddes in a Cartesian co-ordinate system (1966a) are verified by a limiting process. The compared results are also shown in Figs. 8(a), 8(b), and 8(c) for a point load case, a uniform skin friction case, and a linear variation of skin friction case, respectively. They reveal that the presented solutions are in good agreement with those by Geddes’s solutions (1966a) when the medium is isotropic. From Figs. 8(a)–8(d), they also indicate that for a given loading type (the point load, and the uniform, linearly varying, non-linearly varying skin frictions), the depth (\( z \)), and the length of a pile (\( L \)), the trend of these figures is similar; however, the magnitude of the vertical normal stress [1] decreases with the increase of \( z/L \) from 1 to 3, [2] slightly increases with the increase of \( E/E' \) from 1 to 2.37 (Soils 1, 2 and 3), [3] increases with the decrease of \( G'/E' \) from 0.385 to 0.23 (Soils 1 and 4), [4] decreases with the increase of \( G'/E' \) from 0.385 to 0.44 (Soils 1 and 5), [5] is nearly unaffected by the ratio of \( \nu'/\nu \) (Soils 6 and 7), respectively. It is apparent that the vertical normal stress is rather sensitive to the ratio of \( G'/E' \), and relatively insensitive to the ratios of \( E/E' \), and \( \nu'/\nu \). Particularly, the decreases of the ratio of \( G'/E' \) decisively influence this stress. Figure 9 shows the effect of loading types on the non-dimensional vertical normal stress along the line of action. Since the calculated tendency for each soil is alike, only the vertical normal stress for Soil 4 (\( E/E' = 1.0, G'/E' = 0.23, \nu'/\nu = 1 \)) is presented. From Fig. 9, it can be seen that the magnitude of vertical normal stress decreases with the increase of \( z/L \) from 1 to 3 for each loading case, and the order induced stress obeys: the point load case > the non-linear variation of skin friction case > the linear variation of skin friction case > the uniform skin friction case. The results of these two figures (Figs. 8 and 9) also reflect the fact that the magnitude of vertical normal stress intensely depends on the type and degree of soil anisotropy, and the loading types.

From Figs. 8 and 9, it can be found that the values for vertical normal stress are not shown for any \( z/L < 1 \), as these will represent a tension stress in the soil at the depth \( z \) above the pile tip. In many instances, the subsoil transfers tension not at all or only partially because the gravity effects would produce a downward flow of the soil mass to eliminate them (Feda, 1978; Bowles, 1996). It may be possible to superpose the analytical solutions of the gravity-induced stresses (for a cross-anisotropic material with no lateral displacements assumption):

\[
\sigma_{zz}^* = \sigma_{zz}^G = \rho g z \frac{\nu'}{1 - \nu'} \frac{E}{E'}, \quad \sigma_{zz}^G = \rho g z, \\
\tau_{xy}^* = \tau_{xy}^G = \tau_{xy}^0 = 0;
\]

where \( \rho \) is the density, and \( g \) is the gravitational acceleration, (Amadei et al., 1987) to solve it.

Employing the illustrative examples, the results imply that the displacement and stress in isotropic/cross-anisotropic soils due to end-bearing, and three skin friction can be easily calculated by the proposed solutions. Also, the presented four loading cases could provide a general solution for the estimation of displacements and stresses along a pile for any compound forms of loading by superposition.
CONCLUSIONS

The analytical solutions for displacements and stresses in a Cartesian co-ordinate system subjected to a vertical point load \((F)\) acting at \(z = h\) in the interior of a cross-anisotropic half-space are newly expressed by several elementary functions. Integrating the elementary functions, the point load solutions can be extended to derive the solutions for displacements and stresses in a cross-anisotropic half-space due to various loading types. The loading types include uniform skin friction, linear variation of skin friction, and non-linear variation of skin friction of an axially loaded pile, respectively. These solutions indicate that the degree and type of material anisotropy, and the loading types influence the displacements and stresses. The presented analytical solutions of stresses for a point load, a uniform skin friction, and a linear variation of skin friction are the same as Geddes’s solutions (1966a) when the medium is isotropic.

Based on the results of a parametric study for London clay, the following interesting conclusions can be drawn from this paper.

1. Except for the uniform skin friction case, the magnitude of the vertical surface displacement [1] decreases
6. For a given soil type, the order induced vertical normal stress obeys: the point load case > the non-linear variation of skin friction case > the linear variation of skin friction case > the uniform skin friction case.

7. The estimation of displacements and stresses due to the proposed loading cases in an isotropic/cross-anisotropic half-space is fast and correct since the presentation of the analytical solutions is concise and systematized. Also, with results from the four analyzed cases, a great deal of loads can be treated by superposition.

NOTATION

- \( A_j(i,j = 1 - 6) \): elastic moduli or elasticity constants
- \( d \): infinitesimal element along the \( z \)-axis
- \( E \): Young's modulus in the horizontal direction
- \( E' \): Young's modulus in the vertical direction
- \( F \): force for an end-bearing, and a friction pile
- \( G' \): shear modulus in the vertical plane
- \( i \): complex number \( (= \sqrt{-1}) \)
- \( k, m_1, m_2, T_1, T_2, T_3, T_i \): coefficients
- \( L \): the pile length
- \( l_{xi}-l_{xj} \): integral functions for the displacements due to a linear variation of skin friction
- \( l_{si}-l_{sj} \): integral functions for the stresses due to a linear variation of skin friction
- \( n_{xi}-n_{xj} \): integral functions for the displacements due to a non-linear variation of skin friction
- \( n_{si}-n_{sj} \): integral functions for the stresses due to a non-linear variation of skin friction
- \( P_{xi}-P_{xj} \): elementary functions for the displacements due to a point load
- \( P_{si}-P_{sj} \): elementary functions for the stresses due to a point load
- \( Q \): a total load (force per unit length)
- \( r_{xi}-r_{xj} \): integral functions for the displacements due to a uniform skin friction
- \( r_{si}-r_{sj} \): integral functions for the stresses due to a uniform skin friction
- \( U \): displacement components
- \( u_1, u_2 \): roots of the characteristic equation
- \( u_{xi}, u_{xj}, u_i \): displacements due to a linear variation of skin friction
- \( u_{xi}, u_{xj}, u_i \): displacements due to a non-linear variation of skin friction
- \( u_{xi}, u_{xj}, u_i \): displacements due to a point load
- \( u_{xi}, u_{xj}, u_i \): displacements due to uniform skin friction
- \( v \): Poisson's ratio for the effect of horizontal stress on complementary horizontal strain
- \( v' \): Poisson's ratio for the effect of vertical stress on horizontal strain
- \( \sigma \): stress components
- \( \sigma_{x1}, \sigma_{y1}, \sigma_{z1} \): normal stresses due to a linear variation of skin friction
- \( \sigma_{x2}, \sigma_{y2}, \sigma_{z2} \): normal stresses due to a non-linear variation of skin friction
- \( \sigma_{a1}, \sigma_{y1}, \sigma_{z1} \): normal stresses due to a point load
- \( \tau_{x1}, \tau_{y1}, \tau_{z1} \): shear stresses due to a linear variation of skin friction
- \( \tau_{x2}, \tau_{y2}, \tau_{z2} \): shear stresses due to a non-linear variation of skin friction
- \( \tau_{x3}, \tau_{y3}, \tau_{z3} \): shear stresses due to a point load
- \( \tau_{x4}, \tau_{y4}, \tau_{z4} \): shear stresses due to a uniform skin friction
REFERENCES


7) Geddes, J. D. (1966b): Tables for the calculation of stresses in a semi-infinite medium due to vertical subsurface loading, *Bulletin No. 35*, Dept. of Civil Engrg, Univ. of Newcastle upon Tyne.


