A THREE-DIMENSIONAL STRESS-STRAIN MODEL OF SAND UNDERGOING CYCLIC ROTATION OF PRINCIPAL STRESS AXES

SATOSHI NISHIMURA and IKUO TOWHATA

ABSTRACT

A new three-dimensional stress-strain model of sand is proposed by using the concept of multiple shear mechanisms which previously has been developed in a two-dimensional manner. The original model exhibited excellent capability for simulating the principal stress axis rotation. By extending this approach with a new dilatancy model, the present model makes it possible to describe a general stress state in three-dimensional space with the whole six degrees-of-freedom of stress taken into account. Modelling of sand behaviours under general drained and undrained conditions was achieved by applying a stress-dilatancy relation to all individual shear mechanisms. It was demonstrated that the model predicts well soil behaviour in rather complex three-dimensional shear loadings as well as in more fundamental two-dimensional ones.

Key words: consolidated undrained shear, constitutive model, cyclic shear, dilatancy, dynamic, sand (IGC: E8/E13)

INTRODUCTION

Development of a constitutive model of soil has always been a central topic in soil mechanics. A milestone was achieved with the application of plasticity theory to soil, as represented by the Cam Clay model (e.g. Roscoe et al., 1963; Schofield and Wroth, 1968). Many subsequent studies followed it by describing stress and strain with their principal values. This method is convenient from a practical viewpoint because it allows a yield surface to be drawn in $p'$-$q$ space (e.g. Cam Clay model, op. cit.) or in a $\pi$ plane (e.g. Lade and Duncan, 1975), which can be illustrated in a two-dimensional plane. While this approach proved to be successful for a certain family of geotechnical problems, it has long been recognized that employment of principal values alone is not sufficient for description of a genuinely general stress state because it pays attention only to two or three of the original six degrees-of-freedom (DOFs) of stress or strain. As a consequence, the principal direction and its rotation tend to be neglected in such a method, despite fact that their significant effects have been revealed by many experimental studies (e.g. Broms and Casbarian, 1965; Arthur et al., 1980; Symes, 1983; Trowhata and Ishihara, 1985a; Miura et al., 1986a; Matsouka and Sakakibara, 1987; Gutierrez et al., 1991). These studies, with the aid of a directional shear cell or hollow cylinder apparatus, unanimously indicated the generation of plastic strain and/or changes in strength parameters by the rotation of principal stress axes.

In view of this problem, some studies formulated a stress-strain relationship in a fixed reference coordinates system by using general stress and strain tensors. Among them, the multi-laminate model proposed by Pande and Sharma (1983) and Pietruszcak and Pande (1987) accounted for behaviour of a soil element by looking into a number of microscopic yielding mechanisms (Calladine, 1971). Their model, with a number of shear mechanism distributed in a reference coordinate system, exhibited a potential to simulate the effects of the rotation of principal stress axes. A similar idea was also employed by Trowhata and Ishihara (1985c) in their so-called "multi-spring model" for simulation of cyclic mobility and the rotation of principal stress axes. This model consists of a number of inelastic springs, each of which represents a shear mechanism in its corresponding direction, as schematically illustrated in Fig. 1. It was assumed therein that strain state would be expressed by the displacement of the central point and that a summa-

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Fig. 1. Schematic illustration of multi-spring model (Towhata and Ishihara, 1985c)

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tion of the generated spring forces would be the stress imposed to an element. The strain was thus related to the stress through a significant number of mechanisms. For example, consider a stress path along the circumference of Mohr's stress circle in Fig. 1. The magnitude of $F = \frac{(\sigma_1 - \sigma_3)}{2} = \frac{(\sigma_1 - \sigma_3)^2}{4 + r^2}$ is held constant, while the direction of $F$ rotates continuously. The behaviour of an individual spring is just one-dimensional reciprocation during the rotational stress path. Thus, the rotation of principal stress axes is reduced to merely a cyclic loading for individual springs. With proper modelling of inelastic cyclic behaviours of the constituent springs, this model successfully reproduces the soil behaviours under the principal stress rotation. These ideas were further developed into a fully-fledged two-dimensional constitutive model by Iai et al. (1992a).

Despite its excellent capability for describing the principal stress rotation, in order to be utilized in further advanced geotechnical problems, the multiple-mechanism constitutive model required various modifications. Firstly, its formulation was still limited to a two-dimensional space. Recent progress in computer performance has made three-dimensional analysis a more realistic and handy tool. In order to carry out such analysis, a relevant three-dimensional constitutive model is indispensable. Three-dimensional multiple mechanism models have been proposed by Akiyoshi et al. (1994) and more recently by Fang (2003). Study of the similarities and differences between these and the present model is left to the readers. Secondly, the original model by Tawhata and Ishihara (1985c) was applicable only to the undrained condition, in which the development of excess pore water pressure was assessed by an experimental correlation with accumulated strain energy (Tawhata and Ishihara, 1985c). It is of significant engineering interest today, however, to analyse seepage in subsoil and post-liquefaction consolidation, which may affect ground response in some ways. In order to take these problems into consideration, the model should be generalized with regard to the drainage condition. The chief goal of the present study is to develop the multiple mechanism concept into a three-dimensional model which is applicable to a general drainage condition. While such generalization was attempted in this study, the main focus was still put on and around undrained cyclic problems and consequent liquefaction.

MODELLING OF SHEAR DEFORMATION BY 3-D MULTI-MECHANISM CONCEPT

General Structure of Model

Firstly, a scheme to extend the former two-dimensional multiple shear mechanism concept to a three-dimensional space is proposed. The essence of the multiple shear mechanism concept is to express two- or three-dimensional shear stress-strain relationships as a summation of one-dimensional stress-strain relationships mobilized in virtual planes, which are oriented to various directions inside a soil element (Fig. 2(a)). This idea is based on the assumption that the plastic shear deformation of sand consists of particles sliding in a variety of directions. Note that the shear in a plane has two degrees of freedom in a three-dimensional space. For example, if shearing in $z$ plane is considered, the direction of the shear is denoted by two independent coordinates such as $x$ and $y$ in Fig. 2(a). Due to this fact, the shear in the virtual plane needs to be further broken down into several one-dimensional shear mechanisms (Fig. 2(b)), which were referred to as "inelastic springs" by Tawhata and Ishihara (1985c). This model, as illustrated in Fig. 1, is equivalent to the original one, but Fig. 2(b) considers shear distortion in $\gamma_{xz} \sim \gamma_{zc}$ space.

The following two issues are crucial for characterising the model behaviours. Firstly, distribution of orientations of the prepared virtual planes (Fig. 2(a)) determines characteristics of modelled shear behaviours with regard to direction. If the orientation of the planes which contain the springs aligned with constant intervals (i.e., constant $\Delta \theta$ in Fig. 1) is regular and omni-directional, the model is rendered isotropic; otherwise, it becomes anisotropic. Secondly, a one-dimensional stress-strain relationship of each constituent shear mechanism prescribes overall shear behaviours of the model. For example, elastic behaviour of individual springs leads to elastic behaviour of an overall soil element. It is now possible to conclude that the present study constructed a three-dimensional multi-mechanism model by distributing a number of two-dimensional models in different
directions in 3-D space. In more specific terms, the proposed model consists of a number of virtual planes, each of which contain several constituent shear mechanisms. Note, however, that the two-dimensional model herein is defined in \( y_{x} - y_{z} \) space, not in \( (e_{x} - e_{z})/2 - y_{z}/2 \) space. Procedures in which strain is related to stress are essentially the same as in the original two-dimensional model. Firstly, shear strains of one-dimensional springs are calculated from strain components in an overall system. Then, the corresponding shear stresses are obtained based on spring characteristics and summed to become the shear stress in the overall system. These processes are discussed in detail in the following.

**Detailed Mathematical Structure of Model**

The shear strain in each one-dimensional shear mechanism (one spring) is obtained by coordinate transformation. The relevant angles used for the coordinate transformation are defined in Fig. 3(a) through (d). Firstly, a new system \( x'y'z' \) appears after the original coordinate system \( xyz \) is rotated by \( \theta \) around \( z \) axis. Then, a system \( x''y''z'' \) is obtained by further rotating the system \( x'y'z' \) by \( \phi \) around \( y' \) axis. The plane \( x'y' \) is assumed to correspond to one of the virtual planes in Fig. 2(a). Since the shear mechanisms are arranged in \( x''y'' \) plane around \( z'' \) axis in the same way as depicted in Fig. 2(b), step-by-step rotations of the system \( x''y''z'' \) around \( z'' \) axis superimpose the \( y'' \) axis on each of the one-dimensional shear mechanisms. Thus, the shear strain \( \gamma_{x''z''} \) is considered to represent shear strain of the constituent shear mechanism. A transformation matrix of strain, \( [T_{s}] \), was presented by Cook (1981) as follows.

\[
[T_{s}] = \begin{bmatrix}
I_{1} & m_{1} & n_{1} & l_{1}m_{1} & m_{1}n_{1} & n_{1}l_{1} \\
I_{2} & m_{2} & n_{2} & l_{2}m_{2} & m_{2}n_{2} & n_{2}l_{2} \\
I_{3} & m_{3} & n_{3} & l_{3}m_{3} & m_{3}n_{3} & n_{3}l_{3} \\
2l_{1} & 2m_{1} & 2n_{1} & l_{1}m_{1} + l_{1}n_{1} & m_{1}n_{1} + m_{1}l_{1} & n_{1}l_{1} + n_{1}m_{1} \\
2l_{2} & 2m_{2} & 2n_{2} & l_{2}m_{2} + l_{2}n_{2} & m_{2}n_{2} + m_{2}l_{2} & n_{2}l_{2} + n_{2}m_{2} \\
2l_{3} & 2m_{3} & 2n_{3} & l_{3}m_{3} + l_{3}n_{3} & m_{3}n_{3} + m_{3}l_{3} & n_{3}l_{3} + n_{3}m_{3}
\end{bmatrix}
\]  

(1)

**Table 1. Direction cosines between \( xyz \) axes and \( XYZ \) axes**

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<td>Y</td>
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<td>Z</td>
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where \( l_{i}, m_{i}, \) and \( n_{i} \) are direction cosines between the axes of the original coordinates system \( XYZ \) and those of rotated system \( XYZ \), as defined in Table 1. Furthermore, a transformation matrix of stress, \( [T_{s}] \), upon rotation of coordinate axes is related to that of strain by

\[
[T_{s}]^{-1} = [T_{s}]^{T} \quad \text{and} \quad [T_{s}]^{-1} = [T_{s}]^{T}
\]

(2)

As explained above, the present operations involve coordinate rotations with regard to three angles. The overall transformation matrix is obtained as a product of the three matrices which respectively represent those coordinate transformation processes.

\[
[T_{s}] = [T_{s,1}][T_{s,0}][T_{s,0}]
\]

(3)

Thus the shear strain of a particular shear mechanism (namely, the shear strain in \( y'' \) direction) is

\[
\{y''\} = [N][T_{s}]\{\varepsilon\}
\]

(4)

where \( [T_{s}] \) stands for the coordinate rotation by which \( \gamma_{x''z''} \) corresponds to the shear strain of \( i \)th \( (i = 1 \sim n) \) and \( n \) is total number of one-dimensional shear mechanisms in the model such as shown in Fig. 1 or Fig. 2(b) shear mechanisms. \( \{\varepsilon\} \) is the strain vector with 6 elements and

\[
\{\varepsilon\} = (\varepsilon_{x} \varepsilon_{y} \varepsilon_{z} \gamma_{xy} \gamma_{xz} \gamma_{x'z'})^{T}
\]

(5)

Note that a stress vector is also defined in the same way. \([N]\) is a 6-6 matrix in which only the 5-5 element is unity and the rest are zero. By multiplying \([N]\), \( \{y''\} \) becomes a vector with 1 DOF. The shear stress of the same mechanism, \( \{\sigma''\} \), is then obtained from \( \{y''\} \) via a one-dimensional shear stress-strain relationship described in the next section. The total shear stress imposed to the soil
is calculated by retransforming and taking a summation (actually, an average) of \( \{ \tau^{(i)} \} \) as
\[
\{ \tau \} = \{ \sigma_x - p \sigma_y - p \sigma_z - p \tau_{xy} \tau_{yz} \tau_{zx} \}^T
= \frac{1}{n} \sum_{i=1}^{n} \left( T_{ii}^{(i)} \right)^{-1} \{ \tau^{(i)} \}
\]
where \( p \) denotes the mean principal stress. Finally, the effective stress vector, \( \{ \sigma' \} \), is obtained by adding the mean effective principal stress, \( p' \), to the 1st ~ 3rd rows of \( \{ \tau \} \). \( p' \) is derived independently from a consolidation model discussed later.

**Geometrical Features of Model**

The distribution of the virtual plane orientations is determined with the aid of an icosahedron which is inscribed in a unit sphere; i.e., vectors directed from the centre of the icosahedron to its apices coincide with the normal vectors of the planes (\( \hat{n}^{(i)} \) in Fig. 3(a)). The use of a regular polyhedron leads to isotropic shear deformation characteristics of the model. On the other hand, it is possible to achieve "inherent anisotropy of the model" by varying parameters of the individual shear mechanisms as will be mentioned later again. In this study, the total number of the planes was increased by using the facets (strictly describing, the gravity centre of the facets) together with the apices. Thus 32 planes, 12 based on the apices and 20 on the additional points, are prepared as shown in Fig. 4. By distributing 6 single-degree-of-freedom shear mechanisms on each of them (6 per semicircle, considering symmetry), 192 one-dimensional shear mechanisms were employed. Note that actual calculation is required for only half of them, considering the symmetry of the icosahedron with regard to the \( xy \) plane.

**1-D SHEAR STRESS-STRAIN RELATION FOR IRREGULAR CYCLIC LOADING**

A shear stress-strain relationship of the constituent one-dimensional shear mechanism is formulated based on the extended Masing's rule along with several modifications and a hyperbolic skeleton curve. This relationship corresponds to behaviours of the schematic "springs" in Fig. 1 and Fig. 2(b). The modifications include a simplification of the rules for formation of hysteresis loops and an adjustment of calculated damping ratio to an appropriate magnitude. The extended Masing's rule (e.g., Finn et al., 1977) requires memorization of all loading reversal points when cyclic strain amplitude is diminished with the number of loading cycles. This is because the information is necessary for choosing the hysteresis loop to be followed when the strain amplitude is subsequently increased. This method is impractical to incorporate in a three-dimensional analysis due to the immense consumption of computer memory.

The proposed rule simplifies this procedure by creating hysteresis loops connecting the following two points as delineated in Fig. 5(a); points of the most recent loading reversal and at the maximum shear stress ratio in the loading history. Thus, the new rule helps save the computer memory, avoiding memorization of all reversal points. As for the damping ratio, it has been reported that a combination of the original Masing's rule and a hyperbolic skeleton curve yielded excessively large damping ratios (200/\( \pi \)% at large strain amplitude), especially...
at a large strain level. The present study solved this problem by reducing the area of a loop, which is equivalent to a damping ratio in soil dynamics definition (see Fig. 5(b)). With these problems in mind, a set of equations for the one-dimensional shear stress-strain model is formulated in what follows.

Firstly, the skeleton curve is given as

\[ R^{(i)} = \frac{k_{\text{max}}y^{(i)}}{1 + \frac{|y^{(i)}|}{\gamma_t}} \quad (7) \]

in which \( R^{(i)} \) is the stress ratio and \( R^{(i)} = \tau^{(i)}/\sigma^{(i)} \). Herein \( \tau^{(i)} \) is the shear stress and \( \sigma^{(i)} \) the normal effective stress to the shear mechanism in interest (= \( \sigma_s^{(i)} \)). Furthermore, \( k_{\text{max}} \) is the non-dimensional stiffness of a shear mechanism at small strain and \( \gamma_t \) is the reference strain. Note that the superscript \((i)\) implies that indicated symbols are of the constituent shear mechanism numbered as \( i \). If \( k_{\text{max}} \) and \( \gamma_t \) are varied for each spring, the present model is able to describe inherent isotropy of sand. An example of such an approach was offered by Miura et al. (1986b). \( k_{\text{max}} \) and \( \gamma_t \) are functions of \( \sigma^{(i)} \) and the square-root rule (shear modulus at small strain is proportional to square root of \( \sigma^{(i)} \)) is assumed.

\[ \lim_{\gamma_t \to 0} \frac{\tau^{(i)}}{y^{(i)}} = \sigma^{(i)}k_{\text{max}}\alpha\sqrt{\sigma^{(i)}} \quad (8) \]

Thus, when \( k_{\text{max}} \) is defined as \( k_{\text{max},0} \) at the reference normal effective stress, \( \sigma^{(0)} \), \( k_{\text{max}} \) at a given effective stress, \( \sigma^{(i)} \), is expressed by

\[ \frac{k_{\text{max}}}{k_{\text{max},0}} = \sqrt{\frac{\sigma^{(i)}}{\sigma^{(0)}}} \quad (9) \]

Furthermore, assuming that the ultimate strength (stress ratio) of the shear mechanisms, which is calculated from Eq. (7) as \( \lim_{\gamma_t \to \infty} R^{(i)} = k_{\text{max}}\gamma_t \), is independent of the effective stress,

\[ \gamma_t = \frac{\sigma^{(i)}}{\sigma^{(0)}} \quad (10) \]

where \( \gamma_{t,0} = \gamma_t \) at \( \sigma^{(0)} = \sigma_s^{(0)} \). A hysteresis loop is expressed by

\[ \begin{align*}
R^{(i)} &= R^{(i)}_{\text{rev}} + k_{\text{max}}(\gamma_t^{(i)} - \gamma_t^{(0)}) \\
&= \frac{k_{\text{max}}y^{(i)} - y_t^{(i)}}{1 + \frac{y_t^{(i)}}{\eta}} + \eta \left[ \frac{k_{\text{max}}y_t^{(i)} - y_t^{(0)}}{1 + \frac{y_t^{(i)}}{2\gamma_t}} - \right. \\
&\left. \frac{k_{\text{max}}y_t^{(0)} - y_t^{(i)}}{1 + \frac{y_t^{(0)}}{\gamma_t}} \right] \\
\end{align*} \quad (11) \]

where \( y_t^{(i)} \) is the going-to-occur strain amplitude defined by \( y_t^{(i)} = y_t^{(0)} - y_t^{(i)} \), with \( y_t^{(m,0)} \) and \( y_t^{(i)} \) being the maximum strain in the past loading history and the strain at the most recent reversal point, respectively (see Fig. 5(a)). \( \eta \) is a factor to reduce the damping ratio. When \( \eta = 1 \), Eq. (11) stands for a hyperbolic loop made by the conventional Masing's rule. Since the terms inside the parenthesis in Eq. (11) represent the deviation of the hysteresis loop from the secant component which corresponds to an equivalent linear behaviour, a smaller value of \( \eta \) leads to a slenderized shape of the hysteresis loop, which is equivalent to reduced damping ratio. However, Eq. (11) alone is not sufficient to portray a closed loop when irregular loading is concerned. In order to ensure the connection between the aforementioned two points (see Fig. 5(a)) for any pattern of loading, Eq. (11) should be modified as follows.

\[ \begin{align*}
R^{(i)} &= R^{(i)}_{\text{rev}} + C^{(i)}k_{\text{max}}(\gamma_t^{(i)} - \gamma_t^{(0)}) \\
&= \frac{k_{\text{max}}y_t^{(i)} - y_t^{(0)}}{1 + \frac{y_t^{(i)}}{2\gamma_t}}.
\end{align*} \quad (12) \]

and

\[ C^{(i)} = \frac{2k_{\text{max}}y_t^{(i)}}{2k_{\text{max}}y_t^{(0)}} \quad (13) \]

It is noted that stress-strain behaviour follows the skeleton curve again when the maximum stress ratio is exceeded, just as the extended Masing's rule stipulates. Finally, the shear stress-strain relationship is obtained by multiplying the stress ratio, \( R^{(i)} \), by \( \sigma^{(i)} \).

Lastly, a remark should be added to the determination of values for \( k_{\text{max}} \) and \( \gamma_t \). When this one-dimensional shear stress-strain model and the icosahedral multiple mechanism distribution are combined, relationships between \( k_{\text{max}} \) and the overall maximum shear modulus of an element, \( G_{\text{max}} \), and between \( \gamma_t \) and the overall shear strength in torsion shear, \( \tau_{\text{max}} \), are computed as follows, respectively. For a detailed discussion, refer to Nishimura (2002).

\[ k_{\text{max}} \approx \frac{5.0G_{\text{max}}}{p'} \quad (14) \]

and

\[ \gamma_t \approx \frac{2.65\tau_{\text{max}}}{k_{\text{max}}p'} \quad (15) \]

**MODELLING OF HARDENING DURING SHEAR LOADING**

Sand is known to exhibit a gradual increase of rigidity when subjected to drained cyclic loading (e.g. Tatsuoka and Ishihara, 1974; Towhata and Ishihara, 1985c; Pradhan et al., 1989; Shahnazari and Towhata, 2000). Throughout this paper, the behaviour in which shear stress becomes larger for a given strain level is referred to as "hardening". Although this nomenclature is not the same as that used in the plasticity theory, the referred phenomenon is in a broad sense analogous to expansion of a yield surface in the framework of the present model.
In fact, if the intersection between a hysteresis loop and the skeleton curve is considered to be a yield point, it moves away from the origin along $R$ axis in Fig. 5(a) as a result of the drained cyclic loading. The hardening can be attributed to a densification and influence of shear history. This effect is presumably due to rearrangement of the particles in the course of shearing into a more resistant packing against further shear (e.g. Matsuoaka, 1974).

In order to model this hardening phenomenon, a correlation between increase of stress amplitude for constant strain amplitude and accumulated volumetric strain was experimentally investigated by Shahnazari and Towhata (2000) as shown in Fig. 6. In this figure, the ordinate indicates the maximum shear stress in each cycle normalized by that in the initial loading for given constant strain amplitudes. It is seen that the relationship in Fig. 6 is rather independent of factors such as strain amplitude and initial density. In the present study, this relationship was employed for modelling of the hardening phenomenon. As will be explained later, this hardening phenomenon automatically leads to the ultimate volumetric strain generated by dilatancy during cyclic loading. The linear relationship in Fig. 6 suggests the following equation.

$$R' = (1 + H_n \varepsilon_\sigma) R$$  
(16)

in which $R$ is the stress ratio amplitude before hardening is considered, $R'$ is that after modification for the hardening and $H_n$ is a coefficient, which corresponds to the gradient of the line in Fig. 6. Based on Eq. (16), the present model reproduces the hardening effect by multiplying the shear stress of each shear mechanism by a factor of $(1 + H_n \varepsilon_\sigma)$.

**INCORPORATION OF STRESS-DILATANCY RELATION TO THE MULTI-MECHANISM CONCEPT**

The former two-dimensional model by Iai et al. (1992a) calculated development of excess pore water pressure under undrained condition by using a correlation between excess pore water pressure and shear work done to sand (Towhata and Ishihara, 1985b). The correlation was unique regardless of loading paths. Use of the correlation therefore enabled simple and reasonable prediction of excess pore water pressure, with no necessity of considering a loading path. This method has several drawbacks, however, when applied to more general situations. The most significant shortcoming is that it cannot deal with drained or partially drained condition. In reality, some extent of water drainage surely occurs even during an earthquake and in a state of complete liquefaction, which are broadly simplified as “undrained condition”. The partial drainage results in recovery of the effective stress in subsoil and/or eventual ground settlement due to consolidation. In view of this problem, the present study attempted modelling of volumetric change with a stress-dilatancy relation in place of the excess pore water pressure prediction in the former approach. The total volumetric strain, $\varepsilon_v$, is assumed to consist of two components; dilatancy-induced volumetric strain, $\varepsilon_v^d$, and consolidation-induced volumetric strain, $\varepsilon_v^c$. The volumetric change under drained condition is related to the variation of excess pore water pressure under undrained condition via a consolidation curve, based on a postulation that undrained condition is equivalent to constant volume condition (e.g. Hinokio et al., 2001).

The approach in the present study is characterised by application of a stress-dilatancy relation to all the constituent shear mechanisms. Thus, the volumetric change of soil is expressed as an average of those contributions. As a specific stress-dilatancy equation, a linear relationship of Taylor type (1948) between the stress ratio and the dilatancy ratio is employed as illustrated in Fig. 7. Such a theoretical linear relationship was revealed by Pradhan and Tatsuoka (1989) to be able to describe experimental results. Let $N_1$ and $R_0$ be constants, $\varepsilon_v^d(0)$ the volumetric strain caused by dilatancy, and $\varepsilon_v^d(1)$ the plastic shear strain defined by

$$d\varepsilon_v^p(0) = d\varepsilon_v^d(0) - d\varepsilon_v^c$$  
(17)

where $\varepsilon_v^d(0)$ is the elastic shear strain and is calculated by dividing shear stress increment by elastic shear modulus (see Appendix). Then the stress-dilatancy equations are (For loading in the positive direction or $d\varepsilon_v^p(1) > 0$)
MODEL OF CYCLIC LOADING

\[ \frac{\tau^{(i)}}{\sigma^{(i)}} = N_1^{(i)} \left( -\frac{de^{(i)}}{d\nu^{(i)}} + R^{(i)} \right) \]  
(18a)

For loading in the negative direction or \( d\nu^{(i)} < 0 \)

\[ \frac{\tau^{(i)}}{\sigma^{(i)}} = N_1^{(i)} \left( -\frac{de^{(i)}}{d\nu^{(i)}} - R^{(i)} \right) \]  
(18b)

A value of \( R^{(i)} \) was varied for initial and subsequent loading cycles in a cyclic problem, since experimental results in Fig. 8 revealed different stress-dilatancy relations for these cases. Thus, the present study employs \( R_{ps}^{(i)} \) and \( R_{ps-s}^{(i)} \) for the initial and subsequent loadings, respectively. With regard to the definition of “initial” and “subsequent” loadings, special care was required so as to not incur transition from the initial loading line to the subsequent loading line by small disturbances of input loading or mere numerical fluctuation. In the present study, a threshold strain parameter \( \gamma_{th} \) was introduced for all of the shear mechanisms. If loading in one direction generates a plastic shear strain (in terms of an individual shear mechanism) which exceeds \( \gamma_{th} \) measured from a point of the last loading reversal, \( R_{ps}^{(i)} \) is switched to \( R_{ps-s}^{(i)} \) after the next loading reversal. Otherwise, the loading in this direction is considered to be still minor and the “initial” stress-dilatancy relation is kept unchanged even after the next loading reversal.

The dilatancy-induced volumetric strain of a soil element is

\[ \frac{de_v}{n} = \frac{1}{n} \sum_{i} \left( \frac{de^{(i)}}{d\nu^{(i)}} \right) \]  
(19)

Here, the reader is reminded that the conventional theoretical stress-dilatancy equations were formulated only for particular deformation modes such as simple shear, triaxial compression or extension. On the other hand, the proposed model can deal with an arbitrary deformation mode, since any stress and strain state is decomposed into a number of virtual one-dimensional shears, as described earlier. Although \( N_1^{(i)} \) and \( R_{ps}^{(i)} \) are constants for the “virtual” shear mechanisms, their values can be determined directly by performing a drained torsion shear from isotropic consolidation test if those parameters are assumed to be identical for all shear mechanisms. This is because the modelled stress-dilatancy relation in a soil element can be analytically derived in case of torsion shear as follows (for a detailed derivation of this relationship, see Nishimura (2002)):

\[ \frac{\tau}{p'} = N_1^{(i)} \left( -\frac{de_v}{d\nu} \right) + 0.378R_{ps}^{(i)} \]  
(20)

Thus \( N_1^{(i)} \) and \( 0.378R_{ps}^{(i)} \) are obtained as constants for an element, not for an individual shear mechanism. The values of these parameters used in later sections are not changed for different densities.

MODELLING OF CONSOLIDATION

Consolidation is modelled by using the conventional linear \( e - \log p' \) curve. In order to express elasto-plastic deformation characteristics of sand, lines of two different gradients were employed, one for normal consolidation and the other for over-consolidation as depicted in Fig. 9. If the bulk modulus of the sand skeleton is given as \( B_0 \) at reference mean effective principal stress, \( p_0' \), the mean effective principal stress, \( p' \), can be readily derived in terms of volumetric strain caused by consolidation, \( e_v \), as

\[ p' = p_0' \exp \left( \frac{B_0}{p_0'} (e_v' - e_v) \right) \]  
(21)

in which the subscript \( y \) means that the indicated values are at a consolidation yield point. Two different values for \( B_0 \) should be prepared in order to describe normal consolidation and over-consolidation. Those are referred to in the present paper as \( B_{vs} \) and \( B_{vo} \), respectively.

As stated earlier, the volumetric strain is considered to be generated by two major mechanisms; dilatancy and consolidation. Namely,

\[ e_v = e_v^d + e_v^c \]  
(22)

Under undrained condition where \( e_v = 0 \),

\[ e_v^c = -e_v^d \]  
(23)
Thus, variation of effective stress under undrained condition is reproduced via the consolidation model (Eq. (21)) by calculating “virtual” dilatancy, ε_d, from the stress-dilatancy relation.

ANALYSIS OF FUNDAMENTAL BEHAVIOURS UNDER DRAINED AND UNDRAINED CONDITIONS

Procedures and Parameters

Based on the constitutive model described heretofore, a series of analyses was conducted on a single element basis. Flow of an analysis is illustrated in Fig. 10 for a strain-controlled case. In case of stress-control or combined control (a boundary condition which involves prescription of both stress and strain components), several additional procedures are required, such as calculation of a compliance matrix or iterations in some cases.

The parameters used in the present study are presented in Table 2. In some of the analysis cases where relative density was different from those listed, parameters were determined by interpolation from this table. B_w is the bulk modulus of water. In case of undrained condition, the bulk modulus of a soil element as a composite of soil grains and pore water was simply represented by B_w, because B_w was around twenty times as great as B_s,0 or B_o,0. The determination method of some of the parameters was discussed previously along with equations (see Eqs. (14), (15), and (20)).

| Table 2. Parameter values used in analysis |
| Symbol | Value |
|\( \rho_s = 100 \text{ [kPa]} \) | \( \rho_s = 100 \text{ [kPa]} \) |
| \( D_r = 75 \% \) | \( D_r = 75 \% \) |
| \( D_r = 57 \% \) | \( D_r = 57 \% \) |
| \( D_r = 38 \% \) | \( D_r = 38 \% \) |
| \( D_r = 22 \% \) | \( D_r = 22 \% \) |
| \( k_{r,0} \) | 3072 |
| \( \gamma_{r,0} \) | 2624 |
| \( \eta \) | 2176 |
| \( \mu \) | 1728 |
| \( B_{r,0} \text{ [kPa]} \) | 76800 |
| \( B_{r,0} \text{ [kPa]} \) | 65600 |
| \( B_{r,0} \text{ [kPa]} \) | 54400 |
| \( B_{r,0} \text{ [kPa]} \) | 43200 |
| \( B_{r,0} \text{ [kPa]} \) | 96000 |
| \( B_{r,0} \text{ [kPa]} \) | 82000 |
| \( B_{r,0} \text{ [kPa]} \) | 68000 |
| \( B_{r,0} \text{ [kPa]} \) | 54000 |
| \( B_{r,0} \) | 200000 |
| \( H_{r,0} \) | 22 |
| \( R_{r,0} \) | 1.3 |
| \( R_{r,0} \) | 1.65 |
| \( N_r \) | 1.15 |
| \( \gamma_{r,0} \) | 0.0001 |

Fig. 11. Computed stress ratio-shear strain and volumetric strain-shear strain relationships in drained simple shear

Analysis of Drained Shear

Firstly, a series of analyses was conducted on drained shear behaviours of sand. Figure 11 shows the stress ratio-shear strain relationships computed for monotonic simple shear, varying initial densities. Herein, the conditions which define the simple shear are: \( \varepsilon_r = \varepsilon_r = 0 \), \( \tau_{r} = \tau_{r} = 0 \), \( \sigma_r = \text{constant} \), and \( \tau_{r} = \text{variable} \). It is observed that the calculated relationships of relatively dense sands exhibit stress peaks while such a feature is absent in the case of looser sands. Furthermore, the relationships seem to converge at large strain level regardless of their initial densities. This behaviour is consistent with the experimental findings, which gave birth to the so called critical state theory. This convergence of ultimate strength is achieved by a combination of the stress-dilatancy model and the hardening mechanism of the one-dimensional shear stress-strain model. If shear deformation occurs at high shear stress ratio, sand becomes
dilatant according to the stress-dilatancy diagram in Fig. 7, leading to reduced shear stiffness (Eq. (16)). If sand is loose and hence shear stress does not become very large, on the contrary, contraction dominates and consequently the hardening takes place. Thus, shear stress-strain relationships ultimately reach an equilibrium point despite their different initial densities and stress responses.

This mechanism also works even when the loading is imposed in a cyclic manner. Figure 12 compares the computed and experimental volumetric changes during cyclic drained simple shear. In both cases, the accumulation of the volumetric strain gradually diminished as the loading proceeded. This behaviour is again reproduced by the combination of the dilatancy and hardening. Densification by volume contraction increases the amplitude of shear stress for a given strain amplitude according to the hardening expressed by Eq. (16). This amplification of shear stress leads to less contractive behaviour via the dilatancy model by Eq. (18), in the same way as discussed above. Equilibrium is therefore automatically reached.

As described previously, the application of former models, for example, those developed by Iai et al. (1992a) was limited to plane strain condition since they were formulated in the two-dimensional space. On the other hand, major advantage of the present model is that it can handle any boundary condition in the three-dimensional space. This feature enables simulation of true triaxial shear, in which three principal stresses are varied independently, making it possible to explore a failure criterion conceived by the present model in the $\pi$-plane. The obtained locus of failure is delineated in Fig. 13 along with other well-known ones. Points of the stress peaks in shear stress-strain relationships for $D_r = 57\%$ were selected in order to depict the locus for the present model. The obtained failure criterion shown by a line with marks seems to be closest to that by Lade and Duncan (1975), which was derived by fitting the data to their true triaxial tests. This fact may suggest that the failure criterion prescribed by the present model also fits the observed failure points.

Analysis of Undrained Shear

Analyses were also conducted with a variety of boundary conditions under undrained condition. As noted in the introductory remarks, the simulation of undrained sand behaviour is of primary interest in this study. The computed shear stress-strain behaviour and stress paths for monotonic simple shear are plotted in Figs. 14 and 15, respectively. When these outputs are compared with corresponding experimental results in the same figures, relatively good agreement between them is seen. Especially, it is important that the model successfully predicted the softening and the static flow for loose sand ($D_r = 22\%$) followed by strain-hardening after quasi-steady state (Verdugo, 1992) for denser sand ($D_r = 38\%$). These analysis results may serve as rare examples since there have been very few application of multi-mechanisms models to monotonic loading problems of loose sand, as far as the authors are aware.

The analysis was further extended to cyclic loading cases, in most of which liquefaction was triggered.
Fig. 15. Effective stress paths in monotonic undrained simple shear

Fig. 16. Shear stress-strain relationships in cyclic undrained simple shear

Fig. 17. Effective stress-paths in cyclic undrained simple shear

Figure 16 compares typical predicted and observed shear stress-strain relationships of loose sand, respectively. The corresponding stress paths are presented in Fig. 17. The test data were excerpted from Tawhata (1982). The agreement between the predicted and observed behaviours is good, if the very unstable nature of a liquefaction problem is taken into account. Based on a number of analysis cases, a liquefaction strength chart is obtained in Fig. 18. The model reasonably reproduced the observed liquefaction strength of sand, although there is some extent of underestimation of liquefaction resistance for smaller stress amplitude.

As has been stated, the proposed model makes it possible to simulate the rotation of principal stress axes. Herein the performance of the model for continuous rotation of principal stress axes is examined. Figure 19(a)
Fig. 20. Loading schemes in cyclic multi-directional shear

Fig. 21. Time history of shear stress and strain in undrained cyclic-monotonic simple shear

illustrates the applied stress path, which follows a circumference of Mohr's stress circle. The analysis was stress-controlled and the input $\sigma_i$ (normal to the plane of principal stress rotation) was equal to $\sigma_0$, in manner similar to the stress condition in a torsion shear apparatus with inner and outer cells opened to each other. This stress path involves continuous rotation of principal stress with no variation in deviator stress, $\sigma_1 - \sigma_0$. The strain increments obtained for this stress path are presented in the form of vectors in Figs. 19(b) and (c). The strain response was rather elastic in the first cycle of loading, with the strain-increment vectors directed tangentially. It became more plastic as indicated by an increase of radial components of the vectors in the second cycle due to development of excess pore water pressure. These observations are in good agreement with what was found in experiments (Ishihara and Tawhata, 1983; Tawhata and Ishihara, 1985a). Thus, the influence of rotation of principal stress axes on deformation characteristics of sand is considered to be properly modelled.

THREE-DIMENSIONAL ANALYSIS OF CYCLIC SHEAR

The analysis described heretofore was performed with relatively simple loading conditions, which can be done with a two-dimensional model. Finally in this section, performance of the model in three-dimensional problems will be demonstrated with three examples. The first one is superposition of cyclic shear stress on constant shear stress in orthogonal directions and the second is cyclic shear of anisotropically consolidated sand with a variety of intermediate principal stresses. Finally, an example of multi-directional shearing will be presented in order to further look into the nature of the model. The stress and strain conditions of these problems are illustrated in Figs. 20(a), (b) and (c), respectively.

Superposition of Cyclic Shear Stress on Static Shear Stress in Orthogonal Direction

The three-dimensional formulation allows superposition of two simple shear modes in orthogonal directions to be analysed. As shown in Fig. 20(a), one of the shear stress, $\tau_{xy}$, was applied in a cyclic manner while the other, $\tau_{xz}$, was held constant in the present example. This condition is associated with the deformation of slopes which simultaneously undergo static shear stress due to gravity and cyclic shear stress by seismic motions. The calculation was conducted under undrained condition and hereafter this shear mode is referred to simply as cyclic-monotonic simple shear. The predicted time histories of strain components in Fig. 21 exhibit an interesting feature. The development of shear strain in the $x$ direction, $\gamma_{xx}$, is promoted by cyclic shear stress in the perpendicular direction, $\gamma_{xy}$, while $\tau_{xx}$ was kept constant. This result can be attributed to increase of pore water pressure, which led to a decrease of the effective stress and hence strength of sand. Such behaviour of sand has been empirically recognized, as suggested by the fact that flow or substantial permanent deformation of real slopes, triggered by an earthquake, generally occurs in the direction of its maximum gradient (Sasaki et al., 1992).

Cyclic Shear of Anisotropically Consolidated Sand

Simulation of the horizontal cyclic shear (i.e. shear stress is applied on the horizontal plane) of anisotropically consolidated ground is of great interest to both soil mechanics and geotechnical engineering. In terms of mechanics, this problem provides an example of the rotation of principal stress axes. On the other hand, most of the structures are engineered on anisotropically consolidated ground, which is a consequence of natural sedimentation. If boundary conditions are such that the lateral strains are constrained to be zero, the calculated deformation is that of infinite ground during an earthquake. If normal strain in only one lateral direction is constrained (plain strain condition), the obtained results may be more relevant, for example, earthquake-induced subsidence of an infinitely long embankment. These problems were dealt with by Iai et al. (1992b) with their two-dimensional models. However, if problems with different geometry, such as subsidence of an axisymmetric oil tank, are to be analysed, the plain strain condition does not apply and need for a three-dimensional formulation arises. Herein, it is attempted to address this issue with the proposed three-dimensional model by varying magnitude of the intermediate stress, which governs the lateral bulging in real geotechnical problems.
The calculation procedures are as follows. Firstly, sand was consolidated isotropically at $p' = 75$ kPa. Then $\sigma_z'$ was taken as the vertical effective stress and increased to 150 kPa under drained condition while $\sigma_2'$ was kept at 75 kPa. At the same time, $\sigma_1'$ was increased to become the intermediate principal stress so that the intermediate principal stress factor, $b = (\sigma_2' - \sigma_3')/(\sigma_1' - \sigma_3')$, took a value of 0 to 0.3, 0.4 and 0.5. At this point, the stress state corresponds to what appears in-situ after complete dissipation of excess pore water pressure due to construction of a structure. Finally cyclic shear stress with a magnitude of 20 kPa was applied in the $zx$ direction under undrained condition while all the normal stresses were held constant. The development of each normal strain component is presented in Fig. 22.

The first thing to be noticed is that the normal strains in every direction developed significantly by application of the cyclic shear stress. This phenomenon is in fact exactly equivalent to the development of $\tau_{zx}$ against the constant shear stress $\tau_{zx}$ in the previous case. From the variation of $\varepsilon_y$ in Fig. 22, the plain strain condition is considered to appear when $b$ is somewhere between 0.3 and 0.4. This result agrees with the empirical fact that $b = 0.3 - 0.5$ in the plain strain condition in most cases. On the other hand, the calculated $\varepsilon_y$ or the vertical normal strains indicate no substantial difference for different $b$. The development of $\varepsilon_y$ is thus opposite to that of $\varepsilon_z$, since $\varepsilon_y + \varepsilon_z = \varepsilon_t$ under undrained condition. Although validity of these results may need experimental substantiation, it is emphasized that the calculated deformation reflects the effects of both principal stress axis rotation and intermediate principal stress.

Figure 23 portrays the stress path for $b = 0$ in $q-p'-\alpha$ space, where $\alpha$ denotes the angle between the major principal stress and the vertical. It is confirmed that the application of the cyclic shear stress is associated with the continuous rotation of principal stress axis. Also depicted is a portion of a state bounding surface as an envelope of failure lines for each $\alpha$. The calculated mean effective principal stress does not cross the state boundary surface at any $\alpha$, as experimental studies have suggested (e.g. Symes, 1983; Shibuya and Hight, 1987).

**Multi-directional Cyclic Shear**

An element was subjected to two-way undrained simple shear in $yz$ and $zx$ directions as illustrated in Fig. 20(c), with the first three cycles in the $zx$ direction, then one cycle in the $yz$ direction and finally one cycle in the $xz$ direction again. The imposed cyclic stress had only a single amplitude in any of these cycles. The computed stress path is shown in Fig. 24 along with the results obtained for a one-way case. It is clearly seen that the generation of excess pore water pressure, being equivalent to the reduction of $p'$, is much less in the subsequent cycles than in the initial cycle. However, the alteration of
the loading direction from the $xz$ to $yz$ direction boosted the rate of pore water pressure development, as comparison of the one-way and two-way stress paths suggests. This phenomenon is very much similar, albeit qualitatively, to the experiment findings by Towhata (1982) for two-way cyclic shear in $\tau_{xz}$ and $(\sigma_{y} - \sigma_{z})/2$ directions. The model in fact tends to overestimate the excess pore water pressure in the subsequent cycles, mainly because of the slender stress-strain loop, which gives more plastic strain than that given by Masing’s rule. Modification of the hysteretic looping rule might be required to obtain more quantitative results.

In analogy to the plasticity theory, this behaviour implies anisotropic hardening of an yield surface. Although there exists no elastic region (as far as shear deformation is concerned) in the present model, the less pronounced generation of pore water pressure in the subsequent cycles corresponds to the behaviour in an elastic region in the conventional elasto-plasticity theory. The promotion of excess pore water pressure due to the loading direction change means further “yielding” in a different direction, which implies that the “yield surface” had developed anisotropically as a consequence of the previous loading. This phenomenon may render the present model more suitable for investigating multidirectional loading problems.

**CONCLUSIONS**

The present study developed a new three-dimensional constitutive model with multiple shear mechanisms. The intention of the multiple mechanism was to describe rotation of principal stress axes. By extending the existing concept, the idea of multiple mechanism was developed into a new constitutive model which is capable of describing arbitrary stress paths, including rotation of principal stress axes in a three-dimensional space under both drained and undrained conditions. The following conclusions can be drawn from the performance of the new model from a set of single element analyses.

In the analyses of drained simple shear, a stress peak for denser sand was reproduced even with the hyperbolic skeleton curve for individual shear mechanisms. This calculated behaviour is the result of the combination of the dilatancy model and volumetric hardening model, which also played a role for the convergence of drained shear stress-strain relationships at large strain. In cyclic loading cases, the convergence of cumulative volumetric strain after many loading cycles was achieved also by this combination.

The analyses on undrained behaviour of sand elucidated the good agreement between the calculated and observed results for both the monotonic and cyclic loadings. The liquefaction strength chart drawn based on those results was reasonably consistent with that obtained by tests, although some extent of underestimation existed for smaller loading amplitude. The effects of principal stress axis rotation were simulated by applying the stress path along a circumference of Mohr’s circle with no change in the magnitude of principal stresses. Unlike an isotropic hardening elasto-plastic model, the present model predicted the plastic deformation, manifesting its ability to express the kinematic nature of sand, which has been substantiated by a number of experimental studies.

The analysis was extended to three-dimensional problems. By simulating the superposition of cyclic shear on monotonic shear, the potential of the model to predict ground deformation of a flow type caused by seismic motion was demonstrated. The analysis of cyclic shear from an anisotropically consolidated state reflected the effects of intermediate stresses, although quantitative comparison with experimental data was set aside for a future study. Finally, the two-way cyclic simple shear was analysed. The results revealed the anisotropic development of the yielding characteristics of the model as observed in tests. This fact suggests the applicability of the model to multi-directional loading problems.

**REFERENCES**

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**APPENDIX**

It is widely recognized that sand exhibits elastic response when subjected to small unloading or reloading. Thus the equivalent elastic stiffness, $k_{eq}$, was defined as stiffness in this elastic region, which appears immediately after the loading reversal. The original Masing’s rule is characterised by the fact that the equivalent elastic stiffness is independent of strain level and given as $k_{max}$ at every strain level (see Eq. (11)). Contrary to this, $k_{eq}$ varies for different strain level in the proposed rule and is expressed by

$$
k_{eq} = \left. \frac{dR}{d\gamma_{e}} \right|_{\gamma'_{r} = \gamma_{r}} = Ck_{max} \left( \frac{1 - \eta}{1 + \frac{\gamma_{v}^{(i)}}{\gamma_{r}}} \right) + \eta
$$

(24)

Note that $C$ and $\gamma_{v}^{(i)}$ are not constants and depend on strain level. In the present study, the elastic shear strain was calculated based on this stiffness.