A VISCO-HYPOPLASTIC CONSTITUTIVE RELATION FOR SOFT SOILS

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ABSTRACT

Soils with soft angular particles are markedly viscoplastic. For constitutive modelling with hypoplasticity, the granulate hardness is replaced by a rate-dependent (argotropic) solid hardness \( h_s \). Use is made of the viscosity index \( I_v \). The state limits of isotropic compression, stationary shearing (critical state) and cyclic shearing are thus argotropic. They are asymptotic solutions of the proposed visco-hypoplastic relation, the core of which is a viscosity factor \( f \), depending on an overconsolidation ratio \( p_c/p' \)). It resembles a theory by Niemunis (1996), but has a wider variety of asymptotic solutions (attractors). For constant effective stress, the familiar volumetric creep is obtained, and a dependency of strain rate ratio on stress ratio as observed. The response to strain rate jumps is obtained realistically, and also the relaxation. The material parameters can be easily determined from standard tests and estimated from index tests. Further explanation is given with a back analysis of Henkel's (1956) triaxial test results. Deviations are due to non-uniform sample deformations. Peak stress conditions are derived from instantaneous state values including strain rate. Extensions are briefly outlined: tensorial formulation, intergranular strain and polar terms. The microphysical background is explained with Persson's (2000) theory of thermally activated shear melting. Realistic dislocation energies are thus obtained from observed \( I_v \) values.

Key words: asymptotic response, constitutive law, creep and relaxation, hypoplasticity, rate dependence, thermal activation (IGC: D5/D6/D7)

INTRODUCTION

The dominant solid particles of soft soils consist of far softer minerals than quartz and are far more angular than grains. Their aggregates are kept together by van der Waals forces which are more or less compensated by electric repulsion. The ionic strength is of importance therefore, and void ratios after consolidation are far higher than for coarse-grained soils. Organic constituents and diatomaceous relicts also make soils softer. Without considering constituents and fabric, a constitutive relation for such soils is proposed in this article.

Hvorslev (1937) has precisely determined the following properties of a remoulded moderately plastic (M) and a highly plastic (H) clay, which are typical of soft soils:
P1: the void ratio \( e \) decreases nearly linearly with the logarithm of effective vertical pressure \( \sigma' \) in the oedometer (Fig. 1(a)), i.e.

\[ e = e_v - C_v \ln (\sigma'/\sigma_v) \] (1)

holds with reference values \( e_v \) and \( \sigma_v \) and a compression index \( C_v \);
P2: Equation (1) suffices for the M-clay (with \( C_v = 0.13 \) and \( e_v \approx 1.2 \) for \( \sigma_v = 10 \) kPa) up to \( \sigma' \approx 1 \) MPa, whereas for H (with \( C_v = 0.43 \) and \( e_v = 3.2 \) for \( \sigma_v = 10 \) kPa) the slope of \( e \) vs ln \( \sigma' \) is slightly lower for \( \sigma' > \approx 100 \) kPa;
P3: the critical void ratio \( e_c \) obtained by drained shearing after normal consolidation (NC) decreases similarly with ln \( \sigma' \), and in its approximation by Eq. (1) \( C_v \) and \( e_v \) (for the same \( \sigma_v \)) are somewhat below the ones from the oedometer;
P4: the residual shear strength \( \tau_c \) after normal consolidation is proportional to \( \sigma' \),

\[ \tau_c = \sigma' \tan \phi_{nc} \] (2)

with a friction angle \( \phi_{nc} = 23^\circ \) independently of \( \sigma' \) for M, whereas for H \( \phi_{nc} = 18^\circ \) is markedly lower with increasing \( \sigma' \);
P5: the peak shear strength \( \tau_p \) of overconsolidated (OC) drained samples is described by

\[ \tau_p = \sigma' \tan \phi' + k_c p_c \] (3)

with \( \phi' \approx 0^\circ \), \( k_c = \tan \phi_{nc} - \tan \phi' \) and an equivalent pressure \( p_c \) calculated from the actual \( e \) with Eq. (1) and \( p_c \) instead of \( \sigma' \);
P6: the average void ratio of originally OC samples sheared up to \( \tau \), is well below the \( e_c \) by P3;
P7: the undrained shear strength increases less than linearly with the shear rate \( \dot{\gamma} \) for strain-control, and \( \dot{\gamma} \) increases more than linearly with \( \tau/\tau_p \) for stress-control. This non-linear viscosity is stronger for H

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CSSM is simplified as against P2 and P7, goes beyond simple shearing and leaves aside P6. Postulating a single critical state for one \( p' \) and for one mode of deformation is a main advantage. The effective cohesion \( c' \) for peak strength is solely attributed to dilatation. Schofield (2002) postulates \( c' = 0 \) for remoulded soil, whereas Hvorslev (1937) proposed an ‘internal pressure’ \( p_c \) via \( c' = k_p p_c \).

Prandtl (1928) showed that all plastic effects are viscous and thermally activated. For metals he derived the empirically known relation:

\[
c = c_c [1 + c_c \ln (\dot{\gamma} / \dot{\gamma}_c)]
\]

for the dependence of cohesion \( c \) on strain rate \( \dot{\gamma} \), wherein the rate factor \( c_c \) and the reference rate \( \dot{\gamma}_c \) are proportional to the absolute temperature \( T \). The same relation was proposed by Murayama and Shibata (1958) for clays, making use of the Rate Process Theory (RPT) for thermally activated non-Newtonian fluids.

Mitchell et al. (1968) made further use of RPT for soft soils, and provided experimental evidence for a, and \( \dot{\gamma}_c \), in Eq. (5) to be proportional to \( T \). Adachi and Okano (1974) were the first to incorporate RPT into elastoplasticity for soils by using the overstress concept of Perzyna. They showed that the rate-dependence of strength and the secondary consolidation can be characterized by the same viscosity factor. This was again found by Leinenkugel (1976) who called \( c_c \) in Eq. (5) viscosity index \( I_c \). He made use of RPT and showed that relaxation leads to a linear decrease with \( I_c \), Eq. (6) (where this agrees with Lacerda and Houston (1973) had observed. For all that viscous effects are rarely allowed for in present-day design calculations.

Kolymbas (1978) proposed a rate-type constitutive relation which was the starting point of hypoplasticity. By using a second-order strain rate, he could model three viscous effects: rate-dependence as by Eq. (5), creep and relaxation. Gudehus and Kolymbas (1979) added a pressure-dependence according to Eq. (1). Later Kolymbas (1991) outlined that pressure and density dependence, called barotropy and pyknentropy, must go together. On this base Bauer (1996) proposed the formula

\[
e = e_0 \exp \left[ - \left( \frac{3p'}{h_c} \right)^n \right]
\]

both for normal compression and critical states. This is more consistent than Eq. (1): an objective granulate hardness \( h_c \), replaces the subjective \( \sigma_c \), the slope \( c_c \) is not the same for different \( e \) and decreases markedly with \( p' \) for \( p' > h_c / \beta \). Gudehus (1996) proposed the same formula also for the lowest void ratio \( e_0 \) obtained by cyclic densification with constant \( p' \) within an improved hypoplastic relation. This implied critical states which was further improved by v. Wolffersdorff (1996) and Bauer (2000).

The success of hypoplasticity in various applications has three main reasons: asymptotic solutions or attractors are physically sound, state variables are clearly separated from material parameters, and the latter are easily determined and related with grain properties (Herle and Gudehus, 1998). Excessive rateheting for alternating processes with low amplitudes was removed with an
intergranular strain tensor (Niemunis and Herle, 1998).

A shortcoming of hypoplasticity is the presumed rate-

independence, in particular for soft soils. Following Leinenkugel (1976) and Kolymbas (1991), I have proposed (Gudehus, 1996) an argotropic, i.e. velocity-
dependent \( h_s \), for Eq. (6), written here as

\[
h_s = h_s[1 + I, \ln (D/D_s)].
\]

(7)

This corresponds to Eq. (5), taking the amount of stretching rate \( D = \|D\| \) instead of \( \gamma \) and replacing \( \gamma \) by \( D \). Rate-dependence and creep can thus be described, but not relaxation. Niemunis (1996, 2002) made a decisive step forward by proposing

\[
\dot{T} = f_s L: (D - D^*)
\]

(8)

for the objective (e.g. Jaumann) stress rate \( \dot{T} \) as function of the stretching rate \( D \). The fourth-order tensor \( L \) is the same as for the linear part of the hypoplastic relation. \( D^* \) is a viscous rate with the same tensorial direction as the non-linear part of hypoplasticity, and is proportional to OCR\(^{-1/1}\), with a stress and void ratio dependent over-
consolidation ratio OCR depending on the state variables and \( \delta \), as introduced above. \( f_s \) is determined so that \( e_0 \) and \( e_\infty \) are the same as in a modified Cam clay theory. Several soft soils Niemunis' model was shown to enable a remarkably good description of rate-dependence, creep and relaxation.

I took over Eq. (8) with the intention to replace the Cam clay part by genuine hypoplasticity. Thus the state limits with \( e_0 \) and \( e_\infty \) are approximated by Eq. (6), in agreement with P2 above, and are argotropic by Eq. (7).

My equations for the evolution of stress components are outlined for cylindrical samples and are formally similar to the ones of rate-independent hypoplasticity: The factor \( D = \|D\| \) is replaced by \( f_s D \), with a factor \( f_s \), so that \( e_0 \) and \( e_\infty \) are attractors which are argotropic via Eq. (7).

Consequences are outlined for creep in CREEP, and for rate-dependence and relaxation in RESPONSE TO STRAIN RATES JUMPS AND RELAXATION. Procedures for determination of parameters are proposed in DETERMINATION OF PARAMETERS.

The theory is then used for a back analysis of the famous Henkel (1956) triaxial test data in BACK ANALYSIS OF TRIAXIAL TESTS BY HENKEL. This is an occasion for some further explanation. Extensions and limitations are briefly outlined in EXTENSIONS, PHYSICAL BACKGROUND AND LIMITATIONS. I leave aside mathematical aspects, but the physical background is explained.

ARGOTROPIC STATE LIMITS

Figure 2 is a plot of asymptotic void ratios for soft soils. The effective mean pressure \( p' \) is normalised by \( h_s \), which is called solid hardness as there are not only grains. The three curves are given by Bauer's (1996) formula

\[
\frac{e_1}{e_\infty} = \frac{e_0}{e_\infty} = \exp \left[ -\frac{3p'\ln(p'/h_s)}{h_s} \right]
\]

(9)

as for rate-independent hypoplasticity by Eq. (6). The turning point at \( p' = h_s/3 \) is in the usual geotechnical pressure range as \( h_s \) is below ca 1 MPa for soft solid particles (cf. P2 in INTRODUCTION), whereas it is above 1 GPa for hard grains (Herle and Gudehus, 1999). In the pressure range from about \( p'/h_s = 0.1 \) to 10 Eq. (9) leads to nearly straight lines in an \( e \) vs \( \ln (p'/h_s) \) plot.

Figure 2 can thus be considered as a substitute of Fig. 1(b). Critical states are not changed by stationary stretching, i.e. with cylindrical symmetry for \( \dot{e}_2/\dot{e}_1 = -1/2 \) and \( \dot{e}_3 = \text{const} \). The stress ratio is constant, i.e. in our case

\[
(\sigma_1 - \sigma_3)^2/(\sigma_1 + \sigma_3)^2 = \sin^2 \phi_c
\]

(10)

holds with a constant critical friction angle \( \phi_c \). As in hypoplasticity without viscosity, critical states are thus defined as asymptotic states for shearing with constant void ratio and constant components of strain rate.

The critical void ratio \( e_c \) depends on \( p' \) via Eq. (9), whereas CSSM presumes

\[
e_c = e_c - \lambda \ln (p'/p_r)
\]

(11)

with an index \( \lambda \) and a reference void ratio \( e_{cr} \) at a reference pressure \( p_r \). \( e_c \) is assumed in CSSM as the asymptotic void ratio for monotonous shearing with constant \( p' \). This definition leads to problems with shear localization and rate-dependence. With our definition any void ratio in the range \( 0 < e < e_{cr} \) can be critical for sufficient uniform shearing. Other than Eq. (11), Eq. (9) is consistent for \( p' \rightarrow 0 \) (\( e_c \rightarrow e_{cr} \)) and \( p' \rightarrow \infty \) (\( e_c \rightarrow 0 \)), but these limits are excluded in the sequel. \( p_r \) in Eq. (11) is subjective, whereas \( h_s \) in Eq. (9) is objective as it is given by the material.

Comparing Eqs. (9) and (11) yields the slope of the \( e_c \) - line at the turning point, \( \lambda = ne_c/2.72 \), with the \( e_c \) for \( p' = h_s/3 \). Soft soils have \( \lambda \) from about 0.1 to 0.5, \( e_c \) at the turning point from 0.8 to 1.5, and \( n \) is between 0.05 and
0.2 (more in DETERMINATION OF PARAMETERS).  

$$D = \sqrt{\dot{\varepsilon}_i^2 + 2a^2 i}$$  

(12)

for cylindrical symmetry. More generally than Eq. (7), I propose the dependency

$$h_i/h_o = 1 + I, \ln (D/D_0) \text{ for } D \geq D_0,$$

$$= \alpha_e \text{ for } D < D_0$$  

(13)

with a bound $D_0$ outlined below. $D$ is a reference rate, conveniently taken as $10^{-6} \text{ s}^{-1}$, which could be replaced by a physically objective value. The viscosity index $I$, ranges from 0.02 to 0.06. The lower bound for $h_i/h_o$ in Eq. (13) is called relaxation factor $\alpha_e$. It is fixed as $\alpha_e = 0.5$ and secures that $h_i$ does not vanish for $D < D_0 = \exp [(\alpha_e - 1)/I]$, following from Eq. (13). $D_0$ is extremely low, and $D < D_0$ is scarcely reached except for relaxation. Comparative calculations have shown that my constitutive relation yields the same results if $\alpha_e$ ranges from 0.4 to 0.6. As outlined in RESPONSE TO STRAIN RATE JUMPS AND RELAXATION, $\alpha_e$ could be determined by relaxation tests. The assumption of a rate-independent $h_i$ for $D < D_0$ by Eq. (13) is sufficient for the range of applications considered in this paper, although the transition to $D < D_0$ is actually smooth.

Stationary shearing with constant void ratio $e$ (e.g. saturated and undrained) leads asymptotically to a critical state with the mean pressure derived from Eq. (9),

$$\rho' = \frac{1}{3} h_i[-\ln (e/e_o)]^{1/\alpha_e}.$$  

(14)

This is shown in Fig. 2 by the paths A and B. As Eq. (10) determines the asymptotic strength $q_i = |\sigma_1 - \sigma_3|$, path A means softening, and B hardening. Critical states can also be reached with constant mean pressure $\rho'$. As shown in Fig. 2, $e$ decreases if it exceeds $e$, for the initial $\rho'$ (C), and increases in the opposite case (D). As in CSSM, $e > e_c$ may be called the ‘wet side’, and $e < e_c$ the ‘dry side’ of the critical line. Note again that uniform shearing with constant strain rate is presumed.

Leinenkugel (1976) found that the undrained shear strength $c_u$ of soft clays in critical states is rate-dependent via

$$c_u = K_e \sigma_i [1 + I, \ln (e_i/e_0)]$$  

(15)

with a reference strain rate $e_0 = 10^{-6} \text{ s}^{-1}$. $\sigma_i$ is Hvorslev's $e$-equivalent pressure, and $K_e$ is a friction factor depending on $\sigma_c$ via Eq. (4) as in CSSM. Equation (15) is also obtained with Eqs. (10), (11) and (13). The rate-independence of $\sigma_c$ was observed with Kaolin by Kuntsche (1982) and with mud by Krieg (2000). Thus the rate-dependence of $c_u$ enters via the mean pressure $\rho'$, as described by Eqs. (13) and (14).

Proportional compressions are defined in hypoplasticity (Gudehus, 1996) by $\varepsilon_i/\varepsilon_o = \text{const} > -1/2$ for cylindrical symmetry. They lead to proportional stress paths, i.e. 

$$\sigma_i/\sigma_1 = \text{const} \text{ depending on } \varepsilon_i/\varepsilon_0.$$  

This property was discovered with sand and was called swept-out of memory (SOM, Gudehus et al., 1977), and initiated hypoplasticity (Kolymbas, 1978). Later it was understood that the SOM behaviour involves a certain void ratio $e_o$. This depends on $p'$ as $e$, via Eq. (9), with a factor $e_\rho$ depending on $\hat{\varepsilon}_i/\varepsilon_i$ and with $e_\rho \geq e_\rho > e_\rho$. An upper bound of $e_\rho$, called $e_\rho$, holds for isotropic compression, i.e. $\dot{\varepsilon}_i = \dot{\varepsilon}_o$.

Krieg (2000) observed a similar behaviour with soft soils and called it extended SOM or ESOM: There is a unique asymptotic $e_i = -p'$-dependence for a given ratio $\dot{\varepsilon}_i/\dot{\varepsilon}_o > -1/2$ and a constant modulus $D$ of strain rate. I postulate that the rate-dependence of $e_i$ is described by Eq. (9) with $h_i$ from Eq. (13) and a factor $e_\rho$ instead of $e_\rho$ depending on $\dot{\varepsilon}_i/\dot{\varepsilon}_o$, or $\sigma_i/\sigma_1$ equivalently. The upper bound $e_i$ is thus higher for a higher rate $D$, but feasible only if $D$ changes by orders of magnitude. It appears that $e_i/e_\rho$ for a given $p'$ is in the narrow range from about 1.1 to 1.2.

An asymptotic void ratio $e_u$ for cyclic shearing with constant $p'$ is postulated again with Eq. (9) and $h_i$ by Eq. (13). It appears that $e_u/e_u$ for a given $p'$ lies in the range from about 0.5 to 0.65. $e_u$ can be understood as lower bound of plastic behaviour as in CSSM. The conventional plasticity limit $w_0$ is a special case of $e_u$. $e < e_u$ can be reached for given $p'$ by compression and decompression so that $e_u$ is not a general lower bound of $e$. $e/e_u$ is excluded in the sequel but could be allowed for as proposed by Niemunis (2002).

THE ARGOTROPIC RATE-TYPE RELATION AND THREE ATTRACTIONS

The effective stress rates $\dot{\sigma}_i$ and $\dot{\sigma}_z$ for cylindrical symmetry are postulated to depend on the strain rates $\dot{\varepsilon}_i$ and $\dot{\varepsilon}_z$ via

$$\dot{\sigma}_i = f_i(L_{11}\dot{\varepsilon}_i + L_{12}\dot{\varepsilon}_z - f_i N_i, f_i, D),$$  

(16)

$$\dot{\sigma}_z = f_z(L_{21}\dot{\varepsilon}_i + L_{22}\dot{\varepsilon}_z - f_z N_z, f_z, D).$$  

(17)

Formally this constitutive relation deviates from the hypoplastic one (e.g. Herle and Gudehus, 1999) only in the factor $f_i, D$, which replaces the modulus of strain rate $D = \sqrt{\dot{\varepsilon}_i^2 + 2\dot{\varepsilon}_z^2}$. The coefficients in Eqs. (16) and (17) depend on the effective stress ratio $K = \sigma_i/\sigma_1$, as for non-viscous hypoplasticity, via

$$L_{11} = \frac{(1 + 2K^2)}{1 + 2K^2} \left( 1 + a^2 \frac{1}{(1 + 2K)^2} \right)$$  

(18)

$$L_{12} = \frac{(1 + 2K^2)}{1 + 2K^2} \left( \frac{2K}{(1 + 2K)} \right)$$  

(19)

$$L_{21} = \frac{(1 + 2K^2)}{1 + 2K^2} \left( \frac{K}{(1 + 2K)^2} \right)$$  

(20)

$$L_{22} = \frac{(1 + 2K^2)}{1 + 2K^2} \left( 1 + a^2 \frac{K^2}{(1 + 2K)^2} \right)$$  

(21)

$$N_i = a \frac{(1 + 2K^2)}{3} \left( \frac{5 - 2K}{1 + 2K} \right)$$  

(22)

$$N_z = a \frac{(1 + 2K^2)}{3} \left( \frac{4K - 1}{1 + 2K} \right)$$  

(23)
\[ a = \frac{\sqrt{3} (3 - \sin \varphi_c)}{2 \sqrt{2 \sin \varphi_c}}. \] (24)

The argotropc density factor \( f_a \) depends on the density index \( I_a = (e - e_c)/(e_c - e_0) \) via
\[ f_a = (1 - I_a)^{\alpha_b} = \left( \frac{e - e_c}{e_c - e_0} \right)^{\alpha_b}, \] (25)
with \( e_c \) and \( e_0 \) by Eq. (9) with Eq. (13), and an exponent \( \alpha_b \) in the range from about 0.10 to 0.25. For the rate \( D = D_{c0} \), \( \alpha_b \) determines the peak angles of friction and dilatancy as in non-viscous hypoplasticity. It was found by comparative calculations that \( \alpha_b = 0.15 \) suffices for soft soils if viscous effects and compression are of major interest. \( e_c \) and \( e_0 \) are rate-dependent via Eq. (13) for the solid hardness \( h_c \). \( I_a \) indicates the position of a state point in the ‘argotropic phase diagram’ Fig. 2 as against \( e_c \) and \( e_0 \), and is now pressure and strain rate dependent.

The factor \( f_b \) in Eqs. (16) and (17) is determined (cf. Herle and Gudehus, 1999) by calculating an isotropic compression with constant strain rate by means of Eqs. (9) and (16):
\[ f_b = \left( \frac{e_c}{e} \right)^{\beta} + \frac{e_c}{n} \left( \frac{3 \rho' + h_c}{h} \right)^{1 - \frac{\beta}{\alpha}} \left[ \frac{3}{a^2 + 1} + 1 + \frac{\beta}{\alpha} \left( \frac{e_0 - e_0}{e_0 - e_0} \right)^{n} \right]^{-1}. \] (26)

The constant \( \beta \) between 1.0 and 2.0 may be fixed as \( \beta = 1.2 \) for soft soils, \( h_c \) in Eq. (26) is rate dependent by Eq. (13), and so is \( f_b \) with \( e_c \) via Eq. (9) with Eq. (13).

The viscosity factor \( f_c \) in Eqs. (16) and (17) depends on an overconsolidation ratio \( p_{c0} / p' \) by
\[ f_c = \exp \left( \frac{p' / p_{c0} - 1}{I} \right) \] for \( p' / p_{c0} \geq \alpha_c \),
\[ = 0 \] for \( p' / p_{c0} < \alpha_c \). (27)

Solving Eq. (27) for \( p' / p_{c0} \) leads to the function given by Eq. (13) so that \( f_c = D / D_0 \) is obtained with \( p' / p_{c0} = (h_c / h_0) \) for \( D \geq D_0 \), and \( f_c = 0 \) for \( D < D_0 = D_{c0} \) exp \([a_{c0} - 1] / I \). This is necessary for getting the three attractors outlined below.

The equivalent pressure therein is defined as
\[ p_{c0} = \frac{1}{3} h_0 \left[ - \ln (e f_c e_{c0}) \right]^{1/n} \] (28)
with a factor
\[ f_c = \frac{e_{c0}}{e_{c0} - 1} \left( 1 - \frac{\lambda_{c0}}{k_c} \right)^m. \] (29)

\( \lambda_c \) is a ratio of stress invariants (Bauer, 2000),
\[ \hat{\lambda}_c = \frac{2}{I_2} \left( 1 + 2K^2 \right) - \frac{3(1 + 2K^2)(1 + 2K)^2}{3(1 + 2K^2) - 3(1 + 2K)^2}, \] (30)
with the stress ratio \( K = \sigma_1 / \sigma_3 \). \( k_c \) is a friction factor,
\[ k_c = (1 - \sin^2 \varphi_c) / (9 - \sin^2 \varphi_c), \] (31)
and the exponent \( m \) is
\[ m = \ln \left( \frac{1}{e_{c0} - 1} \right) \frac{[1/9 k_c]}{[1/9 k_c]}. \] (32)

\( p_{c0} \) is thus void ratio and stress ratio dependent, as was already proposed by Hvorslev (1937). For an isotropic compression with \( \dot{\varepsilon} = \dot{\varepsilon}_c = D / \sqrt{3} \), \( K = \sigma_1 / \sigma_3 \) results to \( \lambda_c = 1 / 9 \) by Eq. (30), yielding \( f_c = e_{c0} / e_{c0} \) by Eq. (29) with Eq. (32). Inserting this \( f_c \) and the \( e \) from Eq. (9) into Eq. (28) yields \( p_{c0} / p' = h_0 / h_{c0} \), so that \( f_c = 1 \) follows from Eq. (27) with Eq. (13). Thus Eq. (16) with Eqs. (18), (19), (20), (21), (22), (23), and (24) and (25) and (26) coincides with Eq. (9) for \( e_c \). This is necessary for an isotropic normal consolidation to work as attractor. For an isotropic compression with \( \dot{\varepsilon}_c = \dot{\varepsilon}_c = D / \sqrt{3} \), \( K = \sigma_1 / \sigma_3 \) is obtained as outlined above, so that an isotropic normal consolidation with a constant rate is reproduced. This is plotted in Fig. 3(a) for \( D / D_0 = 10^{-2} \) and \( 10^2 \), and also for \( \dot{\varepsilon}_c = D / \sqrt{3} \). In accordance with experiments (e.g. Kriegl, 2000), the void ratio is higher for a higher strain rate.

The evolution of stresses during an isotropic compression as calculated by Eqs. (16) and (17) is plotted in Fig. 3(b) for two initial states which deviate from the ones considered above. The stress state is not isotropic at the beginning. One initial state A (e.g.) may be \( \sigma_1 = 100 \) kPa.
and \( \sigma_1 = 50 \text{kPa}, \) i.e. \( p' = 67 \text{kPa}. \) The void ratio may initially be the \( e_i \) for \( p'' = 67 \text{kPa} \) and \( D/D_o = 10^2. \) Isotropic compression with \( \dot{e}_i = \dot{e}_z = \dot{D} = \dot{D}_0 = 0. \) leads asymptotically to the \( e_i \)-line for \( D = D_o, \) and to an isotropic stress path. Another initial state B (e.g.) may be \( \sigma_1' = 154 \text{kPa} \) and \( \sigma_2' = 272 \text{kPa}, \) i.e. \( p'' = 233 \text{kPa}, \) and \( e = 0.87 \) \( e_i \) for the same \( p' \) and \( D = D_o. \) Isotropic compression with \( D = 10^{-2} \) \( D, \) leads asymptotically to the \( e_i \)-line for \( D/D_o = 10^{-2}, \) and to an isotropic stress path.

The response is similar for any proportional compression, i.e. \( \dot{e}_i/ \dot{e}_z = \text{const} > -1/2, \) with constant rate \( D. \) For the oedometer case \( \dot{e}_z = 0, \) e.g., the stress path tends to \( \sigma_1' = K_o \sigma_1, \) and \( e \) tends to the \( e_i \) for \( K_0. \) \( K_0 \) is rate-independent and approximately equal to \( 1 - \sin \varphi. \) (Jaký's formula). The equivalent pressure \( p_e \) obtained by Eqs. (28) to (32) with \( K = K_0 \) coincides with Hvorslev's \( p_e \) in the \( p' \)-range where Eq. (9) can be replaced by Eq. (11). Other asymptotic stress ratios and void ratios are obtained for other strain rates. This means that the ESOM behaviour outlined in ARGOTROPIC STATE LIMITS is reproduced by Eqs. (16) and (17).

Another asymptotic response (or attractor) is obtained for shearing with constant volume, i.e. \( \dot{e}_i + 2 \dot{e}_z = 0, \) and constant rate \( D. \) The stress path tends to one of the two asymptotes described by Eq. (10), viz

\[
\sigma_1' / \sigma_1 = K_o = \tan^2(45^\circ - \varphi_e / 2)
\]

for \( \dot{e}_i > 0, \) and \( \sigma_1' / \sigma_1 = 1 / K_o \) for \( \dot{e}_i < 0 \) (triaxial extension). The void ratio tends to \( e_i \) by Eq. (9) with \( h \) by Eq. (13). The equivalent pressure \( p_e \) coincides with \( p' \) for the asymptotes, as \( f_1 = 1 \) in Eq. (28) is obtained from Eqs. (29) to (32) with \( K = K_0 = 1 / K_o. \) With \( D \neq D_o, \) the viscosity factor \( f_1 = D / D_o \) is obtained for the asymptote as then \( p' / p_e = h_1 / h_0 \) holds. The rate dependent critical state behaviour as outlined in ARGOTROPIC STATE LIMITS is thus described by the proposed constitutive relation.

The lower bound \( e_0 \) of Fig. 2 is an asymptote for stress cycles with constant \( p', \) of course with drainage, but the mathematical proof is difficult (Niemiunis, 2002). Cyclic deformations with constant \( p' \) and constant \( D \) lead to densification until \( e_0 \) is reached. The rate-dependence is again achieved by Eq. (13) in Eq. (9). The limiting case \( K = 0, \) leading to \( e_i \) in Eqs. (28) from (29) and (30), is fictitious and only needed for the interpolation with Eqs. (29) and (32).

Let us compare Eqs. (16) and (17) with (8). Niemiunis' (1996) OCR has in common with my \( p_i / p' \) the dependence on stress ratio so that 1 is obtained both for isotropic first compression and stationary shearing with \( D = D_o. \) A lower bound \( e_0 \) does not occur in Niemiunis' theory, and also not a fictitious upper bound \( \varphi_e = 90^\circ \) for \( e = e_0 \), as non-viscous hypoplasticity. Niemiunis' OCR \( -1/2 \) comes close to my \( [\text{OCR}^{-1} - 1] / I \) and provides a rapid change of creep or relaxation rate (for \( \delta = 0 \) or \( e_0 = 0 \) respectively) with OCR as \( 1 / I, \) exceeds 20 for \( I < 0.05. \) \( f_0 \) and \( f_2 \) in Eqs. (16) and (17) yield the three attractors of hypoplasticity, now with rate-dependent \( h \), whereas Niemiunis has no \( f_0 \) and adapts his \( f_2 \) to a modified Cam Clay theory. Thus his theory is slightly easier, but it has no lower bound of plasticity and no objective measure of hardness. No end of compressive creep or relaxation is obtained with Niemiunis' OCR \( -1/2 \), whereas Eq. (27) leads to \( f_2 = 0 \) for \( D < D_0, \) and thus to two further asymptotes or attractors.

My theory is more objective than the one of Niemiunis, and much more than CSSM. The tensorial extension is straightforward so that arbitrary deformations and reference systems are covered. \( h_e \) is an objective reference pressure whereas the reference pressure \( p_e \) of CSSM, taken over by Niemiunis, is subjective. \( D \) can be shown to be objective, whereas it is taken by Niemiunis as a parameter for adaption. Reference states do not occur in my theory, instead the asymptotic states (attractors) play a key role.

For \( f, D, D \ll 1 \) the third terms in Eqs. (16) and (17) can be neglected against the other terms, and the response becomes hypoplastic. This is the case for even a slight overconsolidation, say \( p_i / p' < \text{ca} \ 1.3, \) and for the onset of a more rapid straining, say by a factor \( > 10^2. \) Both is known from numerous observations.

### Creep

The time-dependent strain increase for constant effective stresses is most easily obtained for critical states, i.e. stationary flow. \( \dot{e}_i / \dot{e}_z = -1 / 2 \) and Eq. (10) have to hold, and \( D = \text{const}. \) For a constant pressure \( p' \) and void ratio \( e_i, \) Eqs. (9) and (13) then yield the rate of stretching

\[
D = \sqrt{3 / 2} \ e_i = D, \ \exp \left[ \frac{3p'}{h_i} \left( -\ln \left( \frac{e_i}{e_{\text{co}}} \right)^{1/2} \right) - 1 \right]^{1/2}.
\]

(34)

The variation of \( D \) with \( p' \) and \( e \) by Eq. (34) is extremely wide due to \( 1 / I \gg 1. \) In fact a small pressure increase without drainage at a critical state causes a big strain rate increase.

Writing Eqs. (16) and (17) with \( \sigma_1' = 0 \) and \( \sigma_2' = 0 \) leads to

\[
\dot{e}_2 = \frac{L_{21} - L_{21} N_1 / N_2}{e_i - L_{12} + L_{22} N_1 / N_2}.
\]

(35)

As the coefficients depend only on the stress ratio \( K = \sigma_1' / \sigma_1' \) and on \( \varphi \) via Eqs. (18) to (24), \( \dot{e}_2 / \dot{e}_i \) by Eq. (35) can be plotted versus \( K \) (Fig. 4). There is a singularity for \( \dot{e}_i = 0 \) with a stress ratio \( K_{\infty} \) determined from \( L_{12} - L_{22} N_1 / N_2 = 0. \) Critical states appear with \( \dot{e}_2 / \dot{e}_i = -1 / 2 \) and \( K = K_0 \) or \( K_p, \) only then the creep is stationary.

The simplest case of creep with compression is the one with \( \dot{e}_2 = 0. \) The stress ratio \( K_0 \) is determined by Eq. (35) from \( L_{11} - L_{21} N_1 / N_2 = 0. \) This \( K_0 \) is not obtained at the beginning of an oedometer or triaxial test due to inevitable initial deviations, but \( K_0 \) tends to \( K \) in case of a good \( \dot{e}_2 = 0 \) control. It is not precisely the same \( K_0 \) as for a compression with \( \dot{e}_2 = 0 \) and \( \dot{e}_i = \text{const}. \)

We presume now a sequence of ESOM states in the sense of ARGOTROPIC STATE LIMITS and THE ARGOTROPIC RATE-TYPE RELATION AND THREE ATTRACTORS and write
Fig. 4. Dependence of strain rate ratio on stress ratio for creep. Calculated with $\varphi = 15^\circ$

$$p' = p_{ct} \left[ 1 + I, \ln \left( \frac{\varepsilon_1}{D_c} \right) \right] \exp \left( \frac{e - e_\infty}{C_e} \right)$$  \hspace{1cm} (36)

by using Eqs. (9), (11) and (13), with $p_{ct}$ instead of $p$, in Eq. (11) for $D = D_c$. Constant pressure, i.e. $dp'/dt = (dp'/d\varepsilon_1)e_1 + (dp'/d\varepsilon_0)e_0 = 0$ leads to the differential equation

$$-\frac{\varepsilon_1}{C_e} + \frac{\varepsilon_1 I_c}{1 + e} = 0$$  \hspace{1cm} (37)

making use of the relation $e = -\varepsilon_1(1 + e)$ for incompressible solid particles. Equation (37) holds precisely for the onset at $t = 0$ with an initial void ratio $e_0 = e_\infty$, and in a good approximation for a subsequent small reduction of $e$. The solution of Eq. (37) can be written as

$$e = e_0 - I, e_c \ln (1 + t D_c)$$  \hspace{1cm} (38)

with the initial values $e_0 = e_0$ and $\varepsilon_1 = D_c$ for $t = 0$. $1/D_c$ is thus an objective time scale as $D_c$ can be shown to be objective. The same solution is obtained with another initial strain rate $D_c$, taking the initial void ratio associated with $D_c$ by Eq. (9), written for $e_0$ as determined by $\varepsilon_0$. Equation (38) is empirically known with $C_e$ instead of $I, C_e$, so that

$$C_e = I, C_e$$  \hspace{1cm} (39)

holds. It was observed that the ratio $C_e/C_e$ is constant (e.g. Mesri and Castro, 1973), which supports the assumption of a constant $I_e$.

Some calculated creep curves for $\dot{\varepsilon}_c = 0$ are plotted in Fig. 5. A soft mud may be first compressed—under full drainage as in the sequel—with $D = \varepsilon_1 = 10^{-2}D_c$ until $p' = 0.3h_c$ is reached (AB). Then $p'$ is kept constant until $\varepsilon_1$ goes down to $10^{-4}D_c$ (BC). The reduction of $e$ tends to the one by Eq. (38), taking $t = 0$ for $B$. $\varepsilon_1$ is then increased to $10^{-2}D_c$ in the short interval $10^{-3}/D_c$, so that $p' = 0.6h_c$ is reached (CD). Afterwards $p'$ is again kept constant (DE). Shortly after the onset the decrease of $e$ is well described by Eq. (38), taking $t = 0$ for D. The transition period is changed if the previous loading (CD) is changed. Only after an extremely long time, $t > ca 10^{20}/D_c$, $e$ decreases and kept constant afterwards. The stress paths (Fig. 6(a)) are nearly straight during the loading interval, and the strain paths (Fig. 6(b)) are straight with ratios $\varepsilon_1/\varepsilon_0$ by Eq. (35). The void ratio decreases with increasing pressure (c) and time (d) nearly as in Fig. 4. For $K < K_{pl}$ (A) $\varepsilon_1$ is bigger than for $K = K_{pl}$ ($e = 0$). For $K > 1$ (B) $\varepsilon_1 < 0$ is obtained. These results are supported by tests with various soft soils by Krieg (2000).

The creep rates are decreasing with time for $K < K < 1/ K_{pl}$ due to compression, and constant in critical states with $K = K_{pl}$ or $1/K_{pl}$. They increase beyond the value by Eq. (34) for $K < K_{pl}$ or $K > 1/k_{pl}$ due to dilation. Such a drained collapse with peak stress states is impeded by seepage forces except for localized shearing (cf. Hvorslev, 1937).
RESPONSE TO STRAIN RATE JUMPS AND RELAXATION

Figure 7 shows a calculated triaxial test with constant void ratio, i.e. $e_1+2e_2=0$, and section-wise different strain rates $D=\sqrt{3}/2\varepsilon$. The initial state is $\sigma_1^0=\sigma_2^0=500$ kPa and $e=0.58$, which is slightly on the dry side of critical (cf. Fig. 2). The dotted lines show the evolution of $\sigma_1$ and $\sigma_2$ for constant $e_1$. They tend to critical states, i.e. $\sigma_1^t=\sigma_2^t=0$ and thus $dq/de=0$. These have $\sigma_1^t/\sigma_2^t=tan^2\left(45^\circ-\varphi_c/2\right)$ independently of $\varepsilon_1$, and increasing $q$ with increasing $D$ according to Eq. (13) in Eq. (9).

The test starts with $D=D_1=10^{-6}$ s$^{-1}$ up to $e_1=3\%$ (0-1). With a jump to $10^2D_1$ the $q-e_1$ curve tends to the curve for $10^2D_1$, (1-2). A jump back to $10^2D_2$ causes a sharp bend and leads to the curve for $10^2D_2$, (2-3), and so on. This is what Leinenkugel (1976) is found by triaxial and Kuntsche (1982) by biaxial tests with kaolinite clay, and more recently Krieg (2000) with muds.

The immediate response to $e_1$-jumps is easily calculated if it starts from a critical state. With $\varepsilon_1=\varepsilon_0$ and $\varepsilon_2=-\varepsilon_0/2$ we can write instead of Eqs. (16) and (17)

$$\sigma_1^t/\sigma_0=(L_{11} - \frac{1}{2} L_{12})\varepsilon-f_i N_1 D,$$  

immediately after the jump. The ratio of 'moduli' for jumps up and down is thus

$$\frac{E^*}{E_{up}^*} = \frac{(1-e_0/\varepsilon_0)/(1-\varepsilon_0/\varepsilon_0)}{1-e_0/\varepsilon_0}.$$  

e.g. $0.9/(-9) = -0.1$ for $\varepsilon_0/\varepsilon_0=10$. This is what Leinenkugel (1976) observed, and later many others.

Figure 8 shows a calculated fully drained oedometer test with section-wise different strain rates $e_1$. The initial state (1) is $\sigma_1^0=0.1h_{0a}$, $\sigma_2^0=K_0\sigma_1$ and $e=1.76$, which suits to $D=10^2D_1$ on an ESOM-line. The dotted lines represent ESOM states (cf. Fig. 2) for $D/D_1=1$, $10^{-2}$ and $10^{-4}$. The test starts with $D/D_1=10^{-2}$ up to $\sigma_1=100$ kPa (2). With a jump to $D=D$, the soil reacts as if it was slightly overcon-
solidated until the ESOM line for \( D = D_i \) is reached (2-3). With a jump back to \( D = 10^{-4} D_i \), a sharp bend is produced, and the ESOM-line for \( 10^{-4} D_i \) is reached afterwards (4). All that was observed with mud by Krieg (2000).

Let us now consider relaxations. With \( \dot{\varepsilon}_1 = \dot{\varepsilon}_2 = 0 \) in Eqs. (16) and (17) the stress rates are

\[
\dot{\sigma}_i = -f_0 f_i N_i f_i D_i, \quad (44)
\]

\[
\dot{\sigma}_j = -f_0 f_j N_j f_j D_j, \quad (45)
\]

Herein \( f_0 \) and \( f_i \), defined by Eqs. (25) and (26), hold for \( h_i = \alpha_i h_0 \) as by Eq. (13). This is the assumed lower bound of the solid hardness for zero strain rate. \( N_i \), \( N_j \) and \( f \), depend on \( \sigma_i \) and \( \sigma_j \) via Eqs. (22) and (23), and Eqs. (27) to (32), so that Eqs. (44) and (45) is a system of ordinary differential equations for \( \sigma_i \) and \( \sigma_j \).

Some typical numerical solutions are plotted in Fig. 9. Three different initial states are presumed, viz

<table>
<thead>
<tr>
<th>State</th>
<th>( \sigma_i ) [kPa]</th>
<th>( \sigma_j ) [kPa]</th>
<th>( e )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0.6</td>
<td>0.6</td>
<td>0.65</td>
</tr>
<tr>
<td>B</td>
<td>0.6</td>
<td>0.6</td>
<td>0.7</td>
</tr>
<tr>
<td>C</td>
<td>0.4</td>
<td>0.2</td>
<td>0.7</td>
</tr>
</tbody>
</table>

Fig. 8. Calculated oedometric test with strain rate control. Calculated with \( \varphi_i = 48^\circ \), \( h_0 = 373 \text{ kPa} \), \( n = 0.21 \), \( I_i = 0.053 \), \( e_{\text{om}} = 1.93 \), \( e_{\text{es}} = 3.08 \), \( e_{\text{p}} = 3.55 \), adapted to a diatomaceous mud (Krieg, 2000).

Fig. 9. Calculated relaxation: stress paths (a) and pressure reduction with time (b). Parameters as for Fig. 8.

Isotropic states remain isotropic (A, B). The relaxation is more rapid for a higher void ratio (e.g., B vs A), as was found e.g. by Zou (1998). Anisotropic states (e.g., C) relax in a similar fashion, with the tendency of the effective stress to become more isotropic. The relaxation
ends with a state of rest (thermodynamic equilibrium) for 
$t D_i > ca 10^{23}$, which can never be observed.

The tendency towards more isotropic stress is easily
shown with Eqs. (44) and (45), which yield
\[ \sigma' = \frac{N_1}{N_2} = \frac{5 - 2K}{4K - 1} \]  
(46)

by using Eqs. (22) and (23). $\sigma'/\sigma; > \sigma_1'/\sigma_1$ holds except
for $K = 1$, i.e. $K$ tends to 1 if it starts with $K > 1$. Comparative
calculations have shown that the mean pressure 
always tends to decrease via
\[ 1 - p'/p_0 = I_n \ln (AD_i/t/I_i) \]  
(47)
with time $t$. Herein $p_0$ denotes $p'$ at the onset and $A$ is a
constant. This means that the curves in Fig. 9 tend to
straight lines with slope $-I$. This was first observed
by Lacerda and Houston (1973), then by Leinenkugel 
(1976), and demonstrates again the importance of $I$. 
The intersection of the straight line Eq. (47) with the line $p'/p_0 = 1$ may be of use to determine $\alpha_r$.

**DETERMINATION OF PARAMETERS**

The first objective of parameter determination is to
match the state limits outlined in ARGOTROPIC
STATE LIMITS. The parameters are $\varphi_0, I, h_n, n, e_0, e_0, e_90$, 
$e_90$, in this order of relevance. Tests with disturbed
samples are required. Apart from precise measurements,
estimation methods for classification are possible. 
The further parameters cannot be determined as precisely.

The critical friction angle $\varphi_c$ is best determined from triaxial compression of saturated samples after primary
consolidation. The critical state is recognized from
stationarity of $\sigma_1$ and $\sigma_2$ with constant $\varepsilon_t$, then Eq. (10)
yields $\varphi_c$. Instead and more easily, a thin soil layer 
between two filter plates can be brought to stationary
shearing with $p_0 = p_0 = 100$ KPa. The friction angle in $\tau_c = \sigma' \tan \varphi_c$, is related with \( \varphi_c \)
\[ \tan \varphi_c = 2/\sqrt{1 + 3/\sin^2 \varphi_c} \]  
(48)
as derived by Bauer (2000). An estimation of $\varphi_c$ from 
the solid constituents is scarcely possible: $\varphi_c$ can be lower
than 10° for pure clay (Hvorslev, 1937) and higher than 50°
for mud with diatomaceous relicts (Krieg, 2000).

The viscosity index $I$, is easily determined from the
increase of the critical deviatoric stress $q_c$ by $\Delta q_c$ with an increase of $\varepsilon_t$ from $D_i$ to $D > D_i$, e.g. 10$D_i$
for undrained triaxial compression after normal
consolidation. As $p_r$ and $\varphi_c$ do not change with the jump, $I = \Delta q_c / q_c \ln (D / D_i)$ follows from Eq. (15). If primary and secondary consolidation 
lines show straight sections with slopes $C_1$ and $C_n$, one can also calculate $I = C_n/C_1$ from Eq. (39). 
However, this is less correct as the rate $\varepsilon_t$ at the end of 
each primary consolidation step is not the same. The 
empirical correlation
\[ I = 0.05 + 0.026 \ln w_l \]  
(49)
of $I$, with the liquid limit $w_l$ can yield a good estimate.

The quantities $h$ and $n$ for the pressure dependence of
asymptotic void ratios by Eq. (9) can be determined from
primary consolidation tests. Triaxial or oedometer results
can be used. In case of strain rate control, $\varepsilon_t$ must be
lower than $ca 0.01 k \sigma_i / \eta_0 h^2$ with permeability $k$ and 
sample height $h$ for avoiding excess pore water pressure.
The straight section of $e - \ln p'$-line has a slope $C_e$. 
This holds in a range from $p_i$ to $p_i$ with $e_i$ and $e_o$. As 
outlined by Herle and Gudehus (1999) one can then determine
\[ n = \frac{\ln (e_o/e_i)}{\ln (\sigma_i'/\sigma'_c)} \]  
(50) 
and
\[ h = 3p' \left( \frac{n e_0}{C_e} \right)^{1/n} \]  
(51)
Herein, $p' = \sigma_i(1 + 2k_0)/3$ can be taken with $K_0 = 1 - 
\sin \varphi_c$, for oedometer tests. $h_n$ is calculated then from $h_n$ 
with the given $D$ by Eq. (13).

For stress controlled tests the $\varepsilon$ at the (conventionally
recognized) near-end of primary consolidation has to be
determined for each step of $\sigma_i$. The $e - \ln p'$-line has to 
be corrected for the $\varepsilon$-influence into the one for $\varepsilon_i = D_i$, 
by Eq. (13) (cf. Fig. 8). A crude estimate of $n$ is possible with
the shape of particles: subangular ones have $n = 0.2$, 
extremely angular ones $n = 0.1$.

A precise determination of the reference void ratios $e_0$ and 
$e_90$ in Eq. (9) is indispensable, whereas an estimate of 
$e_90$ may suffice. $e_0$-values are best determined with NC 
samples in triaxial tests. Thus shear localization is avoided, 
which makes $e_0$-measurements of OC samples 
rather impossible. The exponent $\beta$ in Eq. (26) may
be fixed to 1.2, this was shown by comparative calculations.
$e_0$ is best determined from isotropic triaxial compres-
sion tests, or from shrinkage tests with suction measure-
ment. As outlined above, the $e - \ln (p'/h_n$) curve has to 
be corrected for the strain rate effect by Eq. (13). $e_0$ can 
also be calculated from oedometer test results with a trial 
and error procedure by means of Eq. (16) and with 
estimated further parameters: $K_0 = 1 - \sin \varphi_c$, $\alpha_r = 0.15$ and $e_0/e_0 = 0.65$. $f_i$ is calculated from Eq. (26) with $h_n$ 
instead of $h$, as $f_i = 1$ holds in Eq. (16). $e_0$ has to be varied 
until the best fit of the observed $e - \ln (p'/h_n$) line 
(corrected for $\varepsilon$) is obtained.

Comparative calculations have shown that $\alpha_r = 0.15$ is 
a good estimate for soft soils. For a more precise determination
of $e_0$, triaxial tests with overconsolidated samples are 
needed, or uniaxial tests with suction measurement. 
Peak friction angles $\varphi_0$ have to be calculated with $c' = c$ 
(cf. Schofield, 2002). The relationship between $\phi_0$, $\varphi_1$, and
$r_c = 1 - I_i$ given by Herle and Gudehus (1999) can be used to 
determine $\alpha_r$ from $\phi_0$, $\varphi_1$, and $I_i$. For $I_i$ the average $e$ at 
peak can be taken, as long as localized dilation is negli-
gible, and $e_0/e_90 = 0.65$ may be assumed therein.

A better approach to the lower bound $e_0$ may be achieved as follows. Samples are densified by alternating 
deformations in triaxial or biaxial tests with drainage 
and constant $p'$. The $\varepsilon_f$-amplitude has to be about 0.01 so that 
there is enough contraction in each near-cycle. The end of
densification, with $e = e_0$ by definition, is recognized from a zero net contraction. ($e_0$ cannot exactly be reached as there is always some dilatation.) A subsequent stronger axial shortening leads to very steep failure planes, or to nearly axial splitting in uniaxial tests. This is the lower bound of plasticity in the sense of CSSM. Formally the inclination of slip planes, $\theta_s = 45^\circ + \varphi_p/2$, tends to $90^\circ$ as $\varphi_p$ tends to $90^\circ$ for $f_i = 0$. This unattainable limit is incorporated in Eqs. (29) to (32). A simple example is the test for determining the plasticity limit $w_p$; rolling with reversals causes densification until further shearing leads to splitting. Cavitation occurs at $p_u = 0$ and $p = p_u$, thus $e_d = w_p p / r_u$ holds for $p' = 100$ kPa. Thus $e_{do}$ can be estimated from Eq. (9).

The relaxation factor $\alpha$ may be assumed as $\alpha = 0.5$ for many applications. More precisely it may be determined from relaxation tests by matching with Eq. (47).

BACK ANALYSIS OF TRIAXIAL TESTS BY HENKEL

Data for Weald Clay are taken from the famous paper by Henkel (1956). $I_1 = 0.03$ is obtained from Eq. (49) with $w_1 = 0.43$, $\varphi = 24^\circ$ is determined by extrapolation of the $q$ vs $e_1$ curve given for drained shearing of an NC sample. The same test provides an $e_{\alpha} = 0.15$ and $h_s = 655$ kPa are calculated with Eqs. (50) and (51) from the given isotropic consolidation line. With $D = 2\cdot10^{-7}$ s$^{-1}$ estimated from the given consolidation time, $h_s = 679$ kPa is calculated with Eq. (7) and $D = 1\cdot10^{-8}$ s$^{-1}$ from the $h_s$ above. Equation (9) tends to $e_{\alpha} = 0.19$ from a given $e_1$, with $h_s$ for the given $D_i$ and to $e_{\alpha} = 1.50$ from the given $e_1$ with the associated $D$. $e_{\alpha} = 1.25$ is taken from $e_{\alpha}/e_\alpha = 1.20$ ($w_p$ would lead to $e_{\alpha} = 1.2$). $\beta = 1.2$ and $\alpha = 0.15$ are used without adaption, their variation has practically no influence.

For isotropic loading observed and calculated $e$ vs ln $\sigma'$ curves are practically the same (Fig. 10), no wonder as this was required for calibration. For isotropic reloading the swelling calculated with the same $D$ as for loading is somewhat bigger than observed. Presuming a more rapid swelling which gets slower with the amount of swelling (no indication given by Henkel), a good agreement can be achieved. This is due to the increase of apparent stiffness with higher $D$ because of viscosity: Just after the $e$-reversal $f_i$ in Eq. (16) is yet unchanged, so that the reducing term $f_{\alpha}/N_i D_i$ is smaller against $(L_{\|} - L_{\perp}) D / \sqrt{3}$ for a higher $D = e_\alpha \sqrt{3}$. Filtration due to swelling is not allowed for in this numerical element test as homogeneity is presumed. This does not matter at the onset of unloading. The reduction of $D$ with time was assumed so that the given $e$ is nearly obtained for the final unloading.

As presented by Henkel's Fig. 3, axial loading was calculated for NC samples with $\sigma_1 = 210$ kPa and $e = 0.62$ initially (Fig. 11). With free drainage, $\sigma_1 = 210$ kPa and $e_1 = 7 \cdot 10^{-8}$ s$^{-1}$ taken from Henkel lead to an increase of $\sigma_1$ (a) and a decrease of $e$ (b) close to the experiment. The agreement holds up to $e_1 = 0.20$ where the deformation was no more uniform in the test, but the system is stable and the deviations from uniform stiffness can be shown with Eqs. (16) and (17) to be small. The void ratio distribution becomes less uniform due to drainage, this cannot be allowed for in our element test. Without drainage and with $e_1 = 3.5 \cdot 10^{-6}$ s$^{-1}$ estimated from Henkel's data, the calculated stress deviator is below the observed one and the peak is at a lower $e_1$ (c). Without a detailed analysis of the nonhomogeneous deformation of the sample it may only be stated that the rough plates strengthen the system someway. Deviations of calculated and observed pore pressures (d) for $\sigma_2 = 210$ kPa are probably due to inhomogeneity. For the onset of undrained shearing, $p'/e_1 = 0$ is obtained as in experiments (this is not so with non-viscous hypoplasticity).

The calculated $q$-peak at $e_i = 6\%$ occurs at the stress ratio $K_{up} = \sigma_1 / \sigma_1 = 0.51$ which is well above $K_s = 0.42$. The $\sigma_1 / \sigma_1$ peak, i.e. for $\sigma_1 - \sigma_1 = 0$ with $e_1 = -2e_2$, can be calculated from Eqs. (16) and (17). For a given $f_2$ and $\varphi_s$, it is determined by $f$. Comparative calculations with NC samples yield $K_{up} / K_s = 1.1 \div 1.3$. This is supported by triaxial tests with uniform deformation due to short samples and non-rotating lubricated plates (Schreizer, 1991).

Taking Henkel's Fig. 4, axial loading was also back-analysed with OC samples, with $e_i = 3 \cdot 10^{-7}$ s$^{-1}$ from Henkel's data and an initial state as B in Fig. 10. With drainage the calculated peak deviator is somewhat higher than the observed one, and with about half the $e_1$ (Fig. 12(a)). The agreement is better for the volume changes (b). The lower actual strength is due to plastification and dilation near the end plates which is reflected by an $f$ close to $D / D_i$ in Eqs. (16) and (17). This effect is also visible for axial loading with drainage (c). Up to $e_1 = 2\%$ the $f$ is so small that the term with it does not influence the stiffness. The pore pressures (d) are rather well modelled. It is noted again that drainage prevents the sample from remaining uniform. A more detailed
analysis with finite elements and filtration of pore water is presently made. This will be compared with the paper by Asaoka et al. (1994).

Peak stress ratios depending on the overconsolidation ratio OCR are obtained rather close to the ones given by Henkel, but there is a conceptual difference: the conventional OCR = $\sigma_i / \sigma'$ refers to a previous consolidation pressure $\sigma_i$, e.g. prior to swelling, whereas the $p_i / p'$ in Eq. (27) refers to the actual void ratio and stress components independently of their origin. Peak deviator stresses or stress ratios can be derived from the actual void ratio, stress and strain rate instead of presuming them for a pre-peak OCR by means of $c'$ and $\varphi'$.

EXTENSIONS, PHYSICAL BACKGROUND AND LIMITATIONS

The tensorial visco-hypoplastic relation replacing Eqs. (16) and (17) reads

$$\tilde{T}_i = f_6 (L D + f_4 f_i N D_i)$$

with an objective (e.g. Jaumann, cf. Bauer, 2000) solid effective stress rate $\tilde{T}_i$ and stretching rate $D_i$ $L$ and $N$ depend on the stress direction $\tilde{T}_i = T_i / ||T||$ as without viscosity (Woltersdorff, 1996). Writing for the mean solid partial pressure $p_i = - \text{tr} T_i / 3$ instead of $p'$, the scalar factors $f_6$, $f_i$ and $f'$ are again determined by Eqs. (25) to (32). The invariant in Eq. (30) is

$$\frac{\tilde{I}_6}{\tilde{I}_2} = \frac{\text{tr} \tilde{T}_d^2 / 3 - \text{tr} \tilde{T}_s^2 / 2 + 1 / 6}{(\text{tr} \tilde{T}_s^2 - 1) / 2}.$$  

Equation (52) is needed for solving initial boundary value problems, and had already been used. As with Eqs. (16) and (17) and with Niemunis' (2002) relation Eq. (8), rapid changes of $f_i$ require numerical caution.

Ratcheting is smaller with the viscous extension than without it and can be further reduced by an intergranular strain tensor $H$ with amount $\rho = ||H||$. The evolution relation for $T_i$ can be written similarly as proposed by Niemunis (2002) so that for $\rho = 0$ the response is visco-hypoplastic, and for $\rho = 1$ Eq. (52) is obtained. An interpolation holds for $0 < \rho < 1$.

A polar extension is obtained by generalizing $\|D\|$ with polar terms (Bauer and Huang, 2000). Thus the two additional attractors for polar stresses are argotropic as the ones for non-polar stress. Relaxation of polar stresses is by several orders of magnitude faster than for the non-polar stress. This is reasonable as polar quantities represent stronger fluctuations in narrow shear zones. The numerical investigation of shear banding in soft soils is more difficult than for sand due to lower band thickness and stronger pore pressure effects.

The microphysical background may briefly be outlined. There is no doubt that all viscous effects (rate-dependence, creep and relaxation) are thermally activated. This was fully recognized by Prandtl (1928).
also gave an explanation for the pressure-independence of the coefficient of dry friction, which was independently proposed by Terzaghi (1925): the sliding resistance is proportional to the solid contact area which is proportional to the normal force.

Although Hvorslev (1937) also realized that plasticity and viscosity go together, its thermal activation is not well understood for soft soils until now. A theory of Persson (2000), which was first used for sliding friction, enables a step forward. For stationary uniaxial plastic flow (i.e. thermally activated cold melting)

\[
\sigma = \frac{\sigma_a}{2} \left[ 1 + \frac{kT}{\varepsilon_a} \ln \left( \dot{\varepsilon}/\varepsilon_a \right) \right]
\]

with the reference strain rate

\[
\dot{\varepsilon}_a = \frac{kT}{\varepsilon_a} \frac{\sigma_a}{G} \frac{c_s}{d_b}
\]

is derived. Therein \( T \) denotes the absolute temperature and \( k \) is Boltzmann’s constant, so that at ground temperature \( kT = (1/40) \text{ eV} \) (electron volt) is the thermal energy unit. \( \varepsilon_a \) is the dislocation energy of a so-called stress block of nano-size \( d_b \) with uniaxial strength \( \sigma_a \). An idealized cubical nano-size block is a substitute for a crystallite enclosed by dislocations. With the von Mises criterion

\[
\sigma_a = \frac{d_b^3 \sigma_a^2}{2G}
\]

is obtained wherein \( G \) is the shear modulus. This is the activation energy of cold shear melting. The reference strain rate \( \dot{\varepsilon}_a \) is proportional to the shear wave speed \( c_s = \sqrt{G/\rho} \). Equation (54) does not hold for \( \ln \left( \dot{\varepsilon}/\dot{\varepsilon}_a \right) > \alpha a - 5 \) as then the released energy cannot be radiated off so that \( T \) rises and \( \sigma_a \) drops. A lower bound for Eq. (54) is

\[
- \ln \left( \dot{\varepsilon}/\dot{\varepsilon}_a \right) > 2\dot{\varepsilon}_a/kT, \text{ for lower } \dot{\varepsilon}/\dot{\varepsilon}_a \text{ is linear in } \dot{\varepsilon}.
\]

This is a more precise substitute of Prandtl’s (1928) theory. For application to soft soils, I postulate

\[
p' = N c d_b^3 \dot{\varepsilon}
\]

with the number \( N_c \) of nano-size contact islands of size \( d_b \) per unit area of a wavy plane passing through solid contacts, the average contact pressure \( \sigma \) and the effective spatial mean pressure \( p' \). Equation (57) corresponds to Prandtl’s and Terzaghi’s assumption outlined above and explains the independence of \( \varphi_n \) on \( p' \), \( T \) and \( \dot{\varepsilon} \). \( p' \) is relevant (principle of effective stress) as \( \sigma_a \) is pressure-independent.

The solid hardnes \( h_i \) is evidently proportional to \( \dot{\varepsilon} \). For comparing Eqs. (7) and (54) I assume the latter to hold likewise for averages \( \dot{\sigma} \) and \( \dot{\varepsilon} \), and that

\[
\dot{\varepsilon} = \kappa_0 D
\]

holds with a kinematic factor \( \kappa_0 \) somewhere around \( 10^3 \). This leads to the activation energy

\[
\dot{\varepsilon}_a = \frac{kT}{I_1} \left[ 1 + \ln \left( \frac{kT}{\varepsilon_a} \frac{\sigma_a}{G} \frac{c_s}{d_b} \kappa_D^{-1} \right) \right].
\]

With \( \sigma_a/G \approx 10^{-2} \), \( c_s/d_b \approx 10^{12} \text{ s}^{-1} \) and \( D_i = 10^{-6} \text{ s}^{-1} \), Eq. (59) can be simplified into
\[ \varepsilon_a \approx (kT/L)(1+12L) \]  
\[ (60) \]

for the range 0.02 < L < 0.07. Equation (60) yields dislocation energies \( \varepsilon_a \) from about 1 to 3 eV for soft to hard minerals which seems to be realistic (hard steel has 5 eV). Combination of Eqs. (55) and (58) yields an objective reference strain rate

\[ D_0 = \frac{1}{\kappa_0} \frac{kT \sigma_a c_s}{G d_b} \]  
\[ (61) \]

\( D_0 \) has the order of magnitude 10^9 s^-1, so that for convenience \( D_0 = 10^{-15} D_0 \) could be used. Equation (61) is only a crude estimate and not apt for actually determining \( D_0 \), but it yields arguments for a constant \( D_0 \); \( \sigma_a / G \) and \( c_s / \varepsilon_a \) are roughly the same for different soil materials. The proportionality with respect to \( T \) was already found by Prandtl (1928) and corroborated for soils by Mitchell et al. (1968). Little can be said about \( \kappa_0 \). Fortunately comparative calculations have shown that the behaviour of quite different soils can be described with visco-hypoplasticity using a constant \( D_0 \). Persson (2000) considers also relaxation and derives a time-dependence like Eq. (47), but more specific consequences for soils could not yet be worked out. His theory can at least help to understand the viscosity index \( I_1 \) and the reference rate \( D_0 \). It cannot be used, of course, to derive the proposed constitutive relation.

Electrocapillary effects are important for soft soils, which is not directly reflected by the present theory. \( \varphi_c, h \), and \( I \) are not affected as they depend only on the solid, and presumably also \( n, \beta, \phi_0 \), and \( \alpha_c \), \( \varepsilon_\alpha, \varepsilon_\beta \) and \( \varepsilon_\phi \) are higher in the vicinity of the isoelectric point as macro-pores are then stable due to van der Waals forces. The latter cannot be represented by an internal pressure as presumed by Hvorslev (1937); this would reduce and not increase the limit void ratios. For lack of a theory of electrocapillarity (soft soils are not covered only by DLVO) one has to determine the parameters with the same ionic strength as in situ.

Capillary effects may only be mentioned here. Gas bubbles smaller than the solid particles make the pore fluid more compressible. Larger bubbles weaken the soil much more as their vicinity is far more deformed than the overall soil. This may be calculated with visco-hypoplasticity. Gas pockets and channels are not properly understood even without viscosity and electrocapillarity.

For mathematical and numerical aspects the reader is referred to Niemunis (2002). A problem is the assumed permanence of solid particles. This is properly not given: soil particles can be dis- and reintegrated.

CONCLUSIONS

Formally viscosity is introduced into hypoplasticity by replacing the factor \( D \) by \( f_D \), and by using a solid hardness \( h \), depending on the amount of strain rate \( D \). The viscosity index \( I_1 \) appears in \( f_I = \exp [(p'/p_0 - 1)/I_1] \) and in \( h = h_\alpha[1 + I_1 \ln (D/D_0)] \). \( h_\alpha \) and \( D_0 \) are reference values. The equivalent pressure \( p_e \) depends on the void ratio \( e \) as the mean effective pressure \( p_e \) for normal consolidation with \( D = D_0 \), and also on the stress ratio \( K = \sigma_1 / \sigma_\alpha \) for cylindrical symmetry. Three special cases are combined: \( K = 1 \) for isotropic compression with \( e = e_0 \), \( K = \tan^2 (45^\circ \pm \varphi_o/2) \) for critical states with \( e = e_c \), and \( K = 0 \) or \( \infty \) for \( e = e_\infty \).

Three asymptotic solutions (attractors) are obtained for deformations with constant \( D_0 \):

- proportional compression, in particular isotropic (\( e \)) or oedometric,
- critical states with constant \( e = e_0 \), depending on \( p' \),
- densification to \( e = e_\infty \) depending on \( p' \) by cycles with constant \( p' \).

This is a substitute of Critical State Soil Mechanics (CSSM). In the precursor of the present theory, Niemunis (1996) combined hypoplasticity with a Cam Clay variant, but did not incorporate the lower bound of plasticity \( e_0 \). His overconsolidation ratio OCR is similar to my \( p_0 / p' \), but without the limit \( K = 0 \) or \( \infty \) for \( e = e_\infty \). By introducing a lower bound \( f_D = 0 \) for \( p'/p_0 < \alpha \) with a constant \( \alpha = 0.5 \), my theory enables two further attractors for compressive creep and relaxation. These equilibrium states are approached in extremely long times.

The asymptotic properties enable an easy determination of the material parameters:

- rate-independent critical friction angle \( \varphi_c \),
- viscosity index \( I_1 \), correlated with \( w_\alpha \),
- reference solid hardness \( h_\alpha \) and compression index \( n \),
- reference void ratio \( e_0 \) for isotropic compression,
- reference critical void ratio \( e_\infty \),
- reference minimal void ratio \( e_\infty \), related with \( w_\alpha \).

As shown by comparative calculations for soft soils three further parameters not listed above can be fixed.

Various evolutions of state---i.e. of void ratio, effective stress and strain rate—are realistically obtained. Initial states have to be carefully defined. Several known properties are reproduced:

- the stationary creep rate increases exponentially with the stress deviator,
- volumetric creep tends to \( e_0 - I_c \ln (1 + tD) \),
- for creep with constant \( \sigma_1 \) and \( \sigma_1, \varepsilon_\alpha / \varepsilon_\phi \) is determined by \( \sigma_1 / \sigma_1 \),
- the differential stiffness, e.g., \( \sigma_1 / \varepsilon_\phi \), is markedly higher just after a sudden increase of \( D \), and lower just after a reduction of \( D \),
- relaxation tends to isotropic stress and to \( -I_1 \ln (tD) \).

Further explanations are given by means of a back analysis of Henkel's (1956) triaxial tests with Weald Clay. Changes of strain rate play a strong role for differential stiffness. This is nearly hypoelastic, i.e. determined only by stress and void ratio, just after a sudden increase of \( D \) by two or more orders of magnitude, and if \( p_0 / p' \) exceeds ca 1.3. Strength values defined by peaks depend on \( D, p_0 / p' \) and drainage conditions. They are derived realistically, so that conventional \( c' \) and \( \varphi' \) values are no more needed.

The tensorial extension of the proposed relation is straightforward and was used already for finite element calculations. Following Niemunis (2002) an intergranular strain tensor can be introduced for getting wider elastic
ranges. The incorporation of polar quantities for shear localization is straightforward. A physical interpretation is given with Peresson’s (2000) theory of thermally activated cold melting. The dislocation energy of nano-size contact islands can be realistically related with $I_c$. An objective reference strain rate is obtained. The conceptual uncertainty of flow units and activation energies in the Theory of Rate Processes is thus removed.

The proposed constitutive concept is mathematically and physically sound, realistic and feasible.

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NOTATION

$q_s$: critical friction angle for triaxial compression
$p_o/p'$: overconsolidation ratio
$n$: exponent for pressure-dependence of characteristic void ratios by Eq. (9)
$h$: solid hardness for pressure dependence of characteristic void ratios via Eq. (9)
$D_o$: value of $h$ for modulus of strain rate $D = D_o$, by Eq. (13)
$D_c$: reference strain rate, taken as $10^{-4}$ s$^{-1}$ or determined with Eq. (61)
$I_c$: viscosity index for strain rate dependence of $h$, by Eq. (13)
$e_v$: void ratio for first isotropic compression with constant strain rate
$e_0$: value of $e_v$ for $p' = 0$ by Eq. (9)
$e_2$: critical void ratio for constant volume deformation with constant strain rate
$e_{ad}$: value of $e_v$ for $p' = 0$ by Eq. (9)
$e_{as}$: asymptotic void ratio for cyclic shearing with constant $p'$ and constant strain rate
$e_{a0}$: value of $e_{a0}$ for $p' = 0$ by Eq. (9)
$D_o$: modulus of strain rate below which $h = a_0$ holds by Eq. (13)
$a_0$: relaxation factor, related with $D_o$ by Eq. (13)
$p_o$: equivalent pressure by Eqs. (28) to (32)
$f_v$: viscosity factor in the constitutive relation by Eq. (27)
$H$: strain rate dependent density factor by Eq. (33)
$e_{p0}$: exponent for $f_v$ by Eq. (25)
$e_{p0}$: exponent for $f_v$ by Eq. (26)

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