A SIMPLE CONSTITUTIVE MODEL FOR THE SEISMIC ANALYSIS OF SLOPES AND ITS APPLICATIONS

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ABSTRACT

Several constitutive models of varying complexity have been proposed for use in estimating the seismic response of slopes. The main purpose of this study is to propose and demonstrate a new 3-D simple cyclic loading model that can be applied to slope design. Both the $G - \gamma$, $h - \gamma$ relationships and the shear strength parameters $c$ and $\phi$ can be appropriately accommodated by the model. This is useful because while the $G - \gamma$ and $h - \gamma$ relationships of soil have been frequently used for the prediction of seismic response, $c$ and $\phi$ are more commonly used in conventional calculations of slope stability based on global factors of safety. Incorporating both parameterizations in a seismic design strategy is therefore likely to prove beneficial. In the first half of the paper, the motivation behind a new model and its theoretical derivation are outlined. Several applications of the model to commonly encountered slope design problems are addressed in the second half. It is shown that the proposed model is very effective in simulating the non-linear dynamic behavior of slopes during strong earthquake.

Key words: constitutive equation of soil, damping, earthquake, elasto-plasticity, finite element method, slope (IGC: E6/E8/E13)

INTRODUCTION

Constitutive models derived from mechanical theory are indispensable for obtaining modeling results with high accuracy. However, with models possessing a large number of independent parameters, it may be difficult to adequately determine all of the necessary input parameters. Accordingly, simple models based on the practicality are often effective in design. In this paper, a simple constitutive model that can be applied to the seismic design of slopes is proposed. The approach we discuss is intended to be used specifically for estimating residual slope deformation induced by earthquakes. This focus seems reasonable, because it is often slope deformation rather than catastrophic collapse that causes damage. Soil liquefaction is not addressed in this paper, although it may be one of the important research subjects, because the factors that compose liquefaction phenomena are complicated and exceed the application range of this model.

The principal constitutive elements governing the residual deformation of a slope are the seismic amplification and soil failure. The amplification caused by a slope is strongly influenced by dynamic deformation characteristics of the soil, represented most commonly, for instance, by the $G - \gamma$ and $h - \gamma$ relationships. These relationships are routinely used in seismic design codes, despite their known limitations, which have been pointed out, for example, by Yoshida (1995). Likewise, the failure of soil is typically described in terms of its shear strength, and conventional slope stability analyses based on limit equilibrium methods have usually relied on the Mohr-Coulomb parameters $c - \phi$. Given these observations, the joint use of $G - \gamma$, $h - \gamma$ relationships and the $c - \phi$ shear strength parameterization in slope design seems satisfactory.

So many researches that examine seismic behavior of soil structures on the basis of numerical analysis have been performed before. The modified Ramberg-Osgood (R-O) model (Jennings, 1964) and the modified Hardin-Drnevich model (H-D; Hardin and Drnevich, 1972) are two commonly cited examples of simple cyclic loading models. Tatsuoka and Shibuya (1992) proposed the so-called the Generalized Hyperbolic Equation Model that can change its skeleton curve more flexibly than the H-D model.

Here the fundamental characteristics of the H-D and R-O models are reviewed. The R-O model has an exponential type of hysteresis curve whose configuration changes flexibly; this makes it straightforward to fit its predicted $G - \gamma$ and $h - \gamma$ relationships to observations. However, it is impossible to describe the shear strength of soil $\tau_s$, because the skeleton curve does not have the upper bound value of a hyperbolic curve. The H-D model, on the other hand, has a hyperbolic hysteresis curve and the shear strength problem is ameliorated. However, the
Fig. 1. Modified hysteresis loops for the correction of $h-\gamma$ relationships

deficiency of the H-D model is that it does not accurately describe $h-\gamma$ relationships. In fact, by applying Masing’s rule to the H-D model, any flexibility that permits variations in the $h-\gamma$ relationship is entirely lost. In response to this inadequacy, simple $h-\gamma$ correction methods have been proposed by several researchers (e.g., Ishihara et al., 1985; Iai, 1991; Murono, 1999). A shared result of these corrections is an improvement in the fit of calculated $h-\gamma$ relationships to those observed, by reducing the original hysteresis loops. The correction methods have a common characteristic in that the unloading shear deformation modulus $G_0$ is different from the initial or purely elastic value, as shown in Fig. 1(a). This often brings about underestimation of the unloading shear deformation modulus after initial shearing as shown in the figure. Therefore, it is undesirable that the soil stiffness $G_0$ and the damping ratio $h$ depend upon each other directly. To address this point in the proposed model, the $h-\gamma$ relationships and the shear modulus $G_0$ do not depend on each other directly, as shown in Fig. 1(b), and the unloading shear modulus can be controlled independently. By using such modeling, it becomes possible to handle the influence of the initial shear stress with a more clear form.

FORMULATION OF A MODEL FOR SEISMIC ANALYSIS

Firstly, 3-D formulation of the proposed model is explained. Stress tensors used in subsequent formulations are expressed with total stress components. It should be noted that the application to more complicated problems such as liquefaction is difficult at present, because the formulations of the model are very simple. Many additional parameters should be introduced into the model, to simulate the strict dilatancy effect of loose sand that is very important for the analysis of liquefiable soils.

Definitions of Hysteresis Curves

The skeleton curve of the stress-strain relationship is described in terms of two constants, $G_0$ and $\tau_0$, with a hyperbola

$$\tau = \frac{G_0 \gamma}{1 + \frac{G_0 \gamma}{\tau_0}} \tag{1}$$

as in the H-D model. $G_0$ can be made dependent on the initial stress level,

$$G_0 = G_{0r} \left( \frac{p}{p_r} \right)^m \tag{2}$$

where $G_{0r}$, $p_r$, and $m$ are constants, $p$ is the mean principal stress. $m$ has no dimension. $G_{0r}$ and $p_r$ have the same dimension as stress. These values are determined so that the model should be fitted to the observed soil properties. Concretely speaking, it becomes $G_0 = G_{0r}$ at the time of $p = p_r$.

The shear strength $\tau_0$ in Eq. (1), defined as the interception of the asymptote of the hyperbola, is given by

$$\tau_0 = c \cdot \cos \phi + \frac{\sigma_1 + \sigma_2}{2} \sin \phi \tag{3}$$

The ratio $\tau_0/G_0$, referred to as $\gamma_0$, in this paper, is termed the reference strain. The maximum shear stress $\tilde{\tau}$ and strain $\tilde{\gamma}$ in 3D formulations are expressed in terms of tensor invariants as

$$\tilde{\tau} = \sqrt{J_2} \sin \left( \frac{\pi}{3} + \theta \right) \tag{4a}$$

$$\tilde{\gamma} = 2 \cdot \sqrt{J_2} \sin \left( \frac{\pi}{3} + \theta^* \right) \tag{4b}$$

where

$$\theta = \frac{1}{3} \cos^{-1} \left( \frac{3 \sqrt{2} J_3}{2 J_2^{3/2}} \right) \quad \left( 0 < \theta < \frac{\pi}{3} \right) \tag{4c}$$

$J_2$ and $J_3$ are second and third invariants of deviatoric stress tensors, respectively. The superscript asterisks in Eq. (4b) denote strain invariants.

The representative shear stress and strain, $\tau$ and $\gamma$, used in Eq. (1) are defined as variables that show the difference from the initial reference point to the present point on the $\tau-\gamma$ coordinate. $\tilde{\gamma}$, corresponds to the initial strain.

$$\tau = \tilde{\tau} (\tilde{e}_i) \tag{5a}$$

$$\gamma = \tilde{\gamma} (\tilde{e}_i - \tilde{e}_{i0}) \tag{5b}$$

On the other hand, unloading on the skeleton curve corresponds to $\gamma < 0$, at which point the stress state moves along the hysteresis loop given by

$$\tilde{\tau} = \frac{G_0 \tilde{\gamma}}{1 + b \tilde{\gamma}} \tag{6}$$

$G_0$, $b$ and $n$ are constants and $a$ is dependent on other
variables, as seen in Eq. (8). Even if \( b \) and \( n \) change, the initial inclination of the curve can be kept constant at the value of \( G_b \), as seen in Fig. 1(b). Therefore, the elastic shear modulus \( G_b \) is not dependent on the shear strain level \( \gamma \). It should be noted that Eq. (6) becomes consistent with a simple hyperbolic equation such as Eq. (1) only when \( b = 0.5 / y_0 \). Under those conditions, the proposed model is equivalent to the H-D model, as seen in APPENDIX.

In an analogous manner to the skeleton curve, the representative shear stress and strain used in Eq. (6), \( \dot{\tau} \) and \( \dot{\gamma} \), are defined as

\[
\dot{\tau} = \tau (\sigma_{ij} - \sigma_{ijb}) \quad (7a)
\]

\[
\dot{\gamma} = \gamma (\epsilon_{ij} - \epsilon_{ijb}) \quad (7b)
\]

where \( \sigma_{ijb} \) and \( \epsilon_{ijb} \) always agree with the last unloading point. In this paper, these are called as the reference stress and strain tensors, respectively.

Figure 2 illustrates the conceptual figures for the hysteretic loops. In fact, this is not expressing strict physical relationships, because the values of \( \dot{\tau} \) and \( \dot{\gamma} \) are always unnegative as Eqs. (4a) and (4b) show. Only the definition of the reversal of loading direction and the outlines of hysteretic loops should be paid attention here. The point that is equal to the values of \( \sigma_{ijb} \) and \( \epsilon_{ijb} \) is the reference point \( (\gamma_{b}, \tau_{b}) \) indicated in Fig. 2(a). \( (\gamma_{b}, \tau_{b}) \) is given with \( (\dot{\gamma} (\epsilon_{ijb}), \dot{\tau} (\sigma_{ijb})) \). The hysteretic loop is connected to the goal point located on the opposite side of the skeleton curve, represented as \( (\gamma_{b}, \tau_{b}) \) in Fig. 2(b). If the symmetry to the origin is considered, \( (\gamma_{b}, \tau_{b}) \) is given with \( (\gamma_{b}, -\tau_{b}) \). The value of \( a \) in Eq. (6) is calculated using Eq. (8) based on the condition that the current loop passes through the two points denoted as \( (\gamma_{b}, \tau_{b}) \) and \( (\gamma_{b}, \tau_{b}) \).

\[
\begin{align*}
\gamma_{b} &= \frac{1}{\gamma_{b}} (\gamma_{b} + \gamma_{0} \gamma_{b}) \quad (8)
\end{align*}
\]

where

\[
\begin{align*}
\gamma_{b} &= |\gamma_{b} - \gamma_{b}| \\
\gamma_{b} &= |\gamma_{b} - \gamma_{b}|
\end{align*}
\]

If the loading direction is reversed again, i.e., \( \dot{\gamma} < 0 \), before the skeleton curve is reached, the present point becomes the new reference point \( (\gamma_{b}, \tau_{b}) \), as shown in Fig. 2(c). The new values of \( \gamma_{b} \) and \( \tau_{b} \) are defined as the difference in \( (\gamma, \tau) \) between the new and old reference points as illustrated in Fig. 2(d), and the value of \( a \) is updated. This procedure is performed each time the loading direction is reversed.

During the calculation, the hysteretic loop needs to be updated. Because the shape of a hysteretic loop on the \( \tau - \gamma \) plane is prescribed from the geometrical idea like Fig. 2, the dramatic change of the stress level brings about the instability of numerical calculation. The change of the stress level influences the change of the shape of the hysteretic curve directly. \( \gamma_{b} \) is assumed to change in proportion to \( \tau_{b} \) as shown by Eq. (10b), because \( \gamma_{b} \) is the parameter that operates the size of the area enclosed by the skeleton curve on the \( \tau - \gamma \) plane in the present stress level. On the other hand, the \( G - \gamma, h - \gamma \) relationships should be operated freely in an optional stress level. To simplify the definition procedure of the \( G - \gamma, h - \gamma \) relationships, the reference stress tensor \( \sigma_{ijb} \) is defined on the \( \tau - \gamma \) plane in the present stress level. The necessary formulas are as follows:

\[
\begin{align*}
\sigma_{ijb} &= \delta_{ij} \rho \\
\tau_{b} &= \frac{\tau_{b}}{\tau_{b}} \tau_{b} \\
\frac{b (\gamma_{0} \gamma_{b})}{\gamma_{c}} &= \frac{b (\gamma_{0} \gamma_{b})}{\gamma_{c}}
\end{align*}
\]

The reference point of the skeleton curve is adjusted so that the hysteretic loop is connected to the skeleton curve continuously as shown in Fig. 3. The renewed reference strain used in Eq. (5b) is given with the following equation.
\[ e_{ij} = \left( 1 - \frac{\ddot{\tau}(\sigma_{ij})}{G_0 \gamma (1 - \dot{\tau}(\sigma_{ij})/\tau_{i})} \right) e_{ij} \] (11)

It may be the inclination of modeling close to Pyke's hysteresis rule (1979), in the meaning such that the skeleton curve is always defined on the basis of the difference from the shear strength line.

**Elasto-plastic Formulations**

Elasto-plastic formulations are necessary to perform the calculation under the condition of multidimension. The relationship between increments of stress and strain can be expressed as follows:

\[ \sigma_{ij} = \begin{bmatrix} D_{ijkl} \frac{\partial g}{\partial e_{kl}} - \frac{\partial f}{\partial \sigma_{kl}} D_{mkl} \delta_{ij} \\ -\frac{\partial f}{\partial H} D_{ijkl} \frac{\partial g}{\partial e_{kl}} + \frac{\partial f}{\partial e_{kl}} D_{ijkl} \frac{\partial g}{\partial \sigma_{kl}} \end{bmatrix} \] (12a)

\[ D_{ijkl} = \frac{E}{(1 + \nu)(1 - 2\nu)} \delta_{ij} \delta_{kl} + \frac{E}{2(1 + \nu)} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \] (12b)

\[ G_0 = \frac{E}{2(1 + \nu)} \] (12c)

\[ \begin{align*}
H &= \gamma = \gamma(e_{ij} - e_{ij}) & \text{on skeleton curve} \\
H &= \gamma = \gamma(e_{ij} - e_{ij}) & \text{on hysteresis loops} 
\end{align*} \] (12d)

\( E \) and \( \nu \) are Young's modulus and Poisson's ratio, respectively. \( H \) is a hardening parameter as shown by Eq. (12d). \( f \) and \( g \) are yield function and plastic potential, respectively.

\( f \) for the skeleton curve and the hysteresis loops are defined as the difference between the right and left sides of Eqs. (1) and (6), respectively.

\[ f = \dot{\tau}(\sigma_{ij}) - \frac{G_0 \gamma(e_{ij} - e_{ij})}{1 + \nu} \] for skeleton curve (13a)

\[ f = \ddot{\tau}(\sigma_{ij} - \sigma_{ij}) - \frac{a \gamma(e_{ij} - e_{ij})}{1 + \nu} + \frac{G_0 \gamma(e_{ij} - e_{ij})}{1 + \nu} \] for hysteresis loops (13b)

The details of the loading criteria will be described again. The pure elastic range does not exist in this model. The loading continues while being \( \gamma > 0 \) on the skeleton curve. It is neutral at the time of \( \gamma = 0 \). On the other hand, immediately after \( \gamma < 0 \), the hysteresis path transfers to the hysteresis loop and the loading continues while \( \gamma > 0 \). A new hysteresis loop is defined at the time of \( \gamma < 0 \), and the definition of \( \dot{\gamma} \) is renewed after that.

One of two types of plastic potential functions \( g \) can be incorporated into the model. Although the consideration of the dilatancy characteristics is always important, the dilatancy effect may be disregarded for simplification sometimes in design. For example, the total stress analysis under the constant volume condition is often applied to the dynamic response analysis of a saturated clayey ground. In those cases, a von Mises-type plastic potential is employed usually. Consequently,

\[ g = \sqrt{J_2(\sigma_{ij})} \] for skeleton curve (14a)

\[ g = \sqrt{J_2(\sigma_{ij} - \sigma_{ij})} \] for hysteresis loops (14b)

where \( J_2(\sigma_{ij}) = J_2 \). As mentioned before, the simulation of liquefaction phenomena based on this model is difficult, because the total stress formulations are adopted and the softening caused by the increase of pore water pressure is not considered.

On the other hand, the consideration of the dilatancy effect is relatively indispensable in the analysis of the ground consisting of unsaturated soil. It is similar even in the case of sand under drained conditions. In these cases, an alternative plastic potential functions \( g \) is required. Here the conventional Rowe's stress-dilatancy relationship (Rowe, 1962) is adopted, for which

\[ \frac{\sigma_1}{\sigma_3} = K_v \left( 1 - \frac{e_p^V}{e_p^V} \right) \] (15)

\( e_p^V \) is plastic volumetric strain. \( K_v \) is a constant that corresponds to the stress ratio at the phase transformation. The plastic potential \( g \) is formulated on the basis of Eq. (15). According to such a viewpoint that Mohr-Coulomb's failure criterion supports, the intermediate stress component is not used in formulations, due to the simplification. Thus, Eqs. (16a) and (16b) are adopted as the plastic potentials in accordance with the study by Barden (1969).

\[ g = \frac{\sigma_1}{\sigma_3} \] for skeleton curve (16a)

\[ g = \frac{\sigma_3}{\sigma_3} \] for hysteresis loops (16b)

The principal axes of stress increment are assumed to be reversed simply at the time of unloading. In other words, the direction of \( \dot{\sigma} \) before unloading becomes the direction of \( \dot{\sigma} \) immediately after unloading. That is, these positions in Eq. (15) reverse. Such an interpretation has already been mentioned by Pradhan et al. (1989a) in their experiments. In the proposed model, the following assumptions are introduced in reference to their discussions.

\[ \begin{align*}
\dot{\sigma}_1 &= \dot{\sigma}_1(\sigma_{ij} - \sigma_{ij}) + \dot{\sigma}_1(\sigma_{ij}) \\
\dot{\sigma}_3 &= \dot{\sigma}_1(\sigma_{ij} - \sigma_{ij}) + \dot{\sigma}_1(\sigma_{ij}) 
\end{align*} \] (17)

where \( \dot{\sigma}_1(\sigma_{ij}) = \sigma_1 \), \( \dot{\sigma}_1(\sigma_{ij}) = \sigma_1 \). Again, \( \sigma_{ij} \) is the stress values at the time of the last unloading point.

Because the direction of the principal stresses changes continuously during the history, Eq. (17) should be recognized as only the first approximation of the phenomena. Incidentally, these assumptions do not contradict with other formulations in the proposed model, because the relative stress values to the components at the last unloading point are used in the definition of the hysteresis loop as well.

**Demonstration of the Description Performance of Dilatancy Effect**

To investigate how well the model can simulate empiri-
Table 1. Material constants for the simulations of torsional simple shear tests

<table>
<thead>
<tr>
<th></th>
<th>( E ) ([G_0]) (kPa)</th>
<th>( \nu )</th>
<th>( c ) (kPa)</th>
<th>( \phi ) (deg)</th>
<th>( K_v )</th>
<th>( b \cdot \gamma_{c0} )</th>
<th>( n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loose</td>
<td>150000 (57700)</td>
<td>0.30</td>
<td>0.35</td>
<td>3.5</td>
<td>0.60</td>
<td>1.2</td>
<td></td>
</tr>
<tr>
<td>Dense</td>
<td>180000 (69200)</td>
<td>0.30</td>
<td>0.50</td>
<td>3.5</td>
<td>0.80</td>
<td>1.5</td>
<td></td>
</tr>
</tbody>
</table>

Fig. 4. Simulations for drained torsional simple shear tests of Toyoura sand

(a) Experiment (after Pradhan et al. 1989)  
(b) Proposed model

Table 1 lists the material constants for the simulations of torsional simple shear tests. The table includes the modulus of elasticity \( E \) \([G_0]\), Poisson's ratio \( \nu \), friction angle \( \phi \), critical shear strain \( \gamma_{c0} \), and a parameter \( n \). The values are given for loose and dense Toyoura sand.

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The damping characteristics are represented by the parameters \( b \) and \( n \). These parameters are determined based on the observed \( h - \gamma \) relationships due to changes in \( b \) and \( n \), and the hysteretic loops are illustrated in Fig. 5. To facilitate comparison between cases with different shear strain amplitudes \( \gamma_{a} \), the values of the vertical axis in the hysteresis loop figures have been normalized. It can be seen that increases in \( b \) or \( n \) decrease \( h \). In cases where \( b \cdot \gamma_{c0} = 0.5 \), the result is that \( h - \gamma \) curve corresponds to that of the H-D model, as seen in APPENDIX.

A reference chart for determining the parameters \( b \) and \( n \) has been prepared for convenience, as shown in Fig. 6. The solid and dotted curves show the value of \( h \) at strain amplitudes of \( \gamma = \gamma_{c0} \) and \( \gamma = 10\gamma_{c0} \), respectively.

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**APPLICATIONS TO TYPICAL PROBLEMS**

**1-D Shear Vibration Cases (Undrained Total Stress Analysis)**

Several test cases using the dynamic finite element method have been investigated to verify the accuracy of the proposed model, and elucidate its essential characteristics.

\[
[M] \{\ddot{u}\} + [C] \{\dot{u}\} + \{P\} = -[M]\{\ddot{U}\} \tag{18}
\]

where \( \{P\} \) is the nodal force vector equivalent to total stress acted inside each element. If the system behaves as an elastic body, \( \{P\} = [K]\{u\} \). \([M]\), \([C]\) and \([K]\) are the mass, damping and stiffness matrices, respectively. \( \{u\} \) and \( \{U\} \) are the relative displacement vector at each node.
Fig. 6. Chart to determine the parameters $b$ and $n$

Fig. 7. 1D shear vibration problem

and the absolute displacement vector at the base. Each matrix and vector are turned separation on the basis of finite element method. Newmark’s $\beta$ method is used to achieve time integration of Eq. (18). Such an analytical model is applied in not only 1D analyses shown in this paragraph but also in 2D analyses in the following paragraphs.

The first configuration considered is that depicted in Fig. 7, where a 10 m-thick homogeneous soil layer overlies (elastic) bedrock. In order to incorporate wave propagation in an infinite media, a viscous boundary (Lysmer and Kuhlemeyer, 1969) is inserted at the soil-bedrock interface. Seismic shaking is represented by a sinusoidal horizontal acceleration with various frequencies, duration of 20 cycles and an amplitude of 2.0 m/s$^2$. The effect of viscous damping based on Rayleigh damping is adopted. That is,

$$[C] = \alpha[M] + \beta[K]$$  \hspace{1cm} (19)

The parameters for Rayleigh damping ($\alpha=0.172$ and $\beta=0.00174$) are chosen so that $h$ due to viscous damping is temporarily maintained at approximately 3%.

Material properties used in each case are listed in Table 2. In this case, undrained and unliquefiable conditions are assumed and von Mises-type plastic potential is employed, then, $K_0$ is neglected. The above assumptions with the total stress analysis are often used
Table 2. Material constants for the simple ground and slope analyses

<table>
<thead>
<tr>
<th>Model</th>
<th>$E$ [GPa] (kPa)</th>
<th>$v$</th>
<th>$c$ (kPa)</th>
<th>$\phi$ (deg)</th>
<th>$K_{ov}$</th>
<th>$b \cdot \gamma_{ov}$</th>
<th>$n$</th>
<th>$\gamma_1$ (kN/m$^3$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proposed</td>
<td>29800 [10000]</td>
<td>(0.49)</td>
<td>100.0</td>
<td>0</td>
<td>1.60</td>
<td>1.60</td>
<td>18.0</td>
<td></td>
</tr>
<tr>
<td>HD</td>
<td>29800 [10000]</td>
<td>(0.49)</td>
<td>100.0</td>
<td>0</td>
<td>18.0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>RO</td>
<td>29800 [10000]</td>
<td>(0.49)</td>
<td>100.0</td>
<td>0</td>
<td>$\alpha = 207^\circ, \beta = 1.78$</td>
<td>18.0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Rayleigh damping

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.172</td>
<td>0.00174</td>
</tr>
</tbody>
</table>

Viscous boundary (only for 1D analysis)

<table>
<thead>
<tr>
<th>Impedance ratio $\kappa$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0849</td>
</tr>
</tbody>
</table>

Fig. 9. Calculated acceleration response at ground surface

Fig. 10. 2D simple homogeneous slope

(a) Skeleton curves ($\tau - \gamma$ relationships)

(b) $G / G_0 - \gamma$ relationships

(c) $h - \gamma$ relationships

Fig. 8. Hysteresis characteristics of each model

in design, although the judgment of the adoption of the assumptions should be done prudently. The initial shear stress in the ground is assumed to be zero in this case. $G_0$ is assumed to be invariable during cyclic loading. The cases of the H-D and R-O model are also analyzed for the purpose of comparative examination. As those models and the proposed model show a nearer mechanical characteristics, the adjustment of material constants is carried out. The hysteresis characteristics of the soil are illustrated in Figs. 8(a), (b) and (c). In this case, the proposed model agrees with the H-D model in terms of its skeleton curve, i.e., $\tau - \gamma$ relationships, while it agrees with the R-O model in the $h - \gamma$ relationships. The H-D and R-O models cannot simulate the mechanical properties perfectly that can be explained by the proposed model.

Figure 9 shows the calculated acceleration response at the ground surface. It is found that the first and second natural vibration periods are approximately 0.9 s and 0.2 s, respectively. The H-D model predicts a smaller ground surface response, while the proposed and R-O models agree more closely and predict a larger response of up to 5.5 m/s$^2$. This suggests that the $h - \gamma$ relationships have a large influence on the acceleration response near the first resonance frequency.

2-D Simple Slope Cases (Undrained Total Stress Analysis)

Analysis of a linear 2-D slope, as shown in Fig. 10, has also been carried out using soil parameters the same as in the previous analyses as seen in Table 2. The undrained and unliquefiable conditions are assumed and the total stress method is applied. Based on the volume constant condition, $v$ is taken as 0.49. $E$ (or $G_0$) is assumed to be constant during the seismic shaking and von Mises-type plastic potential is used. The input horizontal acceleration consists of 10 sinusoidal waves with an amplitude of 2.0 m/s$^2$. The calculation of the initial stress distribution was based on the shear strength reduction method (Zienkiewicz et al., 1975).

Variations in the residual displacement as a function of
frequency of input waves are shown in Fig. 11. The R-O model predicts a smaller response to that suggested by the proposed and H-D models, whose results match closely. This suggests that the shear strength $\tau$ has a large influence on the residual displacement of slope after an earthquake.

The results of the simple 1-D and 2-D analyses can be summarized as follows. In cases where the acceleration response is important, the $h - \gamma$ relationships is indispensable as shown by the 1-D example. On the other hand, in problem where the plastic deformation is vital, the shear strength is indispensable as shown by 2-D example. The proposed model is equipped with the partial characteristic of both the H-D and R-O model.

APPLICATIONS TO VARIOUS SLOPE ANALYSES

Content of the Case Studies

The explanation with regards to the fundamental characteristics of the proposed model has been covered in foregoing chapters. In this chapter, the appropriateness of the proposed model is examined through the FE simulations of model experiments and field measurements. The comparisons with other simple constitutive models are also performed on the basis of the estimated residual displacement of the slope after an earthquake.

The modeling of actual phenomena is summarized in the following manner. There are (a) the total collapse with a slip surface, (b) the large deformation caused by non-liquefaction mechanism without a clear slip surface and (c) the large deformation with liquefaction of the ground, as the rough classification regarding the typical types of embankment failure induced by earthquake. The consideration of the shear strength of soil is necessary for the simulation of the type (a), because most of the residual displacement is produced by the movement of the sliding body. In addition, the strain-softening characteristics of soil and the shear band effect may be also significant to improve the prediction accuracy. The R-O model is unsuitable for such an analysis clearly, as indicated in the foregoing chapters. On the other hand, the phenomenon of the type (b) is often observed on the occasion of relatively smaller deformation in comparison with the type (a). In such cases, most of the residual deformation is produced by the accumulation of plastic deformation of soil as a continuum body. The estimation of the volume change due to the dilatancy effect of soil is very important. And as it has been explained already, the type (c) is not able to be treated by the proposed model. The characteristics of liquefiable soil must be considered in such cases.

Each analytical example corresponding to the type (a) and (b) is shown in this chapter. The former is the simulation of centrifuge test of a simple slope consists of silty clay, and the latter is the simulation of the real failure of an embankment at the time of Hyogo-ken Nanbu Earthquake, respectively. The comparison of the results based on the proposed model and the H-D model is performed in these two cases. After that, the simulation of a series of the centrifuge model tests of an embankment on the weak
Table 3. Material constants for the simulations of a centrifuge test of a simple slope

<table>
<thead>
<tr>
<th>Material</th>
<th>$E/\gamma_0$ (kPa)</th>
<th>$\gamma$</th>
<th>$c$ (kPa)</th>
<th>$\phi$ (deg)</th>
<th>$K_{cv}$</th>
<th>$b - \gamma_0$</th>
<th>$n$</th>
<th>$\gamma_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Silt</td>
<td>17000 [6071]</td>
<td>0.40</td>
<td>11.1</td>
<td>33.3</td>
<td>3.24</td>
<td>8.0</td>
<td>1.5</td>
<td>15.7</td>
</tr>
</tbody>
</table>

Rayleigh damping

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.172</td>
<td>0.00174</td>
</tr>
</tbody>
</table>

clayey ground with soil improvement is shown as an example of the application to the actual design. These details are shown in the following paragraphs.

Case of Total Failure with Clear Slip Surface

The first case is a failure with a clear slip surface. Sato et al., which is summarized by Wakai et al. (2001b), performed a shaking table test of a slope under centrifugal acceleration of 20G. In the experiment, the slope was made of homogeneous unsaturated silty clay (DL clay). The prototype height and inclination of the slope were 7.0 m and 45°, respectively. The input acceleration is close to regular sinusoidal waves whose amplitude gradually increases.

Table 3 shows the soil parameters used in the analysis. $E$ was assumed to be $2 \times E_{50}$ on the basis of a hyperbolic stress-strain curve that is adopted in the proposed model. $E_{50}$ was determined by triaxial tests. The stress dependency of $E_{50}$ is not so obvious in this time, and $G_{dr}$ is assumed to be constant during and after an earthquake. The above assumption is used even in the remaining exercises. Strictly speaking, of course, the stress dependency of $G_{dr}$ during an earthquake exists. The dependency expressed with Eq. (2) should be considered in the cases with sufficient soil data according to this matter. $v$ was assumed to be 0.4. The strength parameters $c$ and $\phi$ were determined based on the peak strength of triaxial tests. The strainsoftening characteristics of soil observed in the triaxial tests actually are not considered in the analysis. The residual strength is significant to improve the prediction accuracy and this may be a future subject. The value of $K_{cv}$ was measured as the ratio of $(\sigma_1/\sigma_3)$ and $(-2\varepsilon_r/\varepsilon_r)$ in triaxial compression tests directly. The parameters for Rayleigh damping $\alpha$ and $\beta$ are chosen so that $h$ due to viscous damping is maintained at approximately 3%, although this is an assumption. $b\gamma_{cv}$ and $n$ were determined as the analytical $h - \gamma$ curves fit with the measurements in undrained cyclic triaxial compression tests shown in Fig. 12, on the basis of trial and error. Figure 6 is convenient in this process. It is seen that the two curves agree well with each other. Because the soil is silty, the influence of the rate effect is thought to be very small.

In the analysis, the calculation of the initial stress distribution was based on the shear strength reduction method. The shaking table acceleration history measured at the time of the experiment was used as the input waves in the analysis. The observed and calculated residual deformations after shaking are shown in Fig. 13. The overall pattern of deformation of the observed results is simulated extremely well by the numerical analysis. The slip surface, which is represented in the model as the area of highest shear strain, is clearly observed in the slope. Figure 14 shows the time histories of the input acceleration, and resulting vertical and horizontal displacements near the top of the slope. In Fig. 14(b), the analytical results based on the H-D model are shown as well, to compare with the proposed model results. Because the plastic potential of both models made the same in this case, the difference of the assumptions for the hysteresis loops brings about all of the difference of the calculated results. A good agreement between the analysis and the test can be seen in the figure. The agreement of the final displacement values is very important from a design point of view. On the other hand, as for the results based on the H-D model, the residual displacement is evaluated to a little smaller value, which is a result of the dangerous side in the design. This is because the damping ratio was overestimated. It is concluded that the proposed model is effective in the simulation of failure processes that include strain localization and total slope failure.

Surface Failure of an Embankment with Large Settlement

As a second example of our numerical model's applicability to various slope analysis problems, the real failure of railroad embankment in the city of Kobe at the time of Hyogo-ken Nanbu Earthquake is considered. The details of the damage have been reported by Tateyama (2000). The configuration of the embankment is shown in Fig. 15. 2D plane strain analysis was adopted here. The upper part, which the railway tracks rest on, consists of sandy soil (Layer B), and both side ends are supported with concrete retaining walls. A cohesive soil layer (Layer Ac) lies beneath the upper part.

The material parameters used in this analysis are shown in Table 4. Most of these were determined from laboratory tests. $E$ of Layer B is assumed to be proportional to the square root of the initial confining pressure, and also assumed to be constant during the earthquake. $\phi$ of Layer B was determined based on the peak strength of
(a) Centrifuge test performed by Sato et al. (referred Wakai et al. 2001b)

(b) Calculated residual deformation after earthquake

Fig. 13. Simulation for centrifuge test of a simple slope

(a) Input horizontal acceleration history

(b) Displacement at the top of the slope

Fig. 14. Comparison of time histories of displacement

The residual deformation of the embankment as sketched during the subsequent repair work (after Tateyama, 2000) is reproduced in Fig. 17(a). Settlement of about 50 cm to 1 m is observed in the upper part of the embankment. Figures 17(b) and (c) show the calculated residual deformation of the system and the distribution of shear strain after the earthquake, respectively. It is
Fig. 15. FE mesh for the simulation of a damaged railroad embankment

(a) Soil in Layer B
(b) Soil in Layer Ac

Fig. 16. Dynamic deformation characteristics of soils (after Tateyama, 2000)

<table>
<thead>
<tr>
<th>Material</th>
<th>$E [G_0]$ (kPa)</th>
<th>$v$</th>
<th>$c$ (kPa)</th>
<th>$\phi$ (deg)</th>
<th>$K_\phi$</th>
<th>$b \cdot \gamma_0$</th>
<th>$n$</th>
<th>$\gamma_r$ (kN/m$^3$)</th>
</tr>
</thead>
</table>
| B        | $E = 2(1 + v) \cdot G_0$, \[
\sigma_0 = \frac{\sigma_1 + \sigma_3}{2}\right]^{0.5}
G_0 = 1500$ kPa; $\sigma_1, \sigma_3$ (kPa); Initial stresses.|
|          | 0.30           | 0   | 36.      | 3.5         | 1.8     | 2.4            |    | 16.1              |
| A         | 96040 [34300]  | 0.40 | 70.      | 0           | (Undrained) | 5.0     | 2.0 | 17.4              |
| Walls     | $3.0 \times 10^7 [1.3 \times 10^7]$ | 0.13 | $10^{28}$ | —           | —       | —              |    | 18.0              |

Rayleigh damping

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.172</td>
<td>0.00174</td>
</tr>
</tbody>
</table>

noted that the surface of the embankment slid right and left during the earthquake. The retaining walls lean slightly. These results are consistent with field observations. In Fig. 17(b), the calculated residual displacement values based on the H-D model are described as well, to compare with the proposed model results, similar to the previous analytical case. As shown in the figure, the residual displacement given by the H-D model is a little smaller in value than the one given by the proposed model. The proposed model is conceivable as a safer side
in design.

Since the proposed model is very simple, many of complicated physical properties for soils are disregarded in the analysis. Unlike the simulations of model experiments, the analysis of a field case requires many additional assumptions for modeling. It is thought that complicated mechanical behaviors and inhomogeneous nature of soils brought about errors. When the above is considered, the agreement between the observed and calculated deformation is good, suggesting that the proposed model is effective in modeling real embankment failure.

Case of the Evaluation of Soil Improvement under the Slope

Finally, a series of dynamic centrifuge tests of an embankment performed by Tamoto et al. (1997) are considered. The embankment consists of a mixture of silt and sand constructed on soft clayey ground. The centrifugal acceleration of the tests was 50G. The input acceleration is again a sinusoidal wave. As shown in Fig. 18, three cases are considered; without and with one of two types of soil improvement (M1, M2 and M3). The soil improvement considered here is the application of cement. Each case is composed of four successive steps of experiments which correspond to separate vibration episodes. The amplitude of the input waves gradually increases every each step. The settlement at the top of the slope was measured after each step.

Table 5 shows the soil parameters used in the simulations. Because cyclic loading tests regarding the embankment and the improved soil were not carried out, they are assumed as the elasto-perfectly plastic body where
Table 5. Material constants for the simulations of a series of centrifuge tests of an embankment with and without ground improvement

<table>
<thead>
<tr>
<th>Material</th>
<th>Model</th>
<th>$E$ [$G_0$] (kPa)</th>
<th>$\nu$</th>
<th>$c$ (kPa)</th>
<th>$\phi$ (deg)</th>
<th>$K_s$</th>
<th>$b\cdot\gamma_m$</th>
<th>$n$</th>
<th>$\gamma$ (kN/m$^3$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Clay</td>
<td>Proposed</td>
<td>$E=2(1+\nu)\cdot G_0$</td>
<td>0.49</td>
<td>$c=0.520\times\left(\frac{\sigma_1^c+\sigma_3^c}{2}\right)$</td>
<td>0</td>
<td>(Undrained)</td>
<td>0.80</td>
<td>0.30</td>
<td>7.4</td>
</tr>
<tr>
<td>Embankment</td>
<td>*1</td>
<td>2000. [714.3]</td>
<td>0.40</td>
<td>13.0</td>
<td>6.7</td>
<td></td>
<td></td>
<td></td>
<td>16.7 (\gamma)</td>
</tr>
<tr>
<td>Sand</td>
<td>Proposed</td>
<td>20000. [7143.]</td>
<td>0.40</td>
<td>0.0</td>
<td>40.</td>
<td>3.5</td>
<td>0.80</td>
<td>0.30</td>
<td>7.4</td>
</tr>
<tr>
<td>Improved soil</td>
<td>*1</td>
<td>123600. [41757.]</td>
<td>0.49</td>
<td>235.</td>
<td>0.0</td>
<td></td>
<td></td>
<td></td>
<td>5.3</td>
</tr>
</tbody>
</table>

Rayleigh damping

<table>
<thead>
<tr>
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</tr>
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</table>

*1: Elasto-perfectly plastic model

material parameters are easy to be determined by monotonic loading tests. Mohr-Coulomb's equation is adopted to those failure criterion. As for the clay, undrained condition is assumed and von Mises-type plastic potential is employed. $\nu$ was assumed to be 0.49. Undrained shear strength ratio of the clay ($c_s/p'=0.520$) was measured by triaxial compression tests. In addition, $G_0/c_s=516$ was assumed in reference to Ishihara (1976). Here, $\phi=0$ and $c=c_s$ are used. $b\gamma_m$ and $n$ for the clay were determined based on the observed $h-\gamma$ relationships by utilizing the chart in Fig. 6. Because it seems that the relative importance of the physical properties of the sand in this centrifuge model is low, the parameters of the sand were determined in reference to the past data for sand of the same kind at this time. As for Rayleigh damping parameters $\alpha$ and $\beta$, the same values as in the previous example are adopted.

In the analysis, the calculation of the initial stress distribution was based on the shear strength reduction method. The shaking table acceleration history measured at the time of the experiment was used as the input waves in the analyses. The accumulated residual displacement at each step is shown in Fig. 19. The horizontal and vertical axes in the figure denote the amplitude of the input sinusoidal waves and the accumulated settlement at the top of the embankment, respectively. Although the residual displacement in each case is larger than the corresponding measurements, relation between each case can be well simulated by the FEM. The difference between experiments and analyses is thought to be caused by the rate effect of the clay in the centrifuge test. This may be a subject that should be studied from now on. It should be noted that the results in Cases M1 and M2 are similar to each other, while the results in Case M3 gives a much smaller residual displacement. This suggests that the effects of the soil improvement in Case M2 were small. The residual deformation after earthquake in each case can be evaluated by the proposed model and the optimum depth of soil improvement can be determined by the analyses.

CONCLUSIONS

The important results obtained in this study are summarized as follows:

1. A simple 3-D cyclic loading model that can be applied to the seismic design of slope was proposed. Only seven input parameters are necessary for the proposed model. The seismic design of slope based on the allowable displacement is being introduced, and the proposed model was proved to respond to it.

2. The proposed model incorporates both $G-\gamma$ and $h-\gamma$ relationships and shear strength parameters $c, \phi$, which are indispensable in the design of slope. Based on such a simple concept that is easy to understand, a dramatic change in the existing design system is avoided, and it brings about the improve-
ment of prediction accuracy in design.

(3) In cases where \( b \cdot \gamma_{\text{cr}} = 0.5 \), the resultant \( h - \gamma \) curve of the proposed model corresponds to that of the H-D model. It is very easy to control the analytical \( h - \gamma \) relationships in the proposed model.

(4) The proposed model is equipped with the partial characteristic of both the H-D and R-O model. In cases where the acceleration response is important, the proposed model gives a result close to the R-O model. On the other hand, in cases where the plastic deformation is important, the proposed model gives a result close to the H-D model.

(5) The \( h - \gamma \) relationships and the shear modulus \( G_0 \) do not depend on each other directly in the proposed model. By using such modeling, it becomes possible to handle the influence of the initial shear stress with a more clear form.

(6) The undrained or drained condition can be calculated by the model. In cases where drained condition is assumed, the continuous increase of volumetric contraction observed in the loose sand is well simulated. On the other hand, the alternate occurrences of the volume contraction and expansion in the dense sand are also well predicted by the model.

(7) As shown by a variety of examples, including cases of total failure with clear slip surface, of surface failure of an embankment with large settlement, and of the evaluation of soil improvement under the slope, the proposed model is very effective in simulating the dynamic non-linear behavior of slopes during strong earthquake shaking.

(8) The residual displacement based on the H-D model is evaluated to a little smaller value than the one based on the proposed model. This is because the damping ratio is overestimated in the H-D model.

(9) The dynamic elasto-plastic FEM makes it possible to evaluate the dynamic response of slopes by considering appropriate stress-strain relationships. If a reasonable constitutive model is used with FEM, the residual displacement of slopes induced by an earthquake can be predicted precisely.

ACKNOWLEDGEMENTS

We wish to express our deep gratitude to several researchers for their generous cooperation in relation to this study. We would particularly like to thank Dr. Masayoshi Sato of the Japan National Research Institute (formerly of Shimizu Corporation) and Dr. Takashi Tazo of Shimizu Corporation, who offered valuable data pertaining to centrifuge experiments of slope deformation and other laboratory tests. We also thank Mr. Osamu Matsuo of the National Institute for Land and Infrastructure Management (formerly of the Public Works Research Institute, Ministry of Construction) for valuable advice and cooperative research into the simulation of centrifuge tests with soil improvement. Valuable information regarding damage to a railroad embankment and the related laboratory tests was provided by Dr. Masaru Tateyama of the Railway Technical Research Institute.

NOTE

\[ a: \text{ Variable which is dependent on other variables; used in Eq. (6) etc.} \]
\[ b: \text{ Material constant which has no dimension; used in Eq. (6) etc.} \]
\[ C apologise: \text{ Damping matrix} \]
\[ E_{\text{aprise}}: \text{ Secant elastic modulus at 50\% of the peak shear strength} \]
\[ f: \text{ Yield function} \]
\[ G_0: \text{ Initial shear deformation modulus; defined by Eq. (2)} \]
\[ G_0: \text{ Material constant which has the same dimension as stress; used in Eq. (2)} \]
\[ g: \text{ Plastic potential} \]
\[ H: \text{ Hardening parameter} \]
\[ h: \text{ Damping ratio} \]
\[ J_2: \text{ Second invariant of deviatoric stress tensor} \]
\[ J_3: \text{ Third invariant of deviatoric stress tensor} \]
\[ \{K\}: \text{ Stiffness matrix} \]
\[ K_0: \text{ Material constant that corresponds to the stress ratio at the phase transformation; used in Eq. (15) etc.} \]
\[ \{M\}: \text{ Mass matrix} \]
\[ m: \text{ Material constant which has no dimension; used in Eq. (2)} \]
\[ n: \text{ Material constant which has no dimension; used in Eq. (6) etc.} \]
\[ \{P\}: \text{ Nodal force vector equivalent to total stress acted inside each element} \]
\[ \rho: \text{ Mean principal stress} \]
\[ \rho: \text{ Material constant which has the same dimension as stress; used in Eq. (2)} \]
\[ \{U\}: \text{ Absolute displacement vector at the base} \]
\[ \{u\}: \text{ Relative displacement vector at each node} \]
\[ \alpha: \text{ Parameter related to Rayleigh damping; used in Eq. (19) etc.} \]
\[ \beta: \text{ Parameter related to Rayleigh damping; used in Eq. (19) etc.} \]
\[ \{e\}: \text{ Plastic component of maximum principal strain} \]
\[ \varepsilon_i: \text{ Initial strain tensor} \]
\[ \varepsilon_{ij}: \text{ Strain tensor at the time of the last unloading point} \]
\[ \varepsilon: \text{ Plastic volumetric strain} \]
\[ \gamma: \text{ Maximum shear strain; defined by Eq. (4b)} \]
\[ \gamma: \text{ Representative maximum shear strain; defined by Eq. (7b)} \]
\[ \gamma: \text{ Variable defined by Eq. (9b)} \]
\[ \gamma: \text{ Reference shear strain} \]
\[ \delta_i: \text{ Variable defined by Eq. (17)} \]
\[ \delta_\gamma: \text{ Variable defined by Eq. (17)} \]
\[ \sigma_{ij}: \text{ Stress tensor at the time of the last unloading point} \]
\[ \mu: \text{ Maximum shear stress; defined by Eq. (4a)} \]
\[ \mu: \text{ Representative maximum shear stress; defined by Eq. (7a)} \]
\[ \mu: \text{ Variable defined by Eq. (9a)} \]

REFERENCES


**APPENDIX**

In the proposed model, in cases where \(b \cdot \gamma_{C_0} = 0.5\), the resultant \(\theta - \gamma\) curve corresponds to that of the H-D model. A proof regarding this matter is as follows:

A hysteresis loop from \((-\gamma_1, -\tau_1)\) to \((+\gamma_1, +\tau_1)\) on the \(\tau - \gamma\) coordinate can be written with

\[
\tau + \tau_1 = \frac{a(y + \gamma_1)^2 + G_0(y + \gamma_1)}{1 + b(y + \gamma_1)}
\]

where

\[
a = \frac{1}{(2\gamma_1)^3 \{2\tau_1(1 + 2b\gamma_1) - 2G_0\gamma_1\}}
\]

The point \((+\gamma_1, +\tau_1)\) is located on the skeleton curve. Therefore,

\[
\tau_1 = \frac{G_0\gamma_1}{1 + \frac{G_0\gamma_1}{\tau_1}} = \frac{G_0\gamma_1}{1 + \frac{\gamma_1}{\gamma_{C_0}}}
\]

From Eqs. (21) and (22),

\[
a = \frac{4G_0\gamma_1^2}{\gamma_{C_0}^2} \left( \frac{b\gamma_{C_0} - \frac{1}{2}}{2\gamma_1^2 \left(1 + \frac{\gamma_1}{\gamma_{C_0}}\right)} \right)
\]

When \(b \cdot \gamma_{C_0} = 0.5\), \(a = 0\) and Eq. (20) becomes

\[
\tau + \tau_1 = \frac{G_0(y + \gamma_1)}{1 + \frac{\gamma_1}{2\gamma_{C_0}}} = \frac{G_0(y + \gamma_1)}{1 + \frac{G_0(y + \gamma_1)}{2\tau_1}}
\]

This is the same as a H-D model’s hysteresis loop.