FORMULAS FOR AN INFINITELY LONG
BERNOULLI-EULER BEAM ON THE PASTERNAK MODEL

HIDEAKI TANAHASHI

ABSTRACT

The Winkler model is the simplest mechanical model of a continuum and the easiest for mathematical treatment. However, it has some shortcomings involving discontinuity of adjacent spring displacements. Many researchers have proposed improved soil models to overcome these problems. These models are often called "two-parameter models", because they have the second parameter which presents the continuity of adjacent springs in addition to the first parameter. Pasternak’s model, the most reasonable and generalized two-parameter model, can account for the actual shearing effect of soils in the vertical direction. Pasternak used his model to show the analytical solutions of an infinitely long beam on the model and compared the displacements and stresses of the beam with those on the Winkler model. However, his work did not involve whole cases and its actual usefulness has not been made clear yet. This paper, therefore, presents a complete set of formulas to calculate the displacements and stresses on an infinitely long Bernoulli-Euler beam on the Pasternak model. We then carried out numerical case studies on mechanical quantities of the beam and the shear layer. The resulting effects of shear stiffness of the Pasternak model on these quantities are discussed in comparison with those of the Winkler model. Then, the applicability and usefulness of the Pasternak model are shown.

Key words: coefficient of subgrade reaction, deformation, elasticity, (mechanical model), plane strain, soil-structure interaction (IGC: E2/E13)

INTRODUCTION

A Bernoulli-Euler beam on the Winkler model frequently plays a significant role as an analytical model of a beam on an elastic continuum. The Winkler model is the simplest mechanical model of a continuum and the easiest for mathematical treatment. It is based on the hypothesis that the pressure at any point on the surface is proportional to the deflection of the point (Winkler, 1867). However, the model has some shortcomings involving the discontinuity of adjacent spring displacements because each spring behaves independently. In order to overcome these shortcomings, many researchers have proposed improved soil models (Filonenko-Borodich, 1940; Hetényi, 1946; Pasternak, 1954; Reissner, 1958; Vlasov et al., 1960; Loof, 1965). These models are called "two-parameter models", because in addition to the first parameter, the coefficient of subgrade reaction, they have the second parameter, which presents the continuity of adjacent springs.

Filonenko-Borodich (1940) considered applying a constant tensile force or an elastic membrane on the Winkler model. Pasternak (1954) introduced a shear interaction between adjacent spring elements. Kerr (1964) regarded Pasternak's mechanical model as a shear layer on the Winkler model and called it the Pasternak model as a generalized foundation model.

Vlasov et al. (1960) proposed a comprehensive analytical method of an elastic continuum based on variational principles. They formulated two parameters to impose the following constraints on vertical and horizontal displacements in a continuum on a rigid base. Vertical displacements are expressed in a linear or an exponential shape function which determines the vertical distribution profile uniquely through the continuum, while horizontal displacements are negligible across the continuum when only vertical loads work. Herein, this constraint condition is called "Vlasov's assumption". The model based on this assumption, usually called the Vlasov model, is mathematically the same as the Pasternak model in the governing equation.

Among two-parameter models, Tanahashi (1994) selected the Pasternak model as the simplest and most reasonable one to predict differential settlements of structures through soil-structure interaction. There were two significant reasons for this decision. First, the Pasternak model can account for the reduction effects of the shear stiffness of soils on differential settlements. Second, the Pasternak model can express surface displacements outside the beams or structures actually to

Professor, Department of Environmental Design, Kyoto Prefectural University, S-1, Hangi-cho, Shimogamo, Sakyo-ku, Kyoto 606-8522, Japan (tana@kpu.ac.jp).
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a greater degree than the Winkler model.

Of course, since the accuracy of mechanical models depends on the parameters, many researchers for several decades have discussed methods to evaluate parameters. However, the present paper mainly focuses on the effects of shear stiffness of the Pasternak model. Consequently, reviews are limited to the researched results concerning the effects of the shear stiffness of an elastic layer with a finite depth.

Pasternak (1954) himself suggested plate loading tests to evaluate the two parameters, but did not consider any substantial values of the two parameters. Vlasov et al. (1960) gave a significant base on which to determine parameters (refer to APPENDIX). Using Vlasov’s assumption, Jones et al. (1977), Nogami et al. (1985) and Vallabhan et al. (1988, 1991) modified or further developed Vlasov’s method. According to their results, surface displacements tend to be underestimated in comparison with the finite element method (FEM), even when their methods provide sufficiently accurate solutions.

Tanahashi (2000a, 2000b) proposed the parameters of the Pasternak model for practical calculations as “nondimensional characteristic value $yH \approx 4 - 6$” with closed-form formulations of surface displacements of an elastic layer with a finite depth subjected to uniformly distributed vertical loads. Such formulations are approximate but, since their error ratios are within 10%, they are accurate enough. Applicable conditions are limited to the cases of the length-depth ratio $L/H > 0.1$ (two-dimensional plane strain condition) or the radius-depth ratio $R/H > 0.1$ (axisymmetric condition) and Poisson’s ratio $\nu \leq 0.3$, where $L$ is the length of a uniformly loaded area, $R$ is the radius of a uniformly loaded circular area and $H$ is the depth of an elastic layer. The proposed nondimensional characteristic values were determined by modifying the exact values according to Vlasov’s assumption, allowing us to fit the surface displacements to those of FEM. As a result, the shear stiffness of the elastic layer was reduced. The applicable condition of Poisson’s ratio $\nu \leq 0.3$ for good approximation stems mainly from the solutions of the Pasternak model itself. It is because the solutions consisting of exponential functions (in the two-dimensional condition) cannot express upward displacements, while the elastic layer behaves upward around the vertically loaded area according to Poisson’s ratio $\nu$.

Pasternak (1954) showed the solutions of an infinitely long beam on the Pasternak model and compared them to those on the Winkler model. However, his work was limited to the case of $p < 1$, i.e., $G < 2kEI/B$ (refer to NOTATION). Fletcher et al. (1971) discussed one-, two- and three-parameter models by the estimation of the parameters. They however did not treat any substantial values of the corresponding cases of $p \geq 1$ in the three-parameter model, because such cases are of little interest. Although Scott (1981) also discussed the coefficients of subgrade reaction, he did not treat such cases because of the similar reason. Zhaohua et al. (1983) and Shirima et al. (1992) showed the same governing equation for a beam on the Pasternak model, but, neither of them discussed the cases of $p > 1$, because most practical cases do not correspond to this condition. However, according to the author’s knowledge, such cases are not uncommon, indeed are probable under certain conditions.

Therefore, this paper presents a complete set of formulas of displacements, slopes and stresses of an infinitely long Bernoulli-Euler beam on the Pasternak model. We then carried out numerical case studies of displacements and stresses of the beam and the shearing forces of the Pasternak model. From the results, the effects of the shear stiffness of the Pasternak model on the mechanical quantities are discussed in comparison with the Winkler model. Then, the applicability and usefulness of the Pasternak model are shown.

**EQUATION OF A BERNOULLI-EULER BEAM ON THE PASTERNAK MODEL**

As described above, Pasternak (1954) introduced a shear interaction between adjacent spring elements in order to improve the Winkler model and proposed the second parameter which expresses the stiffness of the vertical shear interaction. Here, we denote the second parameter the shear stiffness $G$, which is the whole shear stiffness of the soil layer per width. It is noted that $G$ is different from the shear modulus of the soil material. On the bases of the interaction, he presented the differential equation of displacements $y$ of an infinitely long Bernoulli-Euler beam on the Pasternak model in the two-dimensional plane strain condition as:

$$Ely'' - GBy'' + kBy = q$$  \hspace{1cm} (1)

where, $y'' = \frac{d^2y}{dx^2}$, $y'' = \frac{d^2y}{dx^2}$.

- $B$: effective width of both the beam and the layer (L),
- $G$: vertical shear stiffness of the elastic layer (FL$^{-1}$),
- $k$: coefficient of subgrade reaction (PL$^{-2}$),
- $E$: Young’s modulus of the beam (PL$^{-2}$),
- $I$: moment of inertia of the cross section of the beam (L$^4$),
- $q$: vertical load on the beam (PL$^{-1}$).

The origin is taken at the loading point in a rectangular coordinate system, the ordinate is displacement $y$ and the abscissa is distance $x$. When $G$ is zero in Eq. (1), the governing equation corresponds to the well-known equation of the Bernoulli-Euler beam on the Winkler model which is frequently used in many mechanical fields. Mechanical quantities in the problem are denoted as,

- $y$: displacement of the beam,
- $\theta$: slope of the beam,
- $M$: bending moment of the beam,
- $Q$: vertical shearing force (consists of shearing force $Q_b$ of the beam and vertical shearing force $Q_v$, which the elastic layer carries).

By defining the following parameters,
FORMULAS FOR BEAM ON PASTERNAK MODEL

\[ \beta = \sqrt[4]{\frac{k B}{4 E I}} \]  
(2a)

\[ \gamma = \sqrt[4]{\frac{k}{G}} \]  
(2b)

\[ \beta: \text{ characteristic value of a Bernoulli-Euler beam on} \]  
the Winkler model (L^{-1}),

\[ y: \text{ characteristic value of the Pasternak model (L^{-1}),} \]  

\[ \rho = (\frac{\beta}{\gamma})^2 = G/2\sqrt{kEI/B} \]  
(2c)

the homogeneous differential equation of (1) yields (3).

\[ y^{iv} - 4\beta^2py'' + 4\beta^4y = 0 \]  
(3)

Substituting \( y = e^{mx} \) into Eq. (3), the characteristic equation is

\[ m^4 - 4\beta^2pm^2 + 4\beta^4 = 0 \]  
(4)

This Eq. (4) yields

\[ m_{1,2,3,4} = \pm \beta\sqrt{2(\rho \pm \sqrt{\rho^2 - 1})} \]  
(5)

There are three possible forms of the general solution of Eq. (3), depending on whether \((\rho - 1)\) is positive, zero or negative.

1) \( \rho < 1 \): As \( m \) consists of two pairs of conjugate complex numbers,

\[ m_{1,2,3,4} = \pm (\phi_1 \pm i\phi_2) \]  

where

\[ \phi_1 = \beta\sqrt{1 + \rho}, \quad \phi_2 = \beta\sqrt{1 - \rho} \]  
(6)

Then the general solution of Eq. (3) is of the form,

\[ y = (A_1 e^{-\phi_1 x} + A_2 e^{\phi_2 x}) \cos \phi_2 x + (A_3 e^{-\phi_1 x} + A_4 e^{\phi_2 x}) \sin \phi_2 x \]  
(7)

where \( A_1 - A_4 \) are integration constants determined by the boundary conditions. \( B_1 - B_4 \) and \( C_1 - C_4 \) are also integration constants in the following.

2) \( \rho = 1 \): \( m_{1,2,3,4} = \pm \sqrt{2} \beta = \pm \phi_1, \quad \phi_2 = 0 \)

Then the general solution of Eq. (3) is of the form,

\[ y = e^{-\phi_1 x}(B_1 + B_2 x) + e^{\phi_1 x}(B_1 + B_4 x) \]  
(8)

3) \( \rho > 1 \): As \( m \) consists of real numbers,

\[ m_{1,2,3,4} = \pm (\phi_1 \pm \phi_2) \]  

where

\[ \phi_1 = \beta\sqrt{\rho + 1}, \quad \phi_2 = \beta\sqrt{\rho - 1} \]  
(9)

Then the general solution of Eq. (3) is of the form,

\[ y = (C_1 e^{-\phi_1 x} + C_2 e^{\phi_1 x}) \cos \phi_2 x + (C_3 e^{-\phi_1 x} + C_4 e^{\phi_1 x}) \sin \phi_2 x \]  
(10)

As for the boundary conditions of the beam on the Pasternak model, two typical cases are considered here. The first case is a continuous Bernoulli-Euler beam as shown in Fig. 1, and the second is a Bernoulli-Euler beam that has a hinge at the concentrated vertical load point, as shown in Fig. 2. Finally, the differential Eq. (1) is solved completely and six mechanical quantities subjected to a vertical concentrated load \( 2P \) are formulated in three cases according to \( \rho \) in Tables 1 and 2. Table 1 shows the results for the beam with no hinge and Table 2 shows those for the hinged beam. In either case, the mechanical quantities of the right side are shown because of the symmetry at the origin. Vertical displacement \( y \) is positive downward. Bending moments are positive for the dishing deflection line and negative for the hogging deflection line.

Since these cases are symmetrical at the loading point, the right side model corresponds to a vertical semi-infinite pile in an elastic soil layer due to a horizontal load at the pile top, if the \( x \) coordinate is taken in the vertical direction.

Pasternak (1954) presented only solution Eq. (7), which as he noted corresponds to most practical cases. However, he did not explain this decision in detail. There may be other practical cases that correspond to Eqs. (8) and (10). Therefore, we present here a complete set of quantities of the Pasternak model.

For reference, Hetényi (1946) solved the equation of a Bernoulli-Euler beam on the Winkler model under a constant tensile axial force \( N \) on the beam in addition to vertical loads. This equation is mathematically the same as Eq. (1) if \( GB \) is replaced by \( N \).

**NUMERICAL CASE STUDIES AND DISCUSSIONS**

In numerical case studies, the actual properties of the system should be determined. The evaluation method of the parameters \( k \) and \( G = 2\gamma \) or \( \beta = \frac{s\alpha}{2} \) and \( \gamma = \alpha \) are essentially the same as Vlasov's method but using different symbols (Vlasov's symbols are given in parentheses).

Jones et al. (1977), Nogami et al. (1985) and Vallabhan et al. (1988, 1991) developed procedures to determine the parameters through Vlasov's method. Nogami et al. (1985), and Vallabhan et al. (1988, 1991) concluded that, in general, surface displacements were underestimated in
Table 1. Solutions of Bernoulli-Euler beam on Pasternak model
(with no hinge, subjected to vertical load 2P)

<table>
<thead>
<tr>
<th>( \varphi )</th>
<th>( 0 \leq \rho &lt; 1 )</th>
<th>( \rho = 1 )</th>
<th>( \rho &gt; 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \varphi_1 = \beta \sqrt{1 + \rho} ), ( \varphi_2 = \beta \sqrt{1 - \rho} )</td>
<td>( \varphi_1 = \sqrt{2} \beta )</td>
<td>( \varphi_1 = \beta_1 \rho + 1 ), ( \varphi_2 = \beta \rho - 1 )</td>
<td></td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\varphi & = \frac{P}{4EI\rho} e^{-\alpha_1 \rho} \left( \cos \varphi_2 x - \sin \varphi_2 x \right) \left( \frac{\varphi_1}{\rho_1} + \frac{\varphi_2}{\rho_2} \right) \\
\theta & = \frac{P}{2EI\rho_1 \rho_2} e^{-\alpha_1 \rho} \sin \varphi_1 x \\
M & = \frac{P}{2} e^{-\alpha_1 \rho} \left( \cos \varphi_2 x - \sin \varphi_2 x \right) \left( \frac{\varphi_1}{\rho_1} \right) \\
Q & = -\frac{P}{2} e^{-\alpha_1 \rho} \cos \varphi_2 x + \frac{\beta^2}{\rho_1 \rho_2} \sin \varphi_1 x \\
Q_h & = -\frac{P}{2} e^{-\alpha_1 \rho} \cos \varphi_2 x - \frac{\beta^2}{\rho_1 \rho_2} \sin \varphi_1 x \\
Q_i & = -\frac{P}{2} e^{-\alpha_1 \rho} \left( \frac{2 \beta^2}{\rho_1 \rho_2} \sin \varphi_1 x \right) \\
\end{align*}
\]

Note: \( Q = Q_h \cos \theta + Q_i dM/dx + GB \theta \)

Table 2. Solutions of Bernoulli-Euler beam on Pasternak model
(with a hinge, subjected to vertical load 2P)

<table>
<thead>
<tr>
<th>( \varphi )</th>
<th>( 0 \leq \rho &lt; 1 )</th>
<th>( \rho = 1 )</th>
<th>( \rho &gt; 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \varphi_1 = \beta \sqrt{1 + \rho} ), ( \varphi_2 = \beta \sqrt{1 - \rho} )</td>
<td>( \varphi_1 = \sqrt{2} \beta )</td>
<td>( \varphi_1 = \beta_1 \rho + 1 ), ( \varphi_2 = \beta \rho - 1 )</td>
<td></td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\varphi & = \frac{P}{2EI\rho} e^{-\alpha_1 \rho} \left( \cos \varphi_2 x - \sin \varphi_2 x \right) \left( 1 + \frac{\beta^2}{\rho_1 \rho_2} \right) \\
\theta & = \frac{P}{2EI\rho_1 \rho_2} e^{-\alpha_1 \rho} \sin \varphi_1 x \\
M & = \frac{P}{2} e^{-\alpha_1 \rho} \left( \cos \varphi_2 x - \sin \varphi_2 x \right) \left( \frac{\varphi_1}{\rho_1} \right) \\
Q & = -\frac{P}{2} e^{-\alpha_1 \rho} \cos \varphi_2 x + \frac{\beta^2}{\rho_1 \rho_2} \sin \varphi_1 x \\
Q_h & = -\frac{P}{2} e^{-\alpha_1 \rho} \cos \varphi_2 x - \frac{\beta^2}{\rho_1 \rho_2} \sin \varphi_1 x \\
Q_i & = -\frac{P}{2} e^{-\alpha_1 \rho} \left( \frac{2 \beta^2}{\rho_1 \rho_2} \sin \varphi_1 x \right) \\
\end{align*}
\]

Note: \( Q = Q_h \cos \theta + Q_i dM/dx + GB \theta \)

Comparison with FEM. We interpret this to mean that such underestimation comes mainly from Vlasov’s assumption, even if the solutions are mathematically accurate enough.

The present author, therefore, earlier attempted to fit the displacements to those of FEM solutions and found that they get close to the FEM solutions by modifying the nondimensional characteristic values \( \gamma H \), which are distributed within narrow values and are almost constant. As a result, Tanahashi (2000a, 2000b) proposed the nondimensional characteristic values \( \gamma H \) with simple formulas of surface displacements in the two-dimensional plane strain condition (Tanahashi, 2000a) and in axisymmetric condition (Tanahashi, 2000b). In the present study, the formula in the two-dimensional plane strain condition is shown in Eqs. (11) and (12), while the recommended nondimensional characteristic values are shown in Table 4.

\[
\begin{align*}
W & = S \left( 1 - e^{-\gamma H} \cos \left( \gamma x \right) \right) : |x| \leq L \\
W & = S \left( 1 - e^{-\gamma H} e^{-\gamma x} \right) : |x| > L \\
S & = \frac{(1 + v)(1 - 2v)E_0 H}{(1 - v)E_0} \\
\end{align*}
\]

where, \( W \): vertical surface displacement of the elastic layer (L), \( S \): fundamental vertical displacement which is the same as that in one-dimensional plane strain condition (L),
FORMULAS FOR BEAM ON PASTERNAK MODEL

\[ E_0: \text{Young's modulus of the elastic layer (PL}^{-1}\text{),} \]
\[ q: \text{uniformly distributed vertical load on the} \]
\[ \text{surface of the elastic layer (PL}^{-1}\text{),} \]
\[ L: \text{half-length of the vertical load q (L),} \]
\[ H: \text{depth of the elastic layer (L),} \]

The values \( \gamma H \) depend on Poisson's ratio and are distributed from 4 to 6, although applicability has some limits for good approximation. Especially, in a long uniformly distributed load on the Pasternak model in the plane strain condition, \( \gamma H = 5.4 \) is recommended for Poisson's ratio \( \nu = 0.3 \) (Tanahashi, 2000a). The present study adopts this value. This means the shear stiffness \( G \) is modified, so that it is lower than the exact value obtained from Vlasov's method. The coefficient of subgrade reaction \( k \) is the same as that obtained by Vlasov's method when the shape function is linear (Tanahashi, 2000a). It is also the same as the coefficient of subgrade reaction of an elastic medium in the one-dimensional plane strain condition in Eq. (13).

\[
k = \frac{q}{S} = \frac{(1 - \nu)E_0}{(1 + \nu)(1 - 2\nu)H}
\]

(13)

Thus, the parameters for the case study are determined as follows.

- Depth of an elastic layer: \( H = 18.0 \text{ m.} \) The width of the layer and beam together: \( B = 1 \text{ m.} \)
- Young’s modulus of the layer: \( E_0 = 14 \text{ MN/m}^2 \).
- Poisson's ratio: \( \nu = 0.3 \).
- Coefficient of subgrade reaction: \( k = (1 - \nu)E_0/(1 + \nu)(1 - 2\nu)H = 1047 \text{ kN/m}^2 \).
- Characteristic value of the Pasternak model: \( \gamma = 5.4/H = 0.30 \text{ m}^{-1} \).

Young's modulus of a concrete beam on the Pasternak model: \( E_c = 2.1 \times 10^3 \text{ kN/m}^2 \).

- Moment of inertia of the beam: \( I = BD^3/12 \).
- Value \( \rho = (\beta/\gamma)^2: 0, 0.25, 1.0, 4.0 \).

Young’s modulus of the elastic layer \( E_0 = 14 \text{ MN/m}^2 \) is obtained from the formula \( E_0 = 1.4N \text{ MN/m}^2 \) which is often used in Japan for normally consolidated sand (Architectural Institute of Japan, 2001). \( N \), which is a blow count of SPT (Standard Penetration Test), is assumed to be 10 for a loose sandy layer. The depth \( D \) of the beam is calculated inversely in order that values \( \rho \) become 0, 0.25, 1.0, 4.0. If these calculations are applied for plates on the Pasternak model, the moment of inertia of the plate is replaced by \( I = BD^3/12(1 - v_p^2) \), where \( v_p \) is Poisson’s ratio of the plate.

As mentioned above, Pasternak (1954) described that practical cases mostly correspond to \( \rho < 1. \) Fletcher et al. (1971), Scott (1981) and others also commented that other cases are of little practical interest. However, there are more than a few cases of \( \rho \geq 1 \), i.e., \( G \geq 2\sqrt{kEI/B} \); indeed such cases are probable when an elastic layer is deep and soft and at the same time the flexural stiffness of a beam is relatively small as shown in Table 3. These conditions are probable on reclaimed land or in areas having deep deposits of loose sand. Of course, it should be considered that these situations may depend on strain levels of the soil because of its nonlinearity.

Numerical calculations were carried out in four cases of \( \rho \) in Table 3, where \( k \) is constant; the same calculations were also carried out where \( G \) and \( \gamma \) are also constant except in the Winkler model. The case \( \rho = 0 \) corresponds to a beam on the Winkler model, as a special case of the Pasternak model. The case \( \rho = 0.25 \) is the same as Pasternak’s example (Pasternak, 1954). The other cases, \( \rho = 1 \) and 4, correspond to relatively shallower beams. The compared results are presented graphically in Figs. 3 – 8 for Table 1, and in Figs. 9 – 14 for Table 2, where PM refer to the Pasternak model and WM to the Winkler model, respectively. Quantities \( \gamma, \beta, M, Q, Q_c \) and \( Q \) are shown for unit load \( P = 1 \text{ kN} \) on the positive side of the symmetry axis.

As shown in Figs. 3 and 5, the maximum bending moment and displacement of the Pasternak model when \( \rho = 0.25 \) are each 10% less than those of the Winkler model (\( \rho = 0 \)), if the same coefficient of subgrade reaction and the same beam stiffness are used. This means that the influence of the shear stiffness of the layer is negligible. On the other hand, if \( \rho > 1 \), \( Q \) is zero at the origin but increases rapidly at a point a little further from the origin. This indicates a large sharing ratio of shearing forces of the elastic layer. Consequently, in such cases the Pasternak model is recommended to apply.

Concerning the case of \( \rho = 0.25 \) in Fig. 2 (beam with a hinge), the maximum displacement of the Pasternak model is 25% less as shown in Fig. 9 and the maximum bending moment is 38% less than that of the Winkler model as shown in Fig. 11, respectively. These facts are in accord with Pasternak’s description (1954).

As shown in Figs. 3, 5 and 8, if the flexural stiffness \( E_I \) decreases and \( \rho \) increases, the displacements of the beam or plate are underestimated, and the bending moments are overestimated in the conventional Winkler model. At

<table>
<thead>
<tr>
<th>Table 3. Calculation cases</th>
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<tbody>
<tr>
<td>Parameters</td>
</tr>
<tr>
<td>( k ) (kN/m³)</td>
</tr>
<tr>
<td>( G ) (kN/m)</td>
</tr>
<tr>
<td>( B ) (m)</td>
</tr>
<tr>
<td>( D ) (m)</td>
</tr>
<tr>
<td>( EI ) (kN/m³)</td>
</tr>
<tr>
<td>( \beta ) (m⁻¹)</td>
</tr>
<tr>
<td>( \gamma ) (m⁻¹)</td>
</tr>
<tr>
<td>( \rho = (\beta/\gamma)^2 )</td>
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<table>
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<tr>
<th>Table 4. Recommended nondimensional characteristic value ( \gamma H )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (0.1 \leq L/H \leq 5 ) for error ratios within 10%)</td>
</tr>
<tr>
<td>( H ): Depth of an elastic layer ( L ): Length of a loading area</td>
</tr>
<tr>
<td>Poisson's ratio ( \nu )</td>
</tr>
<tr>
<td>( \gamma H )</td>
</tr>
</tbody>
</table>
the same time, the maximum shearing forces $Q_1$ are neglected. As a result, the analytical errors become very large in the Winkler model. Such cases are realized when a very thin and soft plate is put on the elastic layer, for example, a pavement or a soil improvement by means of a certain material with small flexural stiffness on the surface. As an extreme case of $\rho \to \infty$, the deflection line gets close to the ground surface displacement of the Pasternak model without any beam on it; this situation is expressed in Eq. (14) (Tanahashi, 2000a).

$$y = \frac{Py}{2k} e^{-\frac{x}{k}}$$  \hspace{1cm} (14)

Equation (14) is the surface displacement solution of the Pasternak model subjected to a vertical line load $P$ in the two-dimensional plane strain condition (Pasternak, 1954; Loof, 1965; Vlasov et al., 1960; Tanahashi, 2000a). However, the Winkler model cannot express such a surface displacement subjected to a vertical line load $P$ in the two-dimensional plane strain condition, and only shows a concentrated vertical displacement, which is far different from the displacement of the elastic continuum.
Fig. 9. Comparison of displacement $y$ (with a hinge) (PM: Pasternak model, WM: Winkler model)

Fig. 10. Comparison of slope $\theta$ (with a hinge)

Fig. 11. Comparison of bending moment $M$ (with a hinge)

Fig. 12. Comparison of shearing force $Q$ (with a hinge)

Fig. 13. Comparison of shearing force $Q_h$ (with a hinge)

Fig. 14. Comparison of shearing force $Q_i$ (with a hinge)

Consequently, the comprehensive analyses using the Bernoulli-Euler beam on the Pasternak model are very significant.

The solutions of vertical piles subjected to horizontal forces under the boundary condition that the pile top does not rotate at all correspond to Table 1 where $P$ is replaced by $H$. On the other hand, the solutions of piles whose tops can rotate freely correspond to Table 2 where $P$ is replaced by $H$.

The conventional formulas of a pile subjected to horizontal forces coincide with the case of a Bernoulli-Euler beam on the Winkler model, when $\rho$ is replaced by zero.

**EFFECTS OF SHEAR STIFFNESS OF THE PASTERNAK MODEL**

*Infinitely Long Beam*

As shown respectively in Figs. 1 and 2, the shearing force $Q$ consists of shearing force $Q_h$ of the beam and
vertical shearing force \( Q \), which the elastic layer itself carries. The Eq. (15) can then be introduced from the elementary flexural beam theory and the basic relation concerning the shearing force equilibrium of the Pasternak model. That is, the vertical shearing force \( Q \) is proportional to the shear stiffness of the elastic layer and the slope \( \theta \) of the beam as,

\[
Q = Q_b \cos \theta + Q_r = \frac{dM}{dx} \cos \theta + GB \theta = \frac{dM}{dx} + GB \theta
\]

(15)

where slope \( \theta \) of the beam is generally small enough, thus \( \cos \theta \approx 1 \).

One can find out that the shearing force \( Q \) is not zero at the maximum or minimum point of the bending moment \( M \), when \( Q \), i.e., slope \( \theta \), is not zero. This is an important feature of beams on the Pasternak model, and distinguishes this model from Winkler’s.

At the origin in Fig. 1, the vertical force \( P \) is equal to \( Q_b \) because both slopes \( \theta \) are zero. As found in Fig. 8, however, \( Q \) increases rapidly to the maximum as the distance increases from the origin. Additionally, the maximum \( Q \) increases if \( \rho \) increases.

However, in Fig. 2, \( P \) is divided into \( Q_b \) and \( Q \), and the sharing ratio depends only on \( \rho \), as shown in Eqs. (16) and (17);

\[
Q_b = \left[ \frac{dM}{dx} \right]_{x=0} = \frac{1}{1+2\rho} P = \eta_b P
\]

(16)

\[
Q_r = \left[ GB \theta \right]_{x=0} = \frac{2\rho}{1+2\rho} P = \eta_r P
\]

(17)

where the sharing ratios of \( Q \) and \( Q \), of \( Q \) to \( Q \) denote \( \eta \), \( \eta \), \( \eta \), respectively as,

\[
\eta_b = \frac{1}{1+2\rho}
\]

\[
\eta = \frac{2\rho}{1+2\rho}
\]

Semi-infinite Beam

Most foundations are finite in length, and in such cases the displacements of the surface outside the foundation edge are of much interest for differential settlements and the interaction behaviors of adjacent foundations. One such case is illustrated in Fig. 15, where the shearing force \( Q \) occurs in the layer’s left-side section. \( Q_b \) is obtained by considering that the maximum displacement \( y \) of the surface due to the line load \( Q \) (refer to Eq. (14)) is equal to the displacement at the beam end \( x = 0 \). Note that \( y \) is expressed in the same equation in each value \( \rho \) referring to Table 2, one side of which corresponds to the case of a semi-infinite long beam.

\[
Q_b y = \frac{P}{2EI\beta}(1+2\rho)
\]

(19)

Thus,

\[
Q_b = \frac{PBk(1+\rho)}{2EI\beta(1+2\rho)} P
\]

(20)

Therefore, the equilibrium of the forces in the vertical direction at the beam end is expressed as,

\[
Q = Q_b + Q_r + Q_s
\]

\[
= \left( \frac{1}{1+2\rho} + \frac{2\rho}{1+2\rho} + \frac{2\sqrt{\rho(1+\rho)}}{1+2\rho} \right) P
\]

\[
= \frac{1+2\rho + 2\sqrt{\rho(1+\rho)}}{1+2\rho} P
\]

(21)

Thus,

\[
Q_b = \frac{1}{1+2\rho + 2\sqrt{\rho(1+\rho)}} Q = \eta_b Q
\]

\[
Q_r = \frac{2\rho}{1+2\rho + 2\sqrt{\rho(1+\rho)}} Q = \eta Q
\]

\[
Q_s = \frac{2\sqrt{\rho(1+\rho)}}{1+2\rho + 2\sqrt{\rho(1+\rho)}} Q = \eta_s Q
\]

(22)

where

\[
\eta_b = \frac{1}{1+2\rho + 2\sqrt{\rho(1+\rho)}}
\]

\[
\eta = \frac{2\rho}{1+2\rho + 2\sqrt{\rho(1+\rho)}}
\]

\[
\eta_s = \frac{2\sqrt{\rho(1+\rho)}}{1+2\rho + 2\sqrt{\rho(1+\rho)}}
\]

(23)

The ratio of shearing components in Eq. (22) depend on the value \( \rho \). In Fig. 16, the ratios \( \eta_b \), \( \eta \), and \( \eta_s \), are presented according to the change in \( \rho \). The ratios are expressed as percentages.

As the figure shows, \( \eta_s \) decreases from 48% to 0%
when $\rho$ decreases from 1 to zero, and $\eta_s$, the shearing component of the Pasternak model decreases from 34% to 0%. On the other hand, $\eta_n$, which is carried by the beam, increases rapidly to 100%. This means the shear effect of the Pasternak model decreases when $\rho$ decreases to zero, and the subgrade gets close to the Winkler model. When $\rho = 0.25$, $\eta_n = 38\%$, $\eta_s = 19\%$ and $\eta_s = 42\%$, which is a little larger than $\eta_n$. If $\rho$ exceeds 1, the shearing force of the beam decreases drastically and $\eta_s$ and $\eta_n$ converge to 50%. As a result, the surface displacements are supposed to get close to those of the Pasternak model itself, which is expressed by Eq. (14).

**CONCLUSIONS**

The principal conclusions of this study are summarized as follows:

1. A complete set of general formulas of an infinitely long Bernoulli-Euler beam on the Pasternak model, with the beam being subjected to a concentrated vertical load were presented in two boundary conditions. One was a continuous beam with no hinge, while the other was a beam with a hinge beneath the loading point.

2. According to the general formulas, numerical case studies of five beam quantities and shearing force of the Pasternak model were carried out in four cases: $\rho = 0, 0.25, 1$ and 4 and were presented in diagrams. In order to evaluate the parameters, the nondimensional characteristic value of the Pasternak model which the author proposed previously was assumed to be $\gamma H = 5.4$. According to the results, differences between the Pasternak model and the Winkler model were discussed. In cases of $\rho \geq 1$, which were determined to be probable, the bending moments and displacements would be reduced significantly. Thus, the Pasternak model was recommended for practical application.

3. At the end of a semi-ininitely long beam on the Pasternak model, the sharing ratios of shearing forces were discussed according to the general formulas and case studies. The sharing ratio of the beam $\eta_n$, the sharing ratio of the Pasternak model under the beam $\eta_s$ and the sharing ratio of the Pasternak model outside the edge $\eta_s$ were considered. When $\rho$ was increased, $\eta_s$ increased and $\eta_n$ decreased significantly. As a result, the rule of the beam became negligible and the system became similar to the Pasternak model with no beam on it. On the other hand, when $\rho$ was decreased to zero, $\eta_n$ increased drastically and the system became similar to a Bernoulli-Euler beam on the Winkler model. Thus, the Bernoulli-Euler beam on the Pasternak and Winkler models was comprehensively analyzed. As a result, the usefulness of the Pasternak model was made clear.

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**NOTATION**

The following symbols are used in this paper:

- $A_1 - A_4$: integration constants
- $B_1 - B_2$: integration constants
- $B$: effective width of the beam and the layer together (L)
- $C_1 - C_4$: integration constants
- $D$: depth of the beam (L)
- $E$: Young’s modulus of the beam material (FL$^{-2}$)
- $E_c$: Young’s modulus of the elastic layer (FL$^{-2}$)
- $G$: shear stiffness of the elastic layer (FL$^{-1}$)
- $H$: depth of the elastic layer (L), or horizontal force on the top of the pile (F)
- $I$: moment of inertia of the cross section of the beam (L$^4$)
- $k$: coefficient of subgrade reaction (FL$^{-2}$)
- $L$: length of the beam or foundation (L)
- $M$: bending moment of the beam (FL$^{-1}$)
- $m_{1,2,3,4}$: solutions of characteristic equation
- $N$: blow count of SPT (Standard Penetration Test), or tensile axial force denoted by Hetényi (F)
- $Q$: vertical shearing force (F)
- $Q_s$: shearing force of the beam (F)
- $Q_t$: shearing force of the elastic layer beneath the beam (F)
- $Q_{s0}$: shearing force of the elastic layer without beam on it (F)
- $P$: vertical line load (FL$^{-1}$)
- $q$: uniformly distributed vertical load (FL$^{-1}$)
- $R$: radius of the circular foundation (L)
- $S$: fundamental vertical displacement which is the same as the vertical displacement in one-dimensional plane strain condition (L)
- $s$: parameter denoted by Vlasov (corresponds to $\sqrt{2} \beta$ in this paper) (L$^{-1}$)
- $r$: parameter denoted by Vlasov (corresponds to $G$ in this paper) (FL$^{-1}$)
- $u$: horizontal displacement in the continuum (L)
- $w$: vertical displacement in the continuum (L)
- $W$: vertical surface displacement of the elastic layer (L)
- $x$: horizontal distance (L)
- $y$: vertical displacement of the beam (L)
- $z$: parameter denoted by Vlasov (corresponds to $\gamma$ in this paper) (L$^{-1}$)
- $e$: characteristic value of a Bernoulli-Euler beam on the Winkler model (L$^{-1}$)
- $x$: characteristic value of the Pasternak model (L$^{-1}$)
- $\eta_n$, $\eta_n$, $\eta_n$: sharing ratios (%) of $Q_n$, $Q$, and $Q_s$ to $Q$, respectively
- $\eta_1$: constant determining the rate of decrease of the vertical displacement in the shape function
- $\theta$: slope of the beam (rad)
- $\phi$: shape function of vertical displacements based on Vlasov’s assumption
- $\psi_1, \psi_2$: $B_{1} \psi_{1} + \rho_{2} \psi_{2}(1-\rho_{1})$ respectively.
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APPENDIX: VLASOV MODEL AND THE SHAPE FUNCTION

Vlasov et al. (1960) proposed a mechanical model of an elastic layer, starting from the continuum and imposing certain constraints (Vlasov's assumption in this paper) on the displacement distribution of an elastic layer as follows for an approximate analysis of the elastic continuum with a finite depth on a rigid base.

1. The horizontal displacement \( u \) is assumed to be zero everywhere in the continuum, because the horizontal displacements are negligible in comparison with the vertical displacements when only vertical loads work.

2. The vertical displacement \( w \) is expressed using a vertical surface displacement \( W(x) \) and a shape function \( \phi(z) \) as,

\[
w(x, z) = W(x)\phi(z)
\]  \( \text{(A1)} \)

where \( \phi(0) = 1, \phi(H) = 0, w(x, 0) = W(x), z \) is vertical axis and \( H \) is the depth of the layer.

On the bases of this assumption, Vlasov et al. introduced following governing equations using variational principles.

\[
kW - 2t \frac{d^2W}{dx^2} = q
\]  \( \text{(A2)} \)

where \( 2t = G \) and,

\[
k = \frac{(1 - \nu)E}{(1 + \nu)(1 - 2\nu)} \int_0^1 \left( \frac{d\phi}{dz} \right)^2 dz
\]

\[
2t = \frac{E}{2(1 + \nu)} \int_0^1 \phi^2 dz
\]  \( \text{(A3)} \)

Equation (A2) is the same as that of the Pasternak model, if \( 2t \) is replaced by \( G \), and yields the solution of Eq. (11).

Vlasov et al. assumed the shape function as, for a shallow layer: \( \phi(z) = 1 - z/H \)  \( \text{(A4)} \)
for a deep layer: \( \phi(z) = \frac{\sinh \eta H(1 - z/H)}{\sinh \eta H} \)  \( \text{(A5)} \)

where \( \eta \) is a constant that determines the rate of decrease of the vertical displacement. For example, in case of \( \eta \rightarrow 0, \) Eq. (A5) coincides with Eq. (A4), and Eq. (A3) yields,

\[
k = \frac{(1 - \nu)E}{(1 + \nu)(1 - 2\nu)} \int_0^1 \left( \frac{d\phi}{dz} \right)^2 dz
\]

\[
2t = \frac{EH}{6(1 + \nu)}
\]  \( \text{(A6)} \)

Then, \( \alpha = \gamma \)

\[
\alpha = \sqrt{\frac{k}{2t}} = \frac{1}{H} \sqrt{\frac{6(1 - \nu)}{(1 - 2\nu)}}
\]  \( \text{(A7)} \)

In general, Eq. (A7) is a function of \( H, \) Poisson's ratio \( \nu \) and \( \eta, \) thus \( \alpha H (= \gamma H) \) is a constant if Poisson's ratio and \( \eta \) are fixed. This fact shows nondimensional characteristic value \( \gamma H \) is very convenient to treat.