Least Order Solution for Output Feedback Stabilization of Linear SIMO Systems

Stanisław K. DUDA** and Ryoziaburo TAGAWA*

A general theory concerning a least possible order d of a linear dynamic feedback for stabilization of linear SIMO systems is presented ($d \leq v_0 - 1$, where v_0 is the observability index of a given system). The method applies to controllable and observable systems of an arbitrary order. A computationally attractive design procedure which provides the output feedback in a form suitable for direct realization and an illustrative example are given.

Key Words: output feedback, stabilization problem, systematic design procedure

1. Introduction

The fundamental result presented by Wonham¹⁾ shows that for a controllable system all poles can be assigned arbitrarily by selecting an appropriate state feedback. Since the Wonham's publication, a great number of reports on pole assignment and stabilization problems have been presented, see references 3-16 and the literature cited there. Significant are results of Brash and Pearson³⁾, Kimura⁴⁾, and Davison and Wang⁵⁾ who derived the order of a dynamic output feedback by which any pole assignment can be realized. However, in spite of the attempts, the stabilization problem by the least order output feedback has not been solved. The stabilization problem is one of the fundamental questions pertaining to the synthesis of linear control systems, and consists of finding a feedback law for a given system so that the

overall closed-loop system is stable.

The purpose of this paper is to introduce a general approach concerning the stabilization problem of linear SIMO (single-input multi-output) systems by a linear dynamic output feedback of the least possible order. In contrast to design methods presented in the literature on control theory we assume no use of the classical stability criteria (for example no use of the Hurwitz, Nyquist or Lienard-Chipart criteria) but we propose a new approach based on a parametric description of stable characteristic polynomials. This method is based on system's description given in a state variable form. We consider a successively higher order state feedbacks in a parametric form. Each time we put constraints on the state feedback which ensure existence of an equivalent output feedback. The least order output feedback is found if the corresponding state feedback system is stable. A solution is given as a finite step algorithm which can be readily implemented on digital computers. The least order output feedback is obtained in a form suitable for direct realization. By duality, the least order, output feedback for multi-input single-output systems can be derived.

Some notational remarks are given. $\mathbf{R}$, $\mathbf{R}^n$ and $\mathbf{R}^{n \times m}$ denote the real field, n -dimensional vector space and the set of $n \times m$ matrices on $\mathbf{R}$ respectively. C' , $\mathbf{C}'$ and $\mathcal{C}'^\perp$ denote the transpose of a matrix C , the image of C' , and the orthogonal complement of a subspace $\mathcal{C}$ in $\mathbf{R}^n$ respectively, $\mathcal{C} \oplus \mathcal{C}'^\perp = \mathbf{R}^n$. The notation $i \in d$ specifies a set $i=0, 1, \dots, d$.

2. Problem formulation

Consider a controllable and observable linear system:

* Faculty of Engineering, Hokkaido University, Sapporo

** On leave from the Institute of Industrial Automatic Control, Warsaw Technical University, Warsaw, Poland

* 北海道大学工学部, 札幌市北区北 13 西 8

** 同上, ワルシャワ工科大学
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$$\dot{x} = Ax + bu, \quad y = Cx \quad (1)$$

where $u \in \mathbf{R}$ is the scalar input, $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^m$ are the state and the output vectors respectively, and A , b and C are constant matrices (maps) of appropriate size. Let us describe a linear dynamic output feedback of the order d in the general algebraic form:

$$u^{(d)} = \sum_{i=0}^{d-1} \alpha_i u^{(i)} + \sum_{i=0}^d Q_i y^{(i)}, \quad \alpha_i \in \mathbf{R}, \quad Q_i \in \mathbf{R}^{1 \times m} \quad (2)$$

Here $u^{(i)}$ and $y^{(i)}$ denote the i -th derivatives of u and y respectively. The problem considered in this paper deals with the following questions:

- (1) What is the least possible order d , such that the output feedback system in eqs. (1) and (2) will be stable? A block-diagram of the output feedback system is depicted in Fig. 1.
- (2) How can one find the least order d ?
- (3) Given d , how can one obtain appropriate Q_i 's ($i \in d$) and α_i 's ($i \in d-1$)?

3. Preliminary results

In the case $C = I_n$ (I_n is the identity matrix) all poles can arbitrarily be assigned by a constant state feedback, Wonham¹). Therefore, a dynamic state feedback of the order g in the general algebraic form:

$$u^{(g)} = \sum_{i=0}^{g-1} k_i u^{(i)} + Fx, \quad k_i \in \mathbf{R}, \quad F \in \mathbf{R}^{1 \times n} \quad (3)$$

Fig. 1 A block-diagram of the output feedback system in eqs. (1) and (2)

Fig. 2 A block-diagram of the state feedback system in eqs. (1) and (3)

can always be applied to the system (1) to achieve stability of the state feedback system. Naturally, the form (3) is not minimal since stability can be obtained by an appropriately chosen gain feedback $u = Fx$. Eq. (3) serves only for theoretical purpose. A block-diagram of the state feedback system is shown in Fig. 2. Let the characteristic polynomial of the state feedback system be denoted:

$$H(s) = s^{n+g} + a_{n+g-1}s^{n+g-1} + \dots + a_1s + a_0$$

$$= \det \begin{pmatrix} A & b & 0 \\ \vdots & \vdots & \vdots \\ 0 & 1 & 0 \\ \vdots & \vdots & \vdots \\ F & k_0, \dots, k_{g-1} \end{pmatrix} \quad (4)$$

Equating the coefficients of the corresponding powers of s on both sides of (4) one can uniquely express F and k_i 's ($i \in g-1$) in terms of a_j 's ($j \in n+g-1$). For single-input systems the correspondence between the state feedback (3) and a set of poles of the state feedback system is unique.

Define the observability subspaces of the system (1) by:

$$CV_\eta = C' + A'C' + \dots + (A')^\eta C' \quad (5)$$

where $\eta = 0, 1, \dots, v_0-1$ shall be referred to as the index of the observability subspaces (v_0 is the observability index).

4. Main result

In this section, the least possible order d is derived by which stabilization of the system (1) is possible. This least order is obtained by using a transformation method of the state feedback (3) into an equivalent output feedback (2).

The output feedback (2) and the state feedback (3) shall be referred to as equivalent if:

- (i) $d = g$.
- (ii) the output feedback system in (1) and (2), and the state feedback system in (1) and (3) have identical characteristic polynomials.

We note that for any output feedback there exists an equivalent state feedback, but the converse statement is not true in general.

Theorem 1:

Let the overall closed-loop system be defined in eqs. (1) and (3). Then there exists a minimal order $\hat{g}(g=\hat{g})$ of the state feedback (3) equal to an index η of the observability subspace $\mathcal{V}_\eta(\hat{g}=\eta)$, such that the state feedback system is stable and $F' \in \mathcal{CV}_\eta$.

Proof: Construct in succession the state feedback (3) of the order $g=0, 1, \dots, v_0-1$. With each succession the coordinates of the constant vector F can be expressed from (5) as linear functions in $a_j's (j \in n+g-1)$. Stability of the state feedback system is sought under the permanent conditions that $F' \in \mathcal{CV}_\eta$ and $g=\eta$. If $F' \in \mathcal{CV}_\eta$, then $F' \perp \mathcal{CV}_\eta^\perp$. The overall system is stable if there exists a solution $a_j's$ of $F' \perp \mathcal{CV}_\eta^\perp$ which results in a Hurwitz polynomial (4). If solutions of $F' \perp \mathcal{CV}_\eta^\perp$ do not include a Hurwitz polynomial (4), a higher order state feedback must be considered. Certainly, because, $F' \in \mathcal{CV}_{v_0-1}$ a minimal order $\hat{g}$ such that $\hat{g}=\eta \leq v_0-1$ exists. $\square$

A solution of the relation $F' \perp \mathcal{CV}_\eta^\perp$ is discussed in section 5.

Theorem 2:

Let the output feedback system be defined in eqs. (1) and (2). Then the least possible order d of the output feedback (2) for stabilization of (1) is given by the following equation:

$$d=\hat{g} \quad (6)$$

Proof: The proof is based on the constructive procedure given by Pearson²⁾. When the output feedback (2) and the state feedback (3) are equivalent we have:

$$\sum_{i=0}^{d-1} \alpha_i u^{(i)} + \sum_{i=0}^d Q_i y^{(i)} = \sum_{i=0}^{g-1} k_i u^{(i)} + Fx \quad (7)$$

and $d=g$.

Since $y=Cx$, the i -th derivative of y is:

$$y^{(i)} = CA^i x + \sum_{k=1}^i CA^{i-k} b u^{(k-1)}, \quad i=1, \dots, v_0-1$$

By substituting this equation in (7) and equating all factors with x one obtains:

$$[Q_0, Q_1, \dots, Q_d] \begin{bmatrix} C \\ CA \\ \vdots \\ CA^d \end{bmatrix} = F \quad (8)$$

Eq. (8) has a solution $Q_i's (i \in d)$ if and only if $F' \in \mathcal{CV}_\eta$ and $\eta=d$. Therefore, it is always possible to stabilize the system (1) when an output

feedback has the order $d=\hat{g}$.

Now, we shall prove that the order $d=\hat{g}$ is the least possible. Assume that there exists an output feedback of an order $d < \hat{g}$ which stabilizes (1). In this case, there must exist an equivalent state feedback which stabilizes (1). Therefore, the relation $F' \in \mathcal{CV}_d$ must hold for $d < \hat{g}$. However, this contradicts the definition of $\hat{g}$. $\square$

With the $Q_i's$ determined, the $\alpha_i's$ follows directly from:

$$\alpha_i = k_i - [Q_{i+1}, \dots, Q_d] \begin{bmatrix} Cb \\ CA^d b \\ \vdots \\ CA^{d-1-i} b \end{bmatrix}, \quad i \in d-1 \quad (9)$$

5. Design Procedure

It has been shown so far that the constant state feedback vector F determines the least order d of the output feedback (2). In order to find an appropriate F we shall utilize the results of Theorem 1. When F is found, the least order d is directly obtained from Theorem 2. The following algorithm is proposed.

Step 1: Set $g=0$.

Step 2: Take $\eta=g$ and calculate a basis for $\mathcal{CV}_\eta^\perp$.

Step 3: By equating coefficients of corresponding powers of s on both sides of (4) express F and $k_i's (i \in g-1)$ in terms of $a_j's (j \in n+g-1)$. The coefficients $k_i's$ and the coordinates of F are linear functions in real variables $a_j's$ to the first power. Step 4: Impose conditions which ensure that F will belong to the subspace $\mathcal{CV}_g$. If $F' \in \mathcal{CV}_g$ then $F' \perp \mathcal{CV}_g^\perp = \text{Span}\{e_1, \dots, e_t\}$ and the inner products $\langle F', e_k \rangle$ must be equal to zero:

$$\langle F', e_k \rangle = 0, \quad k=1, \dots, t \quad (10)$$

where t is the dimension of the subspace $\mathcal{CV}_g^\perp$. Eqs. (10) are linear functions in $a_j's$ to the first power and ensure that the state feedback (3) can be transformed into an output feedback (2).

Step 5: Find a solution $a_j's$ of (10), if such exists, so that (4) is a Hurwitz polynomial. If such solution exists then Theorem 1 is satisfied and the order $\hat{g}=g$. Next, by applying Theorem 2 we obtain $d=\hat{g}$. A solution of (10) with full particulars is discussed below. When $a_j's$ are obtained, exact values of coordinates of the constant vector F and exact values of the coefficients $k_i's (i \in g-1)$ are

easily calculated from the expressions derived in step 3. When given F and k_i 's, the coefficients Q_i 's ($i \in d$) and a_i 's ($i \in d-1$) are calculated from (8) and (9) respectively. This terminates the procedure.

Step 6: If there exists no solution of (10) then increase g by one and repeat the procedure from step 2.

The stability properties of the characteristic polynomial (4) depend on coefficients a_j 's ($j \in n+g-1$), and it is required to find appropriate values of a_j 's under the imposed conditions (10). The necessary (though not sufficient) conditions for (4) to be stable is that a solution of (10) should be strictly positive. An existence of such solution can be verified by classical algebraic methods e.g. by using Cramer's rule. When a strictly positive solution of (10) exists, one has to verify an existence of a stable one. In general, by applying standard stability tests e.g. the Hurwitz criterion, (10) is easily solvable for $t \leq 2$ or low order n . However, for high t , a set of inequalities with a_j 's to a high power is obtained. In effect, one may fail to find a stable solution by using standard stability criterions, for the detailed analysis see Anderson et al⁽⁹⁾.

In this paper we shall tackle the problem in a new way. Without loss of generality one may assume that the linear equations (10) are independent. By applying e.g. Gaussian elimination method, the number of equations (10) can be minimized up to the situation when each equation has a different set of members a_j 's. For example, assume that we have obtained t_1 ($t_1 \leq t$) equations:

$$f_k(a_j) = 0 \quad k=1, \dots, t_1, \quad j \in n+g-1 \quad (11)$$

Now, we propose the following notations for the characteristic polynomial (4):

$$H(s) = \prod_{i=1}^{(n+g)/2} (s^2 + \beta_i s + \delta_i) \quad (12a)$$

if $n+g$ is even, or

$$H(s) = (s + \gamma) \prod_{i=1}^{(n+g-1)/2} (s^2 + \beta_i s + \delta_i) \quad (12b)$$

if $n+g$ is odd.

The characteristic polynomial (12) is stable if and only if $\gamma, \beta_i, \delta_i > 0$. By comparing (4) and (12) the coefficients a_j 's can be expressed as

polynomials in γ, β_i 's and δ_i 's. By substituting these in (11) we obtain nonlinear equations in real variables γ, β_i 's and δ_i 's to the first power:

$$f_k(\gamma, \beta_i, \delta_i), \quad k=1, \dots, t_1 \quad (13)$$

Clearly, (13) has a strictly positive solution in γ, β_i 's and δ_i 's if and only if the output feedback stabilization problem has a solution, and any strictly positive solution of (13) provides a solution of the stabilization problem. We readily note that (13) can be replaced by the following single equality:

$$\sum_{k=1}^{t_1} [f_k(\gamma, \beta_i, \delta_i)]^2 = 0 \quad (14)$$

with the constraints $\gamma > 0, \beta_i > 0$ and $\delta_i > 0$. Eq. (14) can be solved by computers with readily available numerical minimization methods or in simple cases by hand calculations. Since (13) is in real variables to the first power, the iteration process is well defined for any initial data set. If a minimum equal to zero exists, a solution of the output feedback stabilization problem has been found. Basically, the set of solutions is either open or empty, rather than unique. Each solution corresponds to a different characteristic polynomial of the same degree. A method of selecting a characteristic polynomial with best stability properties is of great practical importance and has to be further investigated. When given the solution of (14), the coefficients Q_i 's are calculated from (8), nonuniquely in general. This non-uniqueness does not effect the location of the system's poles, but sensitivity properties to variation of plant parameters may differ, for the analysis refer to Shaw⁽¹⁷⁾. Example 1: Find a dynamic feedback (2) of least order which stabilizes the following system:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 3 & -2 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x \quad (15)$$

Step 1: Set up $g=0$ and denote the unit vectors by e_i ($i=1, \dots, 4$). Then $\mathcal{C}\mathcal{V}_0^\perp = \text{Span}\{e_3, e_4\}$ and $\mathcal{C}\mathcal{V}_1^\perp = \text{Span}\{e_4\}$.

Step 2: By equating the corresponding powers of s on both sides of (4) for $g=0$, we determine that the characteristic polynomial

$$H(s) = s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 \quad (16)$$

can be obtained by the state feedback (3) in the form

$$u = Fx = -[a_0, a_1 + 3, a_2 - 2, a_3]x \quad (17)$$

Step 3: The state feedback (17) can be transformed into an equivalent output feedback if $F' \perp \mathcal{CV}_0^\perp$. By applying (10) we obtain:

$$\langle F', e_3 \rangle = a_2 - 2 = 0$$

$$\langle F', e_4 \rangle = a_3 = 0$$

Step 4: The condition $a_3 = 0$ results in an unstable characteristic polynomial (16). Consequently, the system (15) cannot be stabilized by any choice of an output feedback of the order $d = g = 0$. Therefore, we have to increase the order g by one and repeat the procedure again.

Step 5: By equating corresponding powers of s on both sides of (4) for $g = 1$ we determine that the characteristic polynomial

$$H(s) = s^5 + a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 \quad (18)$$

can be obtained by the state feedback (3) in the form

$$\begin{aligned} \dot{u} &= k_0 u + Fx = -a_4 u - [a_0, a_1 + 3a_4, \\ &\quad a_2 - 2a_4 + 3, a_3 - 2]x \end{aligned} \quad (19)$$

Step 6: The state feedback (19) can be transformed into an equivalent output feedback if $F' \perp \mathcal{CV}_1^\perp$. By applying (10) we obtain:

$$\langle F', e_4 \rangle = a_3 - 2 = 0 \quad (20)$$

Step 7: The remaining coefficients of the characteristic polynomial (18) can be obtained easily by using e.g. the Hurwitz criterion. However, in order to illustrate the design procedure let us assume $H(s)$ in the following form:

$$H(s) = (s + \gamma)(s^2 + \beta_1 s + \delta_1)(s^2 + \beta_2 s + \delta_2) \quad (21)$$

which if compared with (18) yields:

$$\left. \begin{aligned} a_0 &= \delta_1 \delta_2 \gamma \\ a_1 &= \delta_1 \delta_2 + \beta_1 \beta_2 + \delta_1 \beta_2 \\ a_2 &= \beta_1 \delta_2 + \beta_2 \delta_1 + \delta_1 \gamma + \delta_2 \gamma + \beta_1 \beta_2 \gamma \\ a_3 &= \beta_1 \beta_2 + \beta_1 \gamma + \beta_2 \gamma + \beta_1 + \delta_2 \\ a_4 &= \beta_1 + \beta_2 + \gamma \end{aligned} \right\} \quad (22)$$

By substituting (22) in (20):

$$\beta_1 \beta_2 + \beta_1 \gamma + \beta_2 \gamma + \delta_1 + \delta_2 - 2 = 0 \quad (23)$$

We note, that the stable solution is not unique.

Take for example:

$$\gamma = 0.6, \quad \beta_1 = 0.56, \quad \delta_1 = 0.2, \quad \beta_2 = 0.4, \quad \delta_2 = 1 \quad (24)$$

By substituting these values in (22):

$$a_0 = 0.12, \quad a_1 = 0.84, \quad a_2 = 1.4944, \quad a_3 = 2,$$

Fig. 3 A realization of the closed-loop system in eqs. (15) and (27)

$$a_4 = 1.56 \quad (25)$$

Since $g = 1$ satisfies Theorem 1 we have $\hat{g} = 1$, and a dynamic output feedback of the order $d = \hat{g} = 1$ is sufficient for stabilization of the system (15). By substituting (25) into (19) we have:

$$\dot{u} = -1.56u - [0.12, 5.52, 1.3744, 0]x \quad (26)$$

By applying (8):

$$[Q_0, Q_1] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = -[0.12, 5.52, 1.3744, 0]$$

This leads to:

$$Q_0 = [-0.12, -5.52], \quad Q_1 = [-1.3744, 0] \quad (27a)$$

Note that the solution Q_0 and Q_1 is not unique. By substituting (26) and (27a) in (9) we obtain:

$$a_0 = -1.56 - [-1.3744, 0] \begin{bmatrix} 0 \\ 0 \end{bmatrix} = -1.56 \quad (27b)$$

6. Conclusion

A new method of designing the least order, stable output feedback system which has applicability to a wider class of problems has been presented. The least order output feedback is computed only once and is obtained in a form suitable for direct realization. The advantage of this method is that it can be applied to both high order systems and low order systems, while the existing methods e.g. root-loci, methods based on the Hurwitz or Nyquist criteria, etc., give the satisfactory solution for low order systems. It was shown that the least order was the unique property of a given system and was determined by the constant vector from the states. When the least order is found, the corresponding set of characteristic polynomials is usually open. When

the characteristic polynomial is selected, the corresponding set of output feedbacks is nonunique in general. Although the output feedbacks give rise to the same set of poles, the system's sensitivity to variation of plant parameters may differ (see Shaw¹⁷). Therefore, an interesting problem for further study is to find the least order output feedback in respect to sensitivity.

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