L_1 Optimal Controller for MIMO Continuous-Time Systems[†]

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In this paper, an optimal solution of the continuous-time L_1 control problem for 1-block MIMO systems is studied. Firstly, a characterization of an optimal solution is derived. A crucial idea of the approach is the duality of convex sets of finite dimensional vector spaces. This avoids a functional space argument. Secondly, it is shown that confining controllers to be finite dimensional does not hamper the ability to reduce the L_∞ induced norm of the closed loop operator.

Key Words: L_1 optimal control, interpolation condition, convex analysis, polar set, duality

1. Introduction

The problem of $L_1(h)$ control was first formulated by Vidyasager¹⁾. If the set of disturbances is unknown but bounded signals and the goal is to minimize the bound of the resulting output, the control problem is to minimize L_∞ induced norm. The problem is called the $L_1(h)$ optimization because it involves minimizing the $L_1(h)$ norm of the impulse response.

Dahleh and Pearson solved the problem for discrete and continuous-time systems^{2),3)} with the assumption that the system is 1-block. In 3) it was shown that an optimal closed loop system is a sum of delta functions, which implies that the corresponding optimal controller is infinite dimensional even if the plant is finite dimensional. Ohta, Maeda and Kodama showed that for SISO continuous-time systems finite dimensional controllers can achieve the optimal cost arbitrarily closely⁴⁾.

In this paper, 1-block MIMO continuous-time L_1 control problems are studied, and an optimal solution is given using duality theory on convex sets in finite dimensional spaces. In 3), the alignment condition that is a part of the duality theory on functional spaces was used to derive an optimal solution. The

merits of using finite dimensional argument are the following:

- (i) the existence of an optimal solution is directly shown.
- (ii) it is shown that an optimal solution is a scaled convex combination of delta functions that correspond to extreme points of the polar set of a certain convex set.

As a consequence of this result, the result in 4) is extended, and it is shown that 1-block MIMO continuous-time L_1 problems have rational controllers that achieves the optimal cost in the limit.

2. Preliminaries

In this section, the notions of convolution operators and convex sets are reviewed. For the Banach algebra $\mathcal{A}$ that is a class of BIBO stable convolution operators, 5) is a good source for information. For convex sets, see 6) for example.

2.1 Convolution Operator

The Banach space of n -vector-valued continuous-time bounded signals is denoted by L_∞ where n is suppressed for simplicity. For $x \in L_\infty$, the norm is defined by

$$\|x\|_\infty = \max \{ \sup \{ |x_i(t)| : t \geq 0 \} : 1 \leq i \leq n \}.$$

The convolution operator $H : L_\infty \rightarrow L_\infty$ is an input-output model for linear time-invariant systems, and is given by

$$y = Hu, \tag{1}$$

$$y(t) = \int_0^t H_M(t-\tau)u(\tau)d\tau + \sum_{k, t_k \leq t} H_k u(t-t_k),$$

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where

$$H(t) = H_M(t) + H_{AM}(t)$$

$$H_M \in L_1(0, \infty),$$

$$H_{AM}(t) = \sum_{k=0}^{\infty} H_k \delta(t - t_k),$$

$$\{H_k\} \in l_1, t_k \geq 0.$$

The convolution kernel consists of continuous measure H_M and atomic measure H_{AM} where the measure on $t_k, \delta(t - t_k)$, is the pure delay t_k . Notice that H_M and H_{AM} are $q \times r$ -matrix valued functions where q is the number of output and r is the number of input.

The induced norm of the operator H of (1) is shown to be equal to the sum of the L_1 norm of the continuous kernel and the l_1 norm of the atomic kernel. More precisely, for SISO systems,

$$\|H\|_{\text{ind}} = \int_0^{\infty} |H_M(t)| dt + \sum_{k=0}^{\infty} |H_k|$$

$$\stackrel{\text{det}}{=} \|H\|_{\mathcal{A}}$$

and for MIMO systems,

$$\|H\|_{\text{ind}} = \max \{ \sum_{j=1}^r \|H_{ij}\|_{\mathcal{A}}, \mathcal{A}: i=1, \dots, q \}$$

$$\stackrel{\text{det}}{=} \|H\|_{\mathcal{A}}$$

where H_{ij} is (i, j) -th element of H .

For SISO systems, the class of convolution operators is a Banach algebra, and is denoted by $\mathcal{A}$. For MIMO systems, the operator is an $\mathcal{A}$ -valued matrix. For notational simplicity, it is denoted as $\mathcal{A}$ as well in the sequel.

The Laplace transform of the operator H is

$$\begin{aligned} \hat{H}(s) &= \int_0^{\infty} H_M(t) \exp(-st) dt \\ &\quad + \sum_{k=0}^{\infty} H_k \exp(-st_k), \end{aligned}$$

and is called the transfer function. An important subclass of $\mathcal{A}$ is the class $\mathcal{B}$ of finite dimensional systems whose Laplace transform is stable rational functions.

2.2 Convex Analysis

The convex hull of a set V in a finite dimensional vector space is the smallest convex set that contains V , and is denoted by $\text{conv. } V$. The polar set of V is the convex set defined by

$$V^\circ = \{x: x^T y \leq 1, y \in V\}. \quad (2)$$

The following properties, reflexivity and duality, are the key to the construction of an L_1 optimal solution.

Proposition 1.

$$V \subseteq V^{\circ\circ}$$

with the equality holds if and only if V is a closed convex set that contains the origin.

(Proof) From the definition (2), $V \subseteq V^{\circ\circ}$ is obvious. Also the definition implies that a polar set is a closed convex set and contains the origin. This proves the necessity. For the sufficiency, let $0 \neq z \notin V$. Then the Hahn-Banach theorem⁷⁾ implies that there is y such that $y^T x < 1 < y^T z$ for any $x \in V$. Thus from (2), $y \in V^\circ$. Hence again from (2), $z \notin V^{\circ\circ}$.

Q. E. D.

Proposition 2. Suppose that the polar set $V^\circ \subset R^n$ is bounded. Let $c \in R^n$ be a fixed vector. Consider the linear programming

$$\gamma_1 = \max c^T x, x \in V^\circ, \quad (3)$$

and the minimization problem

$$\gamma_2 = \min \{ \gamma: c \in \gamma V^{\circ\circ}, \gamma \geq 0 \}. \quad (4)$$

Then

$$\gamma_1 = \gamma_2.$$

(Proof) Since V° is bounded, $V^{\circ\circ}$ contains an open ball around the origin. Thus the minimum of (4) is well defined. Suppose $c \in \gamma V^{\circ\circ}$, $\gamma \neq 0$. Then $c/\gamma \in V^{\circ\circ}$ implies that $(c/\gamma)^T x \leq 1$ or $c^T x \leq \gamma$ for any $x \in V^\circ$. Hence $\gamma_1 \leq \gamma_2$. Since $\gamma_1 \geq 0$, it suffices to assume that $\gamma_2 > 0$. Let $0 < \gamma < \gamma_2$. Then from the Hahn-Banach theorem $c \notin \gamma V^{\circ\circ}$ implies that there is y such that $y^T z < 1 < y^T c$ for any $z \in \gamma V^{\circ\circ}$. From (2), $y \in (\gamma V^{\circ\circ})^\circ = \gamma^{-1} V^{\circ\circ\circ} = \gamma^{-1} V^\circ$. Hence $\gamma y \in V^\circ$, and it follows that $\gamma \leq \gamma_1$. Since $\gamma < \gamma_2$ is arbitrary, $\gamma_2 \leq \gamma_1$.

Q. E. D.

Usually the computation of the polar set is not practical. However, when a convex set has a particular structure, it is easily obtained. The following proposition computes the polar set for convex sets that appears in L_1 control problems.

Proposition 3. Let A be a possibly infinite index set, and $a(i, \lambda)$, $1 \leq i \leq m$, $\lambda \in A$, be R^n vectors. Define convex sets in R^n

$$P' = \{x: \sum_{i=1}^m |a(i, \lambda)| T_x \leq 1, \lambda \in A\},$$

and

$$\begin{aligned} V' &= \text{conv} \{ \pm a(1, \lambda) \pm \dots \pm a(m, \lambda) : \\ &\quad \lambda \in A, \pm \text{takes any combination} \}. \end{aligned}$$

Then

$$P' = V'^\circ.$$

(Proof) Let $y \in V'^\circ$. From the definition, it follows that

$$y^T \{ \pm a(1, \lambda) \pm \dots \pm a(m, \lambda) \} \leq 1$$

for any choice of the plus and minus sign and $\lambda \in A$. This is equivalent to

$$\sum_{i=1}^m |y^T a(m, \lambda)| \leq 1,$$

or $y \in P'$. Conversely, let

$$z = \pm a(1, \lambda) \pm \dots \pm a(m, \lambda)$$

for fixed $\lambda \in \Lambda$ and fixed set of plus and minus sign. If $x \in P'$, then

$$z^T x = \pm a(1, \lambda)^T x \pm \dots \pm a(m, \lambda)^T x \leq 1.$$

Thus $z \in P^\circ$. Since V' is a convex set generated by such a z , and P° is also a convex set, it follows that $V' \subset P^\circ$. Hence $V^\circ \supset P^{\circ\circ} = P'$, where the equality holds because P' is a closed convex set containing the origin (see Proposition 1). Q. E. D.

3. Problem Definition

Consider the generalized plant P

$$z = P_{11}w + P_{12}u$$

$$y = P_{21}w + P_{22}u$$

where w is the exogenous input, u is the control input, z is the output to be controlled, and y is the observation. Let q be the number of output and r be the number of input.

The objective of the control is to design a feedback controller

$$u = Ky$$

in such a way that (i) the closed loop is stable, and (ii) minimize the induced norm of the resulting operator T_{wz} from w to z . Since the signal is measured by the L_∞ norm, the induced norm is the $\mathcal{A}$ norm.

Making use of the parametrization of stabilizing controllers, under some mild conditions the resulting closed loop operator has the form

$$T_{wz} = T_1 - T_2 Q T_3 \quad (5)$$

where $T_1, T_2, T_3 \in \mathcal{A}$ are fixed operators determined by the plant P , and $Q \in \mathcal{A}$ is the free parameter^{2),3)}. When the plant P is finite dimensional, so are T_1, T_2, T_3 . In this case, choosing $Q \in \mathcal{B}$ makes the controller finite dimensional.

Thus the problem of minimizing the L_∞ induced norm of the closed loop operator from w to z is equivalent to the following model matching problems

$$\gamma_{\text{MM}} = \inf \{ \|T_1 - T_2 Q T_3\|_{\mathcal{A}} : Q \in \mathcal{A} \}. \quad (6)$$

$$\gamma_{\text{RMM}} = \inf \{ \|T_1 - T_2 Q T_3\|_{\mathcal{A}} : Q \in \mathcal{B} \}. \quad (7)$$

where in (7) the controller is confined to be finite dimensional.

The problems (6) and (7) are called 1-block if T_2 and T_3 are square and

$$\det \hat{T}_2(s) \neq 0, \text{ and } \det \hat{T}_3(s) \neq 0$$

hold. A complex number s_0 is called a zero of $\hat{T}(s)$ if $\det \hat{T}(s_0) = 0$.

Assumption 1. The model matching problems (6) and (7) satisfy:

- (i) $\hat{T}_1(s), \hat{T}_2(s), \hat{T}_3(s)$ are rational,
- (ii) 1-block problem, and
- (iii) The zeros of $\hat{T}_2(s)$ and $\hat{T}_3(s)$ are in the open right half plane.

When $\hat{T}_2(s)$ and $\hat{T}_3(s)$ do not have imaginary axis zeros, it is possible to define $\hat{T}_{1,\text{new}}(s), \hat{T}_{2,\text{new}}(s)$, and $\hat{T}_{3,\text{new}}(s)$ so that the assumption (i) holds. For $\hat{T}_{2,\text{new}}(s)$ and $\hat{T}_{3,\text{new}}(s)$, this is done by lumping invertible elements with the free parameter Q . For $\hat{T}_{1,\text{new}}(s)$, construct a rational transfer function that interpolates $\hat{T}_1(s)$ at the right half plane zeros of $\hat{T}_2(s)$ and $\hat{T}_3(s)$.

It is true that many control problems are not 1-block, and the assumption is sometimes restrictive. However, there is a direction of research to apply 1-block problems to solve multi-block problems⁹⁾, where the significance of 1-block problems is obvious.

4. Duality

In this section, the results on the duality in 3) and 9) are included for notational convenience. Specifically, the dual space of $\mathcal{A}$, the interpolation condition, and the characterization of the annihilator are reviewed.

4.1 Dual Problem

The duality theorem can be applied to yield the following proposition because the model matching problem (6) is the minimum distance problem between a point and a subspace.

Proposition 4.³⁾ Consider the model matching problem

$$\gamma_{\text{MM}} = \inf \{ \|T_1 - R\|_{\mathcal{A}} : R \in S \}$$

$$S = \{ R : R = T_2 Q T_3, Q \in \mathcal{A} \} \subset \mathcal{A}, \quad (8)$$

and its dual problem

$$\gamma_{\text{DP}} = \max \{ \langle F, T_1 \rangle : F \in S^\perp, \|F\| \leq 1 \}. \quad (9)$$

Then

$$\gamma_{\text{MM}} = \gamma_{\text{DP}}.$$

4.2 Dual Space

The dual space of the Banach space $\mathcal{A}$ is given in 3). For SISO systems

$$\mathcal{A}^* = L_\infty(0, \infty) \otimes B[0, \infty)$$

where $L_\infty(0, \infty)$ is the space of essentially bounded functions and $B[0, \infty)$ is the space of bounded functions. The evaluation of the linear functional is

$$\langle F, H \rangle = \int_0^\infty F_M(t) H_M(t) dt + \sum_{k=0}^\infty F_{AM}(t_k) H_k$$

where

$$F = (F_M, F_{AM}) \in \mathcal{A}^*,$$

$$F_M \in L_\infty(0, \infty), F_{AM} \in B[0, \infty).$$

The norm in $\mathcal{A}^*$ is given by

$$\|F\| = \max \{\|F_M\|_\infty, \|F_{AM}\|_\infty\},$$

$$\|F_M\|_\infty = \text{ess. sup } |F_M(t)|,$$

$$\|F_{AM}\|_\infty = \sup \{|F_{AM}(t)| : t \geq 0\}.$$

The extension to MIMO systems is straightforward, and the dual space $\mathcal{A}^*$ is

$$\mathcal{A}^* = L_\infty(0, \infty) \otimes B[0, \infty)$$

where $L_\infty(0, \infty)$ is the space of $q \times r$ -matrix valued essentially bounded functions and $B[0, \infty)$ is the space of $q \times r$ -matrix valued bounded functions. Let F_{ij} be (i, j) -th element of $F \in \mathcal{A}^*$ and H_{ij} be (i, j) -th element of $H \in \mathcal{A}$. The evaluation of the linear functional is

$$\langle F, H \rangle = \sum_{i,j=1}^q \langle F_{ij}, H_{ij} \rangle,$$

and the norm is

$$\|F\| = \sum_{i=1}^q \max \{\|F_{ij}\| : j=1, 2, \dots, r\}.$$

4.3 Interpolation Condition

The subspace (8) consists of input-output maps whose “zero structure” is inherited from T_2 and T_3 . In this subsection, what it means by “zero structure” is clarified which leads to the interpolation condition. This is an easy extension to the result in 9).

By Assumption 1 (i), $\hat{T}_2(s)$ and $\hat{T}_3(s)$ are stable rational matrices. Transform $\hat{T}_2(s)$ and $\hat{T}_3(s)$ into the Smith forms:

$$\hat{T}_2(s) = \hat{L}_2(s) \text{diag} \{\hat{\sigma}_{21}(s), \hat{\sigma}_{22}(s), \dots, \hat{\sigma}_{2q}(s)\} \hat{R}_2(s)$$

$$\hat{T}_3(s) = \hat{L}_3(s) \text{diag} \{\hat{\sigma}_{31}(s), \hat{\sigma}_{32}(s), \dots, \hat{\sigma}_{3r}(s)\} \hat{R}_3(s)$$

where $\hat{L}_i(s)$ and $\hat{R}_i(s)$, $i=2, 3$, are unimodular matrices, and

$$\hat{\sigma}_{2i}(s), i=1, \dots, q; \hat{\sigma}_{3i}(s), i=1, \dots, r$$

are the elementary divisors. Notice that by Assumption 1(ii) there are q (resp. r) divisors of the matrix $\hat{T}_2(s)$ (resp. $\hat{T}_3(s)$). Define the set of zeros Σ as

$$\Sigma = \{\zeta : \exists j, \hat{\sigma}_{2j}(\zeta) = 0 \text{ or } \hat{\sigma}_{3j}(\zeta) = 0\}.$$

Notice that by Assumption 1(iii) Σ is a subset of the open right half plane. Define $\kappa(\zeta, i, j)$ as the multiplicity of the zero $\zeta = \Sigma$ of $\hat{\sigma}_{ij}(s)$, $i=2, 3$, $j=1, \dots, q$ or r . Also define vectors of stable rational functions:

$$\hat{L}_2(s)^{-1} = \begin{bmatrix} \hat{\alpha}_1(s)^T \\ \hat{\alpha}_2(s)^T \\ \vdots \\ \hat{\alpha}_q(s)^T \end{bmatrix}, \text{ and}$$

$$\hat{R}_3(s)^{-1} = [\hat{\beta}_1(s) \hat{\beta}_2(s) \dots \hat{\beta}_r(s)].$$

The following characterizes the subspace S in the transformed domain.

Proposition 5.⁹⁾ Let $S = \{R : R = T_2 Q T_3, Q \in \mathcal{A}\} \subset \mathcal{A}$. Then $R \in S$ if and only if

$$(\hat{\alpha}_i^T \hat{R} \hat{\beta}_j)^{(k)}(\zeta) = 0, \quad (10)$$

where

$$\zeta \in \Sigma, i=1, \dots, q, j=1, \dots, r,$$

$$k=0, 1, \dots, \kappa(\zeta, 2, i) + \kappa(\zeta, 3, j) - 1.$$

4.4 Annihilator

Proposition 5 decides elements of the annihilator of S as follows. Since Σ is a subset in the open right half plane, the evaluation of the Laplace transformation as (10) is a bounded linear functional on $\mathcal{A}$. Hence each index (z, i, j, k) corresponds to a linear functional in $S^\perp \subset \mathcal{A}^*$.

To give a representation of the functional, define the index set

$$\Pi = \{\pi : \pi = (z, i, j, k); z \in \Sigma; i=1, \dots, q; j=1, \dots, r; k=0, \dots, \kappa(z, 2, i) + \kappa(z, 3, j) - 1\}.$$

For $\pi = (z, i, j, k) \in \Pi$, the notations

$$z(\pi) = z, i(\pi) = i, j(\pi) = j, \text{ and } \kappa(\pi) = k$$

are used. The definition of the Laplace transform gives:

$$\langle F_\pi, R \rangle = (\hat{\alpha}_i^T \hat{R} \hat{\beta}_j)^{(k)}(z) \quad (11)$$

$$F_\pi = (F_{\pi M}, F_{\pi AM})$$

$$F_{\pi M}(t) = F_{\pi AM}(t)$$

$$= \sum_{0 \leq k_1 + k_2 \leq k} \frac{k!}{(k - k_1 - k_2)! k_1! k_2!} \\ \times \hat{\beta}_j^{(k_2)}(z) \hat{\alpha}_i^{(k_1)}(z)^T (-t)^{k - k_1 - k_2} \exp(-zt).$$

Using F_π , equation (10) is written as

$$\langle F_\pi, R \rangle = 0, \pi \in \Pi \quad (12)$$

To confine the functional to be real valued, it is necessary to take the real part and imaginary part in (11). But this will need one more index to distinguish the functionals. To avoid extra complexity, it is assumed that the zeros are real.

If the zeros of $\hat{T}_2(s)$ and $\hat{T}_3(s)$ are disjoint and simple, the representation (10) is even simplified. Note that $\pi = (z, i, j, k) \in \Pi$ implies $k=0$, and there holds

$$F_\pi = (F_{\pi M}, F_{\pi AM}),$$

$$\begin{aligned} F_{\pi M}(t) &= F_{\pi AM}(t) \\ &= \hat{\beta}_j(z(\pi)) \hat{a}_i(z(\pi))^T \exp(-z(\pi)t). \end{aligned}$$

The functional $\langle F_\pi, R \rangle$ is nothing but the evaluation of the Laplace transform $\hat{a}_i^T \hat{R} \hat{\beta}_j$ at $s=z(\pi)$.

The following describes a subset of the annihilator, and is immediate from (10).

Proposition 6 Let

$$S^\# = \text{span} \{F_\pi : \pi \in \Pi\}.$$

Then

$$S^\# \subset S^\perp.$$

5. Optimal Solution

In this section, an optimal solution to the model matching problem (5) is given, and rational approximability is studied.

5.1 Linear Programming

The dual problem (9) is transformed to a linear programming with infinite number of constraints. Modify the dual problem to get

$$\gamma_{\text{MDP}} = \max \{ \langle F, T_1 \rangle : F \in S^\#, \|F\| \leq 1 \}. \quad (13)$$

By Proposition 6, it is evident that $\gamma_{\text{MDP}} \leq \gamma_{\text{DP}}$.

Notice that the subspace $S^\#$ is finitely generated. Thus the problem (13) becomes a maximization problem on the space of coefficients of the linear combination of F_π

$$F = \sum_{\pi \in \Pi} \omega_\pi F_\pi. \quad (14)$$

Actually, define

$$\lambda_\pi = \langle F_\pi, T_1 \rangle, \quad (15)$$

and let $\mathcal{Q}$ (resp. $\mathcal{A}$) $\in R^{|\Pi|}$ be the vector whose π -th ($\pi \in \Pi$) coordinate is ω_π (resp. λ_π), where $|\Pi|$ denotes the number of the elements in Π .

Define the convex set in $R^{|\Pi|}$

$$P = \{ \mathcal{Q} : \|\sum_{\pi \in \Pi} \omega_\pi F_\pi\| \leq 1 \}. \quad (16)$$

Proposition 7. The set P of (16) is a closed convex set that contains the origin, and satisfies

$$\begin{aligned} P = \{ \mathcal{Q} : \sum_{i=1}^q |\sum_{\pi \in \Pi} \omega_\pi F_{\pi M}(t_i)_{ij}| \leq 1, \\ j_i = 1, \dots, r; 0 \leq t_i \leq t_{\max} \}, \end{aligned} \quad (17)$$

for some $t_{\max} > 0$, where $F_{\pi Mij}$ is (i, j) th element of $F_{\pi M}$.

(Proof) It is obvious that P contains the origin, and is closed and convex. The norm constraint (16) is equivalent to (17) except that there is a time instant $t_{\max} > 0$. The latter is proved in the same way as in 8).

Proposition 8. The problem (13) is equivalent to the following linear programming.

$$\gamma_{LP} = \max \mathcal{A}^T \mathcal{Q}$$

subject to (18)

$$\mathcal{Q} \in P$$

where $\mathcal{A} \in R^{|\Pi|}$ whose π -th ($\pi \in \Pi$) component λ_π is given by (15).

(Proof) Equation (14) defines the map from $R^{|\Pi|}$ to $S^\#$. Equation (16) says that $\|F\| \leq 1$ is equivalent to $\mathcal{Q} \in P$. The objective functions are

$$\begin{aligned} \langle F, T_1 \rangle &= \langle \sum_{\pi \in \Pi} \omega_\pi F_\pi, T_1 \rangle \\ &= \sum_{\pi \in \Pi} \omega_\pi \langle F_\pi, T_1 \rangle \\ &= \sum_{\pi \in \Pi} \omega_\pi \lambda_\pi = \mathcal{A}^T \mathcal{Q}. \end{aligned} \quad \text{Q. E. D.}$$

5.2 Characterization of an Optimal Solution

The duality of Proposition 2 says that an optimal solution of a linear programming is a scaled convex combination of the extreme points of the polar set. The idea is applied to the problem (18), and yields a characterization of an optimal solution to the model matching problem (5). It turns out that an optimal solution of MIMO problems is a finite sum of delta functions just as the SISO case.

Proposition 9. Let W be the set of vectors in $R^{|\Pi|}$ whose π -th ($\pi \in \Pi$) component is of the form

$$\pm F_\pi(t_1)_{j_1} \pm \dots \pm F_\pi(t_q)_{j_q}, \quad (19)$$

where the plus and minus signs take any combination, and

$$j_i = 1, \dots, r; 0 \leq t_i \leq t_{\max}; i = 1, \dots, q.$$

Let V be the smallest convex set containing W , and P is given by (17). Then $P = V^\circ$.

(Proof) Since W is symmetric with respect to the origin, V contains the origin. Equation (19) can be seen as the continuous map from the compact set $2^q \times r^q \times [0, t_{\max}]^q$ to $R^{|\Pi|}$. Thus W is compact, and so is V . Hence $P = V^\circ$ is a consequence of Proposition 3.

Q. E. D.

In view of Propositions 2 and 9, the linear programming (18) has the dual representation.

Proposition 10. Let $\mathcal{A} \in R^{|\Pi|}$ be defined by (15), and W be as in Proposition 9. Then there are vectors $w_p \in W$ and real numbers μ_p ($p = 1, \dots, |\Pi| + 1$) such that

$$\mathcal{A} = \sum_{p=1}^{|\Pi|+1} \mu_p w_p,$$

and

$$\sum_{p=1}^{|\Pi|+1} |\mu_p| = \gamma_{LP}.$$

Proposition 10 gives an optimal solution of the L_1 model matching problem (5). For each $w \in W$, define $\Phi_{AM,w} \in \mathcal{A}$ as follows: if π -th ($\pi \in \Pi$) compo-

nent of the vector w is given by (19), then each row of $\Phi_{AM,w}$ is zero except at (i, j_i) -th component that is equal to $\pm \delta(t - t_i)$:

$$\Phi_{AM,w} = \begin{bmatrix} \cdots & \pm \delta(t - t_1) & \cdots & \cdots \\ \cdots & \cdots & \pm \delta(t - t_2) & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ \cdots & \pm \delta(t - t_q) & \cdots & \cdots \end{bmatrix}, \quad (20)$$

where at i -th row $\pm \delta(t - t_i)$ is at j_i -th column. Notice that

$$\|\Phi_{AM,w}\|_{\mathcal{A}} = 1.$$

Theorem 1. Suppose Assumption 1 holds. Then an optimal solution Q_{opt} of the model matching problem (5) exists, and is given by

$$\Phi_{\text{opt}} = T_1 - T_2 Q_{\text{opt}} T_3, \quad (21)$$

$$\Phi_{\text{opt}} = \Phi_{AM} = \sum_{p=1}^{|\Pi|+1} \mu_p \Phi_{AM,w_p},$$

where μ_p and w_p are given by Proposition 9, and $\Phi_{AM,w}$ is given by (20).

(Proof) To show that $\Phi_{\text{opt}} - T_1 \in S$, the condition (12) has to be checked, i. e.,

$$\langle F_{\pi}, \Phi_{\text{opt}} - T_1 \rangle = 0, \quad \pi \in \Pi. \quad (22)$$

It follows from the evaluation property of delta functions that

$$\begin{aligned} \langle F_{\pi}, \Phi_{AM,w} \rangle &= \pm F_{\pi}(t_1)_{1j_1} \pm \cdots \pm F_{\pi}(t_q)_{qj_q} \\ &= w_{p\pi} \end{aligned}$$

Thus

$$\begin{aligned} \langle F_{\pi}, \Phi \rangle &= \sum_{p=1}^{|\Pi|+1} \mu_p \langle F_{\pi}, \Phi_{AM,w_p} \rangle \\ &= \sum_{p=1}^{|\Pi|+1} \mu_p w_{p\pi} \\ &= \lambda_{\pi} \\ &= \langle F_{\pi}, T_1 \rangle \end{aligned}$$

and this establishes (22). Hence there is $Q_{\text{opt}} \in \mathcal{A}$ in (21). To evaluate the norm of Φ_{opt} ,

$$\begin{aligned} \|\Phi_{\text{opt}}\|_{\mathcal{A}} &= \|\Phi_{AM}\|_{\mathcal{A}} \\ &\leq \sum_{p=1}^{|\Pi|+1} |\mu_p| \|\Phi_{AM,w_p}\|_{\mathcal{A}} \\ &= \sum_{p=1}^{|\Pi|+1} |\mu_p| \\ &= \gamma_{LP} \leq \gamma_{DP} = \gamma_{MM} \end{aligned}$$

This implies that Q_{opt} is an optimal solution.

Q. E. D.

Notice that the importance of Proposition 10 is not for computational purpose, but for the qualitative characterization of an optimal solution. Indeed, Theorem 1 says an optimal solution is a scaled convex combination of vertices of the convex set consisting of delta functions as equation (20). These delta

functions correspond to extreme points of the polar set.

5.3 Finite Dimensional Controllers

In 4), it was shown that for SISO problems the achievable norm of the model matching problem (5) and that of the rational model matching problem (6) are equal, i. e. $\gamma_{MM} = \gamma_{RMM}$. In what follows, it is shown that this is also true for 1-block MIMO problems.

Theorem 2. Consider the model matching problems (5) and (6). Suppose Assumption 1 holds. Then

$$\gamma_{MM} = \gamma_{RMM}.$$

(Proof) Let Φ_{opt} be an optimal solution for (5) whose existence was proved by Theorem 1. Let $1 \leq i_0 \leq q$, $1 \leq j_0 \leq r$, and define $\Phi_{i_0 j_0}$ be (i_0, j_0) -th element of Φ_{opt} . Note that $\Phi_{i_0 j_0}$ is an SISO operator. Fix $\varepsilon > 0$. Consider the SISO interpolation problem

$$\begin{aligned} \gamma_{i_0 j_0} &= \inf \{ \|\Psi\|_{\mathcal{A}} : \langle F_{\pi}, \Psi \rangle \\ &= \langle F_{\pi}, e_{i_0} \Phi_{i_0 j_0} e_{j_0}^T \rangle \}. \end{aligned} \quad (23)$$

Notice that the linear functional F_{π} is the evaluation of the Laplace transformation indexed by $\pi \in \Pi$. Hence the constraint in (23) is equivalent to

$$\begin{aligned} &(\hat{\alpha}_i^T e_{i_0} \tilde{\Psi}_{i_0 j_0} e_{j_0}^T \hat{\beta})^{(k)}(\zeta) \\ &= (\hat{\alpha}_i^T e_{i_0} \hat{\Phi}_{i_0 j_0} e_{j_0}^T \hat{\beta})^{(k)}(\zeta). \end{aligned} \quad (24)$$

Since (24) is an interpolation condition for an SISO model matching problem, there is a finite dimensional SISO system $\Psi_{i_0 j_0}^{(\varepsilon)}$ such that (24) and

$$\|\Psi_{i_0 j_0}^{(\varepsilon)}\|_{\mathcal{A}} < \gamma_{i_0 j_0} + \varepsilon$$

hold. Define the finite dimensional MIMO system $\Psi^{(\varepsilon)}$ as the system whose (i_0, j_0) -th element is $\Psi_{i_0 j_0}^{(\varepsilon)}$. $\Psi^{(\varepsilon)}$ satisfies the interpolation condition. Indeed,

$$\begin{aligned} \langle F_{\pi}, \Psi^{(\varepsilon)} \rangle &= \sum_{i_0=1}^q \sum_{j_0=1}^r \langle F_{\pi}, e_{i_0} \Psi_{i_0 j_0}^{(\varepsilon)} e_{j_0}^T \rangle \\ &= \sum_{i_0=1}^q \sum_{j_0=1}^r \langle F_{\pi}, e_{i_0} \Phi_{i_0 j_0} e_{j_0}^T \rangle \\ &= \langle F_{\pi}, \Phi_{\text{opt}} \rangle \end{aligned}$$

Furthermore,

$$\begin{aligned} \|\Psi^{(\varepsilon)}\|_{\mathcal{A}} &= \max \left\{ \sum_{j_0=1}^r \|\Psi_{i_0 j_0}^{(\varepsilon)}\|_{\mathcal{A}} : i_0 = 1, 2, \dots, q \right\} \\ &< \max \left\{ \sum_{j_0=1}^r (\gamma_{i_0 j_0} + \varepsilon) : i_0 = 1, 2, \dots, q \right\} \\ &= \max \left\{ \sum_{j_0=1}^r \gamma_{i_0 j_0} : i_0 = 1, 2, \dots, q \right\} + r\varepsilon. \end{aligned} \quad (25)$$

Since $\gamma_{i_0 j_0}$ is the optimal norm, it follows that

$$\gamma_{i_0 j_0} \leq \|\Phi_{i_0 j_0}\|_{\mathcal{A}} \quad (26)$$

From (25) and (26), the inequality

$$\|\Psi^{(\varepsilon)}\|_{\mathcal{A}} \leq \|\Phi_{\text{opt}}\|_{\mathcal{A}} + r\varepsilon.$$

holds. This proves the theorem.

Q. E. D.

6. Conclusions

In this paper, an optimal solution of the L_1 optimal control for 1-block MIMO continuous-time systems is studied. First a characterization of an optimal solution is given. The notion of polar sets is the duality of convex sets in finite dimensional vector spaces, and proves useful in this setting. An optimal objective function is a scaled convex combination of delta functions. There exists a sequence of finite dimensional controllers for which achievable norm approaches the infimum value.

It is a current topic of investigation whether the framework of this paper is applicable to the general multi-block L_1 optimal design theory for continuous-time systems.

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