Controllability of Multiple Input Discrete-Time Linear Systems with Positive Controls[†]

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The controllability with input constraints is important in practical problems. Especially, the controllability with positive controls is useful in discussing the vibration control problem of a pendulum system, economic system, etc. And so, this paper presents a necessary and sufficient condition for the controllability to the zero state by positive controls of multiple input discrete-time linear systems. It is derived that the positive controllability of a given system may be reduced to those of its subsystems corresponding to the the Jordan canonical form. Since the sizes of the subsystems are smaller than that of a given system, then the positive controllability test can be simplified very much. Furthermore, the structure of a positive controllable system is revealed. The results in this paper are fundamental and essential in the control system theory.

Key Words : positive controls, controllability to the zero state, constrained controls, linear discrete-time systems, multiple input systems

1. Introduction

Many works concerning the controllability have been done since the concept of controllability was introduced by Kalman^{(1),(2)}. The controllability with input constraints is important in dealing with practical problems. Especially, the controllability with positive controls is useful in discussing the control problem of a pendulum system, economic system, etc^{(3)~(5)}. Evance and Murthy presented the necessary and sufficient condition for the controllability of single input discrete-time linear systems with positive controls⁽⁶⁾.

In general, there are two types of definitions of the controllability in regard to the desired state.

- ① Controllability to any state.
- ② Controllability to the zero state.

Evance and Murthy discussed the former case. On the other hand, the latter case is important in regard to a regulator problem. Yoshida studied the latter

case for single input systems⁽⁷⁾. The purpose of this paper is to give the necessary and sufficient conditions for the controllability to the zero state by positive controls of multiple input discrete-time linear systems.

2. Preliminaries

Consider a multiple input linear discrete time system described by

$$S: \mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k); k=0, 1, 2, \dots \quad (1)$$

where

$$\mathbf{A} \in R^{n \times n}, \mathbf{B} \in R^{n \times m}, \\ \mathbf{x}(k) \in R^n, \mathbf{u}(k) \in R^m$$

The control input is limited to the following condition C.

$$C: u_i(k) \geq 0; i=1, 2, \dots, m \quad (2)$$

where $u_i(k)$ is the i -th component of $\mathbf{u}(k)$.

Definition 1: The control input which satisfies (2) is called a positive control. $\square$

Definition 2: Let $\mathbf{x}(0)$ be any initial state. Then system S is called to be positive controllable, if there exist a positive integer ν independent of $\mathbf{x}(0)$ and a positive control sequence $\{\mathbf{u}(0), \mathbf{u}(1), \dots, \mathbf{u}(\nu-1)\}$ which will bring the system to the zero state from

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$x(0)$.

Definition 3: Let

$$B \triangleq [b_1, b_2, \dots, b_m]$$

$$b_i \in R^n; i=1, 2, \dots, m$$

Then we use the following notations:

$$L^+[B] \triangleq \{x | x = c_1 b_1 + c_2 b_2 + \dots + c_m b_m; \\ c_i \geq 0; i=1, 2, \dots, m\} \quad (3)$$

$$\langle A, B, N \rangle \triangleq [B, AB, \dots, A^{N-1}B] \in R^{n \times Nm} \quad (4)$$

$$E_m \triangleq [1, 1, \dots, 1]^T \in R^m$$

where T denotes the transpose. Furthermore, we use I_n as the $n \times n$ identity matrix.

Definition 4: If $x_i > 0$ or $x_i \geq 0$ for all $i=1, 2, \dots, n$, then $x = [x_1, x_2, \dots, x_n]^T$ is called to be a positive ($x > 0$) or nonnegative vector ($x \geq 0$), respectively.

Definition 5: Let A and B be transformed into

$$A = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix}, B = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix},$$

$$\tilde{A}_{11} \in R^{\tilde{n}_1 \times \tilde{n}_1}, \tilde{A}_{12} \in R^{\tilde{n}_1 \times \tilde{n}_2}, \tilde{A}_{22} \in R^{\tilde{n}_2 \times \tilde{n}_2},$$

$$\tilde{B}_1 \in R^{\tilde{n}_1 \times m}, \tilde{B}_2 \in R^{\tilde{n}_2 \times m},$$

$$n = \tilde{n}_1 + \tilde{n}_2$$

by a nonsingular transformation of the state variable.

Then system $\tilde{S}$ described by

$$\tilde{S}: \tilde{x}(k+1) = \tilde{A}\tilde{x}(k) + \tilde{B}u(k)$$

where $\tilde{x}(k) \in R^{\tilde{n}_2}$ is called the subsystem of S .

3. Positive Controllability

It is clear that the positive controllability is invariant under any nonsingular transformation of the state variable. Thus, system S described by (1) can be decomposed into the following two subsystems²⁾:

$$S_o: x_o(k+1) = A_o x_o(k) + B_o u(k) \quad (5)$$

$$S_r: x_r(k+1) = A_r x_r(k) + B_r u(k) \quad (6)$$

where A_o is a nilpotent matrix, and A_r is a nonsingular matrix;

$$A_o \in R^{n_o \times n_o}, B_o \in R^{n_o \times m}, x_o(k) \in R^{n_o},$$

$$A_r \in R^{n_r \times n_r}, B_r \in R^{n_r \times m}, x_r(k) \in R^{n_r},$$

$$n = n_o + n_r$$

Now we obtain the following theorem.

Theorem 1: System S is positive controllable if and only if subsystem S_r is positive controllable.

(Proof) Necessity: It is clear that if subsystem S_r is not positive controllable, then system S is not positive controllable. Hence the necessity is established.

Sufficiency: If subsystem S_r is positive controllable, then by Definition 2, there exist a positive integer ν_r and a positive control sequence $\{u(0), u(1), \dots, u(\nu_r$

$-1)\}$ which will bring subsystem S_r to the zero state $x_r(\nu_r) = 0$ for any initial state $x_r(0)$. Applying the positive control sequence $\{u(0), u(1), \dots, u(\nu_r-1), 0, 0, \dots, 0\}$ with length $\nu_r + n_o$, to subsystems S_o and S_r leads to $x_o(\nu_r + n_o) = 0$ and $x_r(\nu_r + n_o) = 0$, for any initial states $x_o(0)$ and $x_r(0)$. This means that S is positive controllable by definition 2. This completes the sufficiency. Q. E. D.

From Theorem 1, the positive controllability of S may be reduced to that of S_r .

Theorem 2: System S_r is positive controllable if and only if there exists a finite positive integer N , such that

$$L^+[\langle A_r, B_r, N \rangle] = R^{n_r} \quad (7)$$

where

$$B_r \triangleq [b_{r1}, b_{r2}, \dots, b_{rm}]$$

$$b_{ri} \in R^{n_r}; i=1, 2, \dots, m$$

(Proof) Suppose that $x_r(N) = 0$ is realized by applying a positive control sequence $\{u(0), u(1), \dots, u(N-1)\}$ to system S_r . Then we get

$$A_r^N x_r(0) + A_r^{N-1} B_r u(0) \\ + A_r^{N-2} B_r u(1) + \dots + B_r u(N-1) = 0 \quad (8)$$

Two n_r dimensional vectors, $A_r^N x_r(0)$ and $x_r(0)$, correspond to each other, since A_r^N is nonsingular. Thus it is possible to choose an appropriate positive control sequence $\{u(0), u(1), \dots, u(N-1)\}$ such that the left side of (8) may be zero for any initial state $x_r(0)$, if and only if (7) holds.

Therefore, by definition 2, Theorem 2 is established.

Q. E. D.

Theorem 2 gives a geometrical interpretation of a positive controllable system.

The following two lemmas can be directly obtained from Theorem 2 and Definition 3.

Lemma 1: If system S_r is positive controllable, then for any $x \in R^{n_r}$ there exist a positive integer N independent of x and a nonnegative vector Ω such that

$$\langle A_r, B_r, N \rangle \Omega = x; \Omega \geq 0 \quad (9)$$

Lemma 2: If system S_r is positive controllable, then the following two conditions hold:

① S_r is controllable without input constraints.

That is

$$\text{rank} \langle A_r, B_r, n_r \rangle = n_r$$

② any subsystem of S_r is positive controllable.

Remark 1: Assume that system S_r is positive controllable. And let

$$n_r^* \triangleq \min \{n_r^* | \text{rank} \langle A_r, B_r, n_r^* \rangle = n_r\}$$

Then from Cayley-Hamilton theorem it follows that $1 \leq n_r^* \leq n_r$. $\square$

Lemma 3 : If there exists a finite positive integer N which satisfies (7), then

$$\text{rank} \langle A_r, B_r, N \rangle = n_r \quad (10) \square$$

(Proof) If $\text{rank} \langle A_r, B_r, N \rangle \neq n_r$ for some integer N , then the range space of $\langle A_r, B_r, N \rangle$ which contains $L^+[\langle A_r, B_r, N \rangle]$ is properly included by R^{n_r} . Hence, (7) does not hold.

Therefore, Lemma 3 is established. Q. E. D.

Remark 2: The positive integer N in Lemma 3 and n_r^* in Remark 1 satisfy $n_r^* \leq N$. $\square$

Next, we let

$$\begin{aligned} \langle A_r, B_r, N \rangle &\triangleq [e_1, e_2, \dots, e_{Nm}], \\ e_{mj+i} &\triangleq A_r^i b_i \in R^{n_r}; i=1, 2, \dots, m; \\ &j=0, 1, \dots, N-1 \end{aligned} \quad (11)$$

Then we obtain the following Theorem 3.

Theorem 3: System S_r is positive controllable if and only if there exist positive integers $\{N, h, K_1, K_2, \dots, K_h\}$ and positive scalars $\{c_{K_1}, c_{K_2}, \dots, c_{K_h}\}$ which satisfy the following three conditions;

- ① $c_{K_1} e_{K_1} + c_{K_2} e_{K_2} + \dots + c_{K_h} e_{K_h} = 0$ (12)
- ② $1 \leq K_1 < K_2 < \dots < K_h \leq Nm$ (13)
- ③ $\text{rank} [e_{K_1}, e_{K_2}, \dots, e_{K_h}] = n_r < h \leq Nm$ (14) $\square$

The proof is given in Appendix A.

It is possible to determine whether a given system S_r is positive controllable or not by using Theorem 3.

Lemma 4: Suppose that

$$\text{rank} \langle A_r, B_r, n_r \rangle = n_r \quad (15)$$

Then, system S_r is positive controllable, if and only if there exist a positive integer N and a nonnegative vector Ω such that

$$\langle A_r, B_r, N \rangle \Omega = 0, \Omega \geq 0 \quad (16)$$

where

$$\Omega \triangleq [\omega_1^T, \omega_2^T, \dots, \omega_N^T]^T, \omega_1 > 0 \quad (17)$$

$$\omega_i \in R^m; i=1, 2, \dots, N \quad (18) \square$$

(Proof) Necessity: If S_r is positive controllable, then from Theorem 3, (12)~(14) hold.

Thus from (14), we have

$$e_i = d_{i1} e_{K_1} + d_{i2} e_{K_2} + \dots + d_{ih} e_{K_h}; i=1, 2, \dots, m \quad (19)$$

where d_{ij} 's ($i=1, 2, \dots, m; j=1, 2, \dots, h$) are appropriate scalars. Premultiplying both sides of (12) by a

sufficiently large positive scalar M and adding (19) yield

$$e_1 + e_2 + \dots + e_m + c_{K_1}^* e_{K_1} + c_{K_2}^* e_{K_2} + \dots + c_{K_h}^* e_{K_h} = 0 \quad (20)$$

where

$$c_{K_i}^* = M c_{K_i} - d_{1i} - d_{2i} - \dots - d_{mi} > 0; i=1, 2, \dots, h \quad (21)$$

It can be shown that (20) and (21) are equivalent to (16)~(18).

Sufficiency: From (16) and (17) we get

$$B_r \omega_1 + A_r B_r \omega_2 + \dots + A_r^{N-1} B_r \omega_N = 0 \quad (22)$$

Premultiplying (22) by $A_r, A_r^2, \dots, A_r^{n_r-1}$ and adding these equations yield

$$\langle A_r, B_r, n_r + N - 1 \rangle \Omega^+ = 0, \Omega^+ \geq 0 \quad (23)$$

where

$$\begin{aligned} \Omega^+ &\triangleq [\omega_1^{+T}, \omega_2^{+T}, \dots, \omega_{n_r+N-1}^{+T}]^T, \omega_1^+ > 0, \\ \omega_i^+ &= \omega_i + \omega_2 + \dots + \omega_i > 0; i=1, 2, \dots, n_r, \\ \omega_i^+ &\geq 0; i=n_r+1, n_r+2, \dots, n_r+N-1 \end{aligned} \quad (24)$$

It is easy to see that (23) and (24) are equivalent to (12)~(14) by replacing $n_r + N$ by N .

Hence, system S_r is positive controllable by Theorem 3. Q. E. D.

Lemma 5: System S_r is positive controllable if and only if for any small ϵ , there exist positive integers $\{N, h, K_1, K_2, \dots, K_h\}$ independent of ϵ and positive scalars $\{c_{K_1}, c_{K_2}, \dots, c_{K_h}\}$ which satisfy the following three conditions:

$$\textcircled{1} \quad c_{K_1} e_{K_1} + c_{K_2} e_{K_2} + \dots + c_{K_h} e_{K_h} = \epsilon \quad (25)$$

$$\textcircled{2} \quad 1 \leq K_1 < K_2 < \dots < K_h \leq Nm \quad (26)$$

$$\textcircled{3} \quad \text{rank} [e_{K_1}, e_{K_2}, \dots, e_{K_h}] = n_r < h \leq Nm \quad (27)$$

where $\epsilon \triangleq [\epsilon_1, \epsilon_2, \dots, \epsilon_{n_r}]^T$, and the norm

$$\|\epsilon\| \triangleq \sqrt{\epsilon_1^2 + \epsilon_2^2 + \dots + \epsilon_{n_r}^2} > 0 \quad (28)$$

is sufficiently small. $\square$

(Proof) Sufficiency: We suppose (25)~(28) hold. Then we can select n_r independent vectors $\{e_{w_1}, e_{w_2}, \dots, e_{w_{n_r}}\}$ out of h vectors $\{e_{K_1}, e_{K_2}, \dots, e_{K_h}\}$ in (27). Thus, there exist n_r scalars $\{a_{w_1}^*, a_{w_2}^*, \dots, a_{w_{n_r}}^*\}$, such that

$$a_{w_1}^* e_{w_1} + a_{w_2}^* e_{w_2} + \dots + a_{w_{n_r}}^* e_{w_{n_r}} = -\epsilon \quad (29)$$

Here $a_{w_i}^*$'s ($i=1, 2, \dots, n_r$) are sufficiently small scalars, since $\|\epsilon\|$ is sufficiently small. Adding (25) and (29), we obtain

$$c_{K_1}^* e_{K_1} + c_{K_2}^* e_{K_2} + \dots + c_{K_h}^* e_{K_h} = 0 \quad (30)$$

where

$$c_i^* = c_i > 0; i=K_1, K_2, \dots, K_h \text{ and}$$

$$i \neq W_1, W_2, \dots, W_{n_r} \quad (31)$$

$$c_i^* = c_i + a_i^* > 0; i = W_1, W_2, \dots, W_{n_r} \quad (32)$$

Thus from (30), (26) and (27), S_r is positive controllable by virtue of Theorem 3.

Necessity: If system S_r is positive controllable, then by Theorem 3 there exist positive integers $\{N, h, K_1, K_2, \dots, K_h\}$ independent of ε and positive scalars $\{c_{K_1}, c_{K_2}, \dots, c_{K_h}\}$ which satisfy (12)~(14). Thus, we have

$$\alpha_{W_1}^* e_{W_1} + \alpha_{W_2}^* e_{W_2} + \dots + \alpha_{W_{n_r}}^* e_{W_{n_r}} = \varepsilon \quad (33)$$

where $\alpha_{W_i}^* (i=1, 2, \dots, n_r)$ are sufficiently small scalars. By a similar way discussed in the sufficiency, we obtain

$$c_{K_1}^* e_{K_1} + c_{K_2}^* e_{K_2} + \dots + c_{K_h}^* e_{K_h} = \varepsilon \quad (34)$$

$$c_i^* > 0; i = K_1, K_2, \dots, K_h \quad (35)$$

Replacing c_i^* by c_i , we have (25), (26) and (27).

Q. E. D.

Lemma 6: Suppose that

$$\text{rank} \langle A_r, B_r, n_r \rangle = n_r \quad (36)$$

Then, system S_r is positive controllable, if and only if for any small ε there exist a positive integer N independent of ε and a nonnegative vector Ω such that

$$\langle A_r, B_r, N \rangle \Omega = \varepsilon; \Omega \geq 0 \quad (37)$$

where

$$\Omega \triangleq [\omega_1^T, \omega_2^T, \dots, \omega_N^T]^T, \omega_1 > 0 \quad (38)$$

$$\omega_i \in R^m; i = 1, 2, \dots, N \quad (39) \square$$

(Proof) Necessity: If S_r is positive controllable, then from Lemma 4, (16)~(18) hold. We can introduce (37)~(39) by modifying slightly the proof of Lemma 5.

Sufficiency: We assume that (36)~(39) hold. Then premultiplying (37) by $I, A_r, A_r^2, \dots, A_r^{N-1}$ and adding these equations yield

$$\begin{aligned} & \langle A_r, B_r, n_r + N - 1 \rangle \Omega^+ \\ &= (I + A_r + \dots + A_r^{N-1}) \varepsilon \end{aligned} \quad (40)$$

where Ω^+ is given in (24).

Since ε is sufficiently small, the right side of (40) is sufficiently small. It is easy to see that (40) and (24) are equivalent to (25)~(27) by replacing $n_r + N$ by N . Hence by Lemma 5, system S_r is positive controllable.

Q. E. D.

In order to investigate the structure of positive controllable systems, we decompose system S_r into the following two subsystems S_t and S_q , considering

the eigenvalues.

$$S_t: x_t(k+1) = A_t x_t(k) + B_t u(k) \quad (41)$$

$$S_q: x_q(k+1) = A_q x_q(k) + B_q u(k) \quad (42)$$

where

$$A_t \in R^{n_t \times n_t}, B_t \in R^{n_t \times m}, x_t(k) \in R^{n_t},$$

$$A_q \in R^{n_q \times n_q}, B_q \in R^{n_q \times m}, x_q(k) \in R^{n_q},$$

$$n_r = n_t + n_q,$$

$$\lambda_i(A_t) < 0 \text{ or } \text{Im} \{\lambda_i(A_t)\} \neq 0; i = 1, 2, \dots, n_t,$$

$$\lambda_i(A_q) > 0; i = 1, 2, \dots, n_q \quad (43)$$

and $\lambda_i(A)$ denotes the i -th eigenvalue of A .

Then we may establish the following theorem.

Theorem 4: System S_r is positive controllable if and only if the following two conditions hold:

$$\textcircled{1} \text{ rank} \langle A_r, B_r, n_r \rangle = n_r \quad (44)$$

$$\textcircled{2} \text{ system } S_q \text{ is positive controllable.} \quad \square$$

The proof is given in Appendix B.

From Theorem 4, the controllability of S_r may be reduced to that of S_q .

Next, if system S_q has more than two distinct eigenvalues, then we can decompose system S_q into the following two subsystems S_a and S_b ,

$$S_a: x_a(k+1) = A_a x_a(k) + B_a u(k) \quad (45)$$

$$S_b: x_b(k+1) = A_b x_b(k) + B_b u(k) \quad (46)$$

where

$$A_a \in R^{n_a \times n_a}, B_a \in R^{n_a \times m}, x_a(k) \in R^{n_a},$$

$$A_b \in R^{n_b \times n_b}, B_b \in R^{n_b \times m}, x_b(k) \in R^{n_b},$$

$$n_q = n_a + n_b,$$

$$0 < \lambda_i(A_a) < \lambda_j(A_b); i = 1, 2, \dots, n_a;$$

$$j = 1, 2, \dots, n_b$$

Then we have the following theorem.

Theorem 5: System S_q is positive controllable, if and only if the following three conditions hold

$$\textcircled{1} \text{ rank} \langle A_q, B_q, n_q \rangle = n_q \quad (47)$$

$$\textcircled{2} \text{ system } S_a \text{ is positive controllable.}$$

$$\textcircled{3} \text{ system } S_b \text{ is positive controllable.} \quad \square$$

The proof is given in Appendix C.

From Theorem 5, the controllability of system S_q is reduced to those of its subsystems S_a and S_b .

Furthermore, if system S_q has Q distinct positive eigenvalues, then we can transform system S_q into the following Jordan canonical form:

$$A_q = \text{diag} [A_1, A_2, \dots, A_Q] \in R^{n_q \times n_q},$$

$$B_q = [B_1^T, B_2^T, \dots, B_Q^T]^T \in R^{n_q \times m},$$

$$A_i = \text{diag} [A_{i1}, A_{i2}, \dots, A_{i r(i)}] \in R^{n_i \times n_i},$$

$$B_i = [B_{i1}^T, B_{i2}^T, \dots, B_{i r(i)}^T]^T \in R^{n_i \times m},$$

$$n_1 + n_2 + \dots + n_Q = n_q,$$

$$\begin{aligned} A_{ij} &= J[\lambda_i, n_{ij}] \in R^{n_{ij} \times n_{ij}}, \\ B_{ij} &= [b_{1ij}^T, \dots, b_{i_{ir(i)}i}^T]^T \in R^{n_{ij} \times m}, \\ n_{i1} + n_{i2} + \dots + n_{i_{ir(i)}i} &= n_i, \\ i &= 1, 2, \dots, Q; j = 1, 2, \dots, r(i) \end{aligned} \quad (48)$$

where $J[\lambda, n]$ denotes a Jordan block of order n with eigenvalue λ . Further we can assume without loss of generality that

$$0 < \lambda_1 < \lambda_2 < \dots < \lambda_Q \quad (49)$$

$$1 \leq n_{i1} \leq n_{i2} \leq \dots \leq n_{i_{ir(i)}i} \quad (50)$$

Then, by using Theorem 5 repeatedly, we can directly obtain the following theorem.

Theorem 6: System S_q is positive controllable if and only if system S_i described by

$$S_i: x_i(k+1) = A_i x_i(k) + B_i u(k) \quad (51)$$

is positive controllable for each $i=1, 2, \dots, Q$ $\square$

From Theorem 6, the positive controllability of a system may be reduced to those of its Q subsystems $S_i (i=1, 2, \dots, Q)$ with Q distinct positive eigenvalues. Since the dimension of S_i is smaller than that of S_r , the positive controllability test in Theorem 3 may be simplified very much by using Theorem 7.

Now, for any positive scalar λ and any positive integer F , we consider the following system

$$S^*: x^*(k+1) = A^* x^*(k) + B^* u(k) \quad (52)$$

where

$$A^* \triangleq \lambda I_F, \quad B^* \triangleq [\xi_1, \xi_2, \dots, \xi_m] \in R^{F \times m} \quad (53)$$

Then, we get the following theorem. The proof is straightforward based on Theorem 3.

Theorem 7: System S^* is positive controllable if and only if there exist positive integers $\{h, K_1, K_2, \dots, K_h\}$ and positive scalars $\{c_{K_1}, c_{K_2}, \dots, c_{K_h}\}$ which satisfy the following three conditions:

$$\textcircled{1} \quad c_{K_1} \xi_{K_1} + c_{K_2} \xi_{K_2} + \dots + c_{K_h} \xi_{K_h} = 0 \quad (54)$$

$$\textcircled{2} \quad 1 \leq K_1 < K_2 < \dots < K_h \leq m \quad (55)$$

$$\textcircled{3} \quad \text{rank} [\xi_{K_1}, \xi_{K_2}, \dots, \xi_{K_h}] = F < h \leq m \quad (56) \square$$

Remark 3: In Theorem 7, $F < m$ is necessary for S^* to be positive controllable. Thus, a single input system ($m=1$) having any positive eigenvalues is not positive controllable. This agrees with the former results⁷⁾. $\square$

Lemma 7: System S^* is positive controllable, if and only if for any $x^* \in R^F$, there exists a non-negative vector Ω^* , such that

$$B^* \Omega^* = x^*; \Omega^* \geq 0 \quad (57) \square$$

(Proof) Since $A^* = \lambda I_F$ and $\lambda > 0$, it is easy to show that for any positive integer $N \geq 1$,

$$L^+[\langle A^*, B^*, N \rangle] = L^+[B^*] \quad (58)$$

Thus from Theorem 2 and (58), system S^* is positive controllable, if and only if

$$L^+[B^*] = R^F \quad (59)$$

It is evident that (59) is equivalent to that for any $x^* \in R^F$ there exists a nonnegative vector Ω^* , such that (57) holds.

Therefore, we have Lemma 7. $\square$ E. D.

The following lemma will be shown in Appendix D.

Lemma 8: Let F and G be any positive integers.

Then, system S_c^* described by

$$S_c^*: x^+(k+1) = \Phi_c x^+(k) + \Gamma_c u(k) \quad (60)$$

where

$$\begin{aligned} \Phi_c &\triangleq \begin{bmatrix} A^* & I_F & 0 & \dots & 0 \\ 0 & A^* & I_F & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & A^* & I_F \\ 0 & \dots & 0 & 0 & A^* \end{bmatrix}, \quad \Gamma_c \triangleq \begin{bmatrix} B_c^* \\ B_{c-1}^* \\ \dots \\ B_1^* \end{bmatrix}, \\ \Phi_c &\in R^{FG \times FG}, \quad \Gamma_c \in R^{FG \times m}, \quad B_1^* = B^* \quad (61) \\ B_i^* &\in R^{F \times m}; i=1, 2, \dots, G \quad (62) \end{aligned}$$

is positive controllable, if and only if system S^* described by (52) is positive controllable. $\square$

Finally, we have the following theorem.

Theorem 8: System S_q is positive controllable if and only if system S_i^* described by

$$S_i^*: x_i^*(k+1) = A_i^* x_i^*(k) + B_i^* u(k) \quad (63)$$

where

$$\begin{aligned} A_i^* &\triangleq \lambda_i I_{r(i)}, \\ B_i^* &\triangleq \begin{bmatrix} b_{1i1} \\ b_{1i2} \\ \dots \\ b_{1i_{ir(i)}} \end{bmatrix} \in R^{r(i) \times m} \quad (64) \end{aligned}$$

is positive controllable for each $i=1, 2, \dots, Q$ $\square$

(Proof) Necessity: If system S_q is positive controllable, then from Theorem 6, system S_i is positive controllable for each $i=1, 2, \dots, Q$.

On the other hand, system S_i^* is the subsystem of S_i from (48) and (64). Therefore, from Lemma 1, system S_i^* is positive controllable for each $i=1, 2, \dots, Q$.

Sufficiency: For each $i=1, 2, \dots, Q$, consider a system described by

$$S_i^*: x_i^*(k+1) = A_i^* x_i^*(k) + B_i^* u(k) \quad (65)$$

where

$$A_i^+ \triangleq \begin{bmatrix} A_i^* I_{r(i)} & 0 & \cdots & 0 \\ 0 & A_i^* I_{r(i)} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & A_i^* I_{r(i)} \\ 0 & \cdots & 0 & 0 & A_i^* \end{bmatrix} \in R^{r(i)G(i) \times r(i)G(i)},$$

$$B_i^+ \triangleq \begin{bmatrix} B_{iG(i)} \\ \vdots \\ B_{i2} \\ B_{i1} \end{bmatrix} \in R^{r(i)G(i) \times m} \quad (66)$$

$$B_{i1}^+ = B_i^*, \quad G(i) \triangleq n_{ir(i)} \quad (67)$$

Then it is easy to show that A_i^+ and B_i^+ can be chosen so that system S_i is the subsystem of S_i^+ . Now, if system S_i^+ is positive controllable for each $i=1, 2, \dots, Q$, then system S_i^+ is positive controllable for each $i=1, 2, \dots, Q$, by Lemma 8. Since system S_i is the subsystem of S_i^+ , system S_i is positive controllable, by Lemma 2. Therefore, by Theorem 6, system S_Q is positive controllable. Q. E. D.

Ultimately, the controllability test may be checked by using Theorem 7 and Theorem 8, which are the main features of this work.

4. An Example

Consider a system represented by

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 3 \\ 1 & -2 & 1 \\ 0 & 3 & 4 \\ -1 & 0 & 2 \end{bmatrix} \quad (68)$$

where $n=5$, $m=3$. Then we have

$$A_o = 0, \quad B_o = [2 \quad 1 \quad 0] \quad (69)$$

$$A_r = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}, \quad B_r = \begin{bmatrix} 1 & 2 & 3 \\ 1 & -2 & 1 \\ 0 & 3 & 4 \\ -1 & 0 & 2 \end{bmatrix} \quad (70)$$

where $n_o=1$, $n_r=4$,

$$\text{rank}[B_r, A_r B_r, A_r^2 B_r, A_r^3 B_r] = 4 \quad (71)$$

Further, we obtain

$$A_t = -1, \quad B_t = [1 \quad 2 \quad 3] \quad (72)$$

$$A_q = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}, \quad B_q = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 3 & 4 \\ -1 & 0 & 2 \end{bmatrix} \quad (73)$$

where $n_t=1$, $n_q=3$. Thus, we get

$$A_1 = A_o, \quad B_1 = B_o, \quad Q=1, \quad \lambda_1=3, \quad n_1=3 \quad (74)$$

Next, we have

$$A_{11}=3, \quad B_{11}=[1 \quad -2 \quad 1]b_{111} \quad (75)$$

$$A_{12} = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}, \quad B_{22} = \begin{bmatrix} 0 & 3 & 4 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} b_{212} \\ b_{112} \end{bmatrix} \quad (76)$$

where

$$n_{11}=1, \quad n_{12}=2, \quad r(1)=2, \quad G(1)=2 \quad (77)$$

Finally, we obtain

$$S_i^* : x_i^*(k+1) = A_i^* x_i^*(k) + B_i^* u(k) \quad (78)$$

where

$$A_i^* = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, \quad B_i^* = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} b_{111} \\ b_{112} \\ \xi_1 \quad \xi_2 \quad \xi_3 \end{bmatrix} \quad (79)$$

It is easy to verify that

$$\text{rank}[\xi_1, \xi_2, \xi_3] = 2 \quad (80)$$

$$4\xi_1 + 3\xi_2 + 2\xi_3 = 0 \quad (81)$$

Thus system S_i^* is positive controllable from Theorem 7. Then, system S_q is positive controllable from Theorem 8. Accordingly, system S_r is positive controllable from Theorem 4. Hence, system S is positive controllable from Theorem 1.

5. Conclusions

A necessary and sufficient condition for the controllability to the zero state by positive controls of multiple input discrete-time linear systems is presented based on the Jordan canonical form. The results in this paper reveal the structure of the controllable system with positive controls, and are considered to be fundamental and essential in the control system theory.

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《Appendix A》

Proof of Theorem 3

Sufficiency: We can select n_r independent vectors $\{e_{w_1}, e_{w_2}, \dots, e_{w_{n_r}}\}$ out of h vectors $\{e_{K_1}, e_{K_2}, \dots, e_{K_h}\}$ in (14), where

$$1 \leq W_1 < W_2 < \dots < W_{n_r} \leq Nm \quad (\text{A.1})$$

Let $x_r(0) \in R^{n_r}$ be any initial state. Then, $-A_r^N x_r(0) \in R^{n_r}$. Thus $-A_r^N x_r(0)$ can be expressed as a linear combination of n_r independent vectors $\{e_{w_1}, e_{w_2}, \dots, e_{w_{n_r}}\}$ with real coefficients $\{\alpha_{w_1}, \alpha_{w_2}, \dots, \alpha_{w_{n_r}}\}$; that is

$$\alpha_{w_1} e_{w_1} + \alpha_{w_2} e_{w_2} + \dots + \alpha_{w_{n_r}} e_{w_{n_r}} = -A_r^N x_r(0) \quad (\text{A.2})$$

Premultiplying both sides of (12) by a sufficiently large positive scalar M and adding (A.2) yield

$$\beta_1 e_1 + \beta_2 e_2 + \dots + \beta_{Nm} e_{Nm} = -A_r^N x_r(0) \quad (\text{A.3})$$

where

$$\beta_i = 0; i = 1, 2, \dots, Nm \text{ and } i \neq K_1, K_2, \dots, K_h \quad (\text{A.4})$$

$$\beta_i = Mc_i > 0; i = K_1, K_2, \dots, K_h \text{ and } i \neq W_1, W_2, \dots, W_{n_r} \quad (\text{A.5})$$

$$\beta_i = Mc_i + \alpha_i > 0; i = W_1, W_2, \dots, W_{n_r} \quad (\text{A.6})$$

Now we let

$$u^+(N-j-1) = [\beta_{jm+1}, \beta_{jm+2}, \dots, \beta_{jm+m}]^T; j = 0, 1, \dots, N-1 \quad (\text{A.7})$$

Then applying the positive control sequence $\{u^+(0), u^+(1), \dots, u^+(N-1)\}$ to system S_r , we get $x_r(N) = 0$ for any initial state $x_r(0)$. Thus, from Definition 2, system S_r is positive controllable.

Necessity: If we suppose that system S_r is positive controllable, then by Theorem 2 and Lemma 3, we can select n_r independent vectors $\{e_{w_1}, e_{w_2}, \dots, e_{w_{n_r}}\}$ out of Nm vectors $\{e_1, e_2, \dots, e_{Nm}\}$ in (11). Now let

$$x^* = e_{w_1} + e_{w_2} + \dots + e_{w_{n_r}} \quad (\text{A.8})$$

Then $-x^* \in R^{n_r}$. Since system S_r is positive controllable, there exist a finite positive integer N and nonnegative scalars $\{d_1, d_2, \dots, d_{Nm}\}$ such that

$$-x^* = d_1 e_1 + d_2 e_2 + \dots + d_{Nm} e_{Nm} \quad (\text{A.9})$$

by Lemma 1. Adding (A.8) and (A.9) yields

$$d_1^* e_1 + d_2^* e_2 + \dots + d_{Nm}^* e_{Nm} = 0 \quad (\text{A.10})$$

where

$$d_i^* = d_i \geq 0; i = 1, 2, \dots, Nm \text{ and } i \neq W_1, W_2, \dots, W_{n_r} \quad (\text{A.11})$$

$$d_i^* = 1 + d_i > 0; i = W_1, W_2, \dots, W_{n_r} \quad (\text{A.12})$$

If we assume that all of the $(Nm - n_r)$ scalars d_i^* 's

in (A.11) are zeros, then we obtain

$$d_{w_1}^* e_{w_1} + d_{w_2}^* e_{w_2} + \dots + d_{w_{n_r}}^* e_{w_{n_r}} = 0 \quad (\text{A.13})$$

from (A.10). On the other hand, n_r vectors $\{e_{w_1}, e_{w_2}, \dots, e_{w_{n_r}}\}$ are linearly independent.

Thus from (A.13), we obtain

$$d_i^* = 0; i = W_1, W_2, \dots, W_{n_r} \quad (\text{A.14})$$

This contradicts (A.12). Therefore, at least one of the $(Nm - n_r)$ scalars d_i^* 's in (A.11) is not zero. Now let h be the number of positive d_i^* 's in (A.10) and $\{f_1, f_2, \dots, f_{Nm}\}$ be an appropriate rearrangement of $\{e_1, e_2, \dots, e_{Nm}\}$.

Then, it follows from (A.10) ~ (A.12) that

$$g_1 f_1 + g_2 f_2 + \dots + g_h f_h = 0 \quad (\text{A.15})$$

where

$$g_i > 0; i = 1, 2, \dots, h \quad (\text{A.16})$$

$$\text{rank}[f_1, f_2, \dots, f_h] = n_r < h \leq Nm \quad (\text{A.17})$$

(A.15) ~ (A.17) are equivalent to (12) ~ (14).

Therefore, the necessity is established. Q. E. D.

《Appendix B》

Proof of Theorem 4

Necessity: Since system S_q is the subsystem of S_r , the necessity is clear by Lemma 2.

Sufficiency: Suppose that conditions ① and ② hold. Then, by Lemma 4, there exist a positive integer N_q and nonnegative vector Ω such that

$$\langle A_q, B_q, N_q \rangle \Omega = 0; \Omega \geq 0 \quad (\text{B.1})$$

where

$$\Omega \triangleq [\omega_1^T, \omega_2^T, \dots, \omega_{N_q}^T]^T; \omega_1 > 0 \quad (\text{B.2})$$

$$\omega_i \in R^m; i = 1, 2, 3, \dots, N_q \quad (\text{B.3})$$

Next, by a nonsingular transformation, we have the following equation from (41) and (42).

$$A_r = \begin{bmatrix} A_t & 0 \\ 0 & A_q \end{bmatrix}, B_r = \begin{bmatrix} B_t \\ B_q \end{bmatrix} \quad (\text{B.4})$$

On the other hand, it has been proved in 7) that there exists a polynomial $\phi(z)$ with positive coefficients, such that

$$\phi(z) = \delta_L z^L + \dots + \delta_1 z + \delta_0 \quad (\text{B.5})$$

$$\delta_i > 0; i = 0, 1, 2, \dots, L \quad (\text{B.6})$$

$$\phi(A_t) = 0 \quad (\text{B.7})$$

where L may be designated arbitrarily as far as

$$L \gg n_t \geq 1 \quad (\text{B.8}) \square$$

Thus, from (B.4) ~ (B.8), we obtain

$$\phi(A_r) = \sum_{i=0}^L \delta_i A_r^i = \sum_{i=0}^L \delta_i \begin{bmatrix} A_t & 0 \\ 0 & A_q \end{bmatrix}^i$$

$$\begin{aligned}
&= \begin{bmatrix} \sum_{i=0}^L \delta_i A_i^i & 0 \\ 0 & \sum_{i=0}^L \delta_i A_i^i \end{bmatrix} \\
&= \begin{bmatrix} \phi(A_i) & 0 \\ 0 & \phi(A_q) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \phi(A_q) \end{bmatrix} \quad (\text{B.9})
\end{aligned}$$

Therefore we get

$$\begin{aligned}
&\phi(A_r) \langle A_r, B_r, N_q \rangle \Omega \\
&= \begin{bmatrix} 0 & 0 \\ 0 & \phi(A_q) \end{bmatrix} \begin{bmatrix} \langle A_i, B_i, N_q \rangle \\ \langle A_q, B_q, N_q \rangle \end{bmatrix} \Omega \\
&= \begin{bmatrix} 0 \\ \phi(A_q) \langle A_q, B_q, N_q \rangle \Omega \end{bmatrix} = 0 \quad (\text{B.10})
\end{aligned}$$

considering (B.1) and (B.2). Here, we can designate L as $L \gg N_q$ and $L \gg n_r$, from (B.8). If we let

$$\begin{aligned}
\omega_i^* &\triangleq \delta_{i-1} \omega_1 + \delta_{i-2} \omega_2 + \cdots + \delta_{i-N_q} \omega_{N_q} \in R^m; \\
i &= 1, 2, \dots, L + N_q \quad (\text{B.11})
\end{aligned}$$

where

$$\delta_i \triangleq 0; i < 0, i > L \quad (\text{B.12})$$

then (B.10) may be rewritten as;

$$\langle A_r, B_r, L + N_q \rangle \Omega^* = 0; \Omega^* \geq 0 \quad (\text{B.13})$$

where

$$\begin{aligned}
\Omega^* &\triangleq [\omega_1^{*T}, \omega_2^{*T}, \dots, \omega_{L+N_q}^{*T}]^T, \\
\omega_i^* &\triangleq \delta_0 \omega_i > 0 \quad (\text{B.14})
\end{aligned}$$

Hence, by Lemma 4, system S_r is positive controllable. Q. E. D.

《Appendix C》

Proof of Theorem 5

Necessity: Since system S_a and S_b are the subsystems of S_q , the necessity is clear.

Sufficiency: By a nonsingular transformation, we have

$$A_q = \begin{bmatrix} A_a & 0 \\ 0 & A_b \end{bmatrix}, B_q = \begin{bmatrix} B_a \\ B_b \end{bmatrix} \quad (\text{C.1})$$

from (39) and (40). Next, let

$$B_q E_m \triangleq [x_{a0}^T, x_{b0}^T]^T \in R^{n_q} \quad (\text{C.2})$$

where $x_{a0} \in R^{n_a}$, $x_{b0} \in R^{n_b}$. Then, from Lemma 1, there exist a positive integer N_a and a nonnegative vector Ω_a , such that

$$\begin{aligned}
\langle A_a, B_a, N_a \rangle \Omega_a &= -x_{a0} \in R^{n_a}, \\
0 \leq \Omega_a &\in R^{N_a m} \quad (\text{C.3})
\end{aligned}$$

since S_a is positive controllable. Now, let

$$\Omega_a^* \triangleq [E_m^T, 0]^T + \Omega_a \in R^{N_a m} \quad (\text{C.4})$$

$$x_b^* \triangleq \langle A_b, B_b, N_a \rangle \Omega_a^* \in R^{n_b} \quad (\text{C.5})$$

Further, we consider a sufficiently large integer $N_c \gg 1$. Then, from Lemma 1, there exist a positive

integer N_b and a nonnegative vector Ω_b , such that

$$\begin{aligned}
\langle A_b, B_b, N_b \rangle \Omega_b \\
= -(\lambda_b)^{N_c} A_b^{-N_c} x_b^* \in R^{n_b} \quad (\text{C.6})
\end{aligned}$$

$$0 \leq \Omega_b \in R^{N_b m} \quad (\text{C.7})$$

where $\lambda_b > 0$ is any eigenvalue of A_b , because S_b is positive controllable. Next let

$$N_q \triangleq N_a + N_c + N_b \quad (\text{C.8})$$

$$\Omega_q \triangleq [\Omega_a^{*T}, 0, \lambda_b^{-N_c} \Omega_b^T]^T \in R^{N_q m} \quad (\text{C.9})$$

$$[\omega_1^T, \omega_2^T, \dots, \omega_{N_q}^T]^T \triangleq \Omega_q \quad (\text{C.10})$$

$$\omega_i \in R^m; i = 1, 2, \dots, N_q \quad (\text{C.11})$$

Then, from (C.1)~(C.11), we obtain

$$\begin{aligned}
\langle A_q, B_q, N_q \rangle \Omega_q \\
= \begin{bmatrix} \langle A_a, B_a, N_q \rangle \\ \langle A_b, B_b, N_q \rangle \end{bmatrix} [\Omega_a^{*T}, 0, \lambda_b^{-N_c} \Omega_b^T]^T \\
= [\epsilon_a^T, 0]^T \quad (\text{C.12})
\end{aligned}$$

and

$$\Omega_q \geq 0; \omega_i \geq 0 \quad (\text{C.13})$$

$$\epsilon_a \triangleq (\lambda_b^{-1} A_a)^{N_c} A_a^{N_a} \langle A_a, B_a, N_b \rangle \Omega_b \in R^{n_a} \quad (\text{C.14})$$

Here, all of the eigenvalues of the matrix $\lambda_b^{-1} A_a$ are smaller than unity. Thus the norm of ϵ_a is sufficiently small, because N_c is sufficiently large. Hence, by Lemma 6, system S_q is positive controllable.

Q. E. D.

《Appendix D》

Proof of Lemma 8

Since system S^* is the subsystem of S_c^+ , the necessity is clear by Lemma 2.

We will show the sufficiency by mathematical induction method.

When $G=1$, system S_1^+ coincides with system S^* . Therefore, if S^* is positive controllable then S_1^+ is positive controllable.

Next, we assume that for some positive integer $G \geq 1$, the following two conditions hold:

- ① system S^* is positive controllable.
- ② S_c^+ is positive controllable.

Under these hypotheses, we will show that S_{c+1}^+ is positive controllable by Lemma 6, in the following 5 steps.

Step 1: First, let

$$x_c^* \triangleq [0, I_{FG}] \Gamma_{G+1} E_m \in R^{FG} \quad (\text{D.1})$$

Step 2: Since system S_c^+ is positive controllable by assumption ②, then by Lemma 1, there exist a positive integer N_c and a nonnegative vector Ω_c , such that

$$\langle \Phi_G, \Gamma_G, N_G \rangle \Omega_G = -x_G^* \in R^{FG}; \Omega_G \geq 0 \quad (D.2)$$

$$\Omega_G \in R^{N_G m} \quad (D.3)$$

Step 3: Let

$$\Omega_G^* \triangleq [E_m^T, 0]^T + \Omega_G \in R^{N_G m} \quad (D.3)$$

$$x^o \triangleq [I_F, 0] \langle \Phi_{G+1}, \Gamma_{G+1}, N_G \rangle \Omega_G^* \in R^F \quad (D.4)$$

Step 4: Since system S^* is positive controllable by assumption ①, there exists a nonnegative vector Ω_o , such that

$$B^* \Omega_o = -x^o; 0 \leq \Omega_o \in R^m \quad (D.5)$$

from Lemma 7.

Step 5: Let N be a positive integer such that $N \gg N_G$. Further let

$$\Omega^+ \triangleq [\Omega_G^*, 0, \eta \Omega_o^T]^T \in R^{(N+1)m} \quad (D.6)$$

$$\langle \Phi_{G+1}, \Gamma_{G+1}, N+1 \rangle \Omega^+ \triangleq \varepsilon \in R^{F(G+1)} \quad (D.7)$$

$$\eta \triangleq \lambda^{G-N} ({}_N C_G)^{-1} \quad (D.8)$$

$${}_N C_G = \frac{N!}{G!(N-G)!} \quad (D.9)$$

To complete the proof by Lemma 8, it is enough to show that

$$(i) \quad \Omega^+ \geq 0, \omega_1 > 0 \quad (D.10)$$

$$(ii) \quad \text{rank} \langle \Phi_{G+1}, \Gamma_{G+1}, G+1 \rangle = F(G+1) \quad (D.11)$$

(iii) the norm of ε is sufficiently small.

Firstly, (i) is derived from (D.2), (D.3), (D.5), (D.6), (D.8) and (D.9).

Secondly, we will show (ii) as follows.

From (53) and (61) we get

$$\Phi_{G+1} = \lambda I_{F(G+1)} + N \quad (D.12)$$

where

$$N \triangleq \begin{bmatrix} 0 & I_F & 0 & \cdots & 0 \\ 0 & 0 & I_F & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & I_F \\ 0 & \cdots & 0 & 0 & 0 \end{bmatrix} \in R^{F(G+1) \times F(G+1)} \quad (D.13)$$

Then we have from (61) and (D.13)

$$\begin{aligned} \text{rank} \langle \Phi_{G+1}, \Gamma_{G+1}, G+1 \rangle &= \text{rank} [\Gamma_{G+1}, \Phi_{G+1}, \Gamma_{G+1}, \cdots, \Phi_{G+1}^G \Gamma_{G+1}] \\ &= \text{rank} [\Gamma_{G+1}, \{\lambda I_{F(G+1)} + N\} \Gamma_{G+1}, \cdots, \\ &\quad \{\lambda I_{F(G+1)} + N\}^G \Gamma_{G+1}] \\ &= \text{rank} [\Gamma_{G+1}, N \Gamma_{G+1}, \cdots, N^G \Gamma_{G+1}] \\ &= \text{rank} \begin{bmatrix} B_{G+1}^+ & B_G^+ & \cdots & B_1^+ \\ B_G^+ & B_{G-1}^+ & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ B_2^+ & B_1^+ & \cdots & 0 \\ B_1^+ & 0 & \cdots & 0 \end{bmatrix} \\ &= (G+1) \text{rank} [B_1^+] = (G+1) \text{rank} [B^*] \end{aligned} \quad (D.14)$$

On the other hand, system S^* is positive controllable by assumption ①. Thus from (56) in Theorem 7, $\text{rank} [B^*] = F$. Therefore, we have

$$\text{rank} \langle \Phi_{G+1}, \Gamma_{G+1}, G+1 \rangle = F(G+1)$$

Lastly we will show (iii) as follows.

From (D.6) and (D.7), it follows that

$$\varepsilon = \langle \Phi_{G+1}, \Gamma_{G+1}, N_G \rangle \Omega_G^* + \eta \Phi_{G+1}^N \Gamma_{G+1} \Omega_o \quad (D.15)$$

Furthermore, from (D.3), (D.1) and (D.2), we get

$$\begin{aligned} &[0, I_{FG}] \langle \Phi_{G+1}, \Gamma_{G+1}, N_G \rangle \Omega_G^* \\ &= [0, I_{FG}] \langle \Phi_{G+1}, \Gamma_{G+1}, N_G \rangle \Omega_G + [0, I_{FG}] \Gamma_{G+1} E_m \\ &= \langle \Phi_G, \Gamma_G, N_G \rangle \Omega_G + x_G^* = 0 \end{aligned} \quad (D.16)$$

Thus from (D.4) and (D.16) we have

$$\begin{aligned} &\langle \Phi_{G+1}, \Gamma_{G+1}, N_G \rangle \Omega_G^* \\ &= \begin{bmatrix} I_F & 0 \\ 0 & I_{FG} \end{bmatrix} \langle \Phi_{G+1}, \Gamma_{G+1}, N_G \rangle \Omega_G^* \\ &= \begin{bmatrix} I_F & 0 \\ 0 & I_{FG} \end{bmatrix} \langle \Phi_{G+1}, \Gamma_{G+1}, N_G \rangle \Omega_G^* \\ &= [(x^o)^T, 0]^T \end{aligned} \quad (D.17)$$

On the other hand, from (53), (61) and (D.8), we obtain

$$\eta \Phi_{G+1}^N = \begin{bmatrix} A_G & A_{G-1} & \cdots & A_1 & I_F \\ 0 & A_G & \cdots & A_2 & A_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & A_G & A_{G-1} \\ 0 & 0 & \cdots & 0 & A_G \end{bmatrix} \quad (D.18)$$

where

$$A_i \triangleq \lambda^i ({}_N C_G)^{-1} {}_N C_{G-i} I_F \in R^{F \times F}; i=1, 2, \cdots, G \quad (D.19)$$

If we let

$$\begin{aligned} &\eta \Phi_{G+1}^N \Gamma_{G+1} \Omega_o \\ &\triangleq [\bar{x}_{G+1}^N, \varepsilon_G^o, \cdots, \varepsilon_1^T]^T \in R^{F(G+1)} \end{aligned} \quad (D.20)$$

then we have the following equations from (D.5), (D.18) and (61).

$$\bar{x}_{G+1} \triangleq \varepsilon_{G+1} - x^o \in R^F \quad (D.21)$$

$$\varepsilon_{G+1} \triangleq (A_G B_{G+1}^+ + A_{G-1} B_G^+ + \cdots + A_1 B_2^+) \Omega_o \in R^F \quad (D.22)$$

$$\begin{aligned} \varepsilon_i &\triangleq (A_G B_i^+ + A_{G-1} B_{i-1}^+ + \cdots + A_{G-i+1} B_1^+) \Omega_o \in R^F; \\ &i=1, 2, \cdots, G \end{aligned} \quad (D.23)$$

Substituting (D.17), (D.20) and (D.21) into (D.15), it follows that

$$\varepsilon = [\varepsilon_{G+1}^T, \varepsilon_G^T, \cdots, \varepsilon_1^T]^T \quad (D.24)$$

Since N is sufficiently large, all of the elements of A_i 's ($i=1, 2, \cdots, G$) in (D.19) are sufficiently small. Thus all of the norms of ε_i 's ($i=1, 2, \cdots, G+1$) are sufficiently small by (D.22) and (D.23).

Hence, the norm of ϵ is sufficiently small.

Therefore, $S_{\epsilon+1}^+$ is positive controllable by Lemma 6.

Q. E. D.

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