A Methodology for Multiple Objective Control Design with Robust Performance[†]

Hiroshi ITO*, Hiromitsu OHMORI* and Akira SANO*

In this paper, we consider performance robustness with two objectives of finite-dimensional, linear time-invariant systems having structured norm-bounded uncertainty, where two performance objectives are given by $\mathcal{H}^\infty$ norm constraints on transfer functions associated with a single closed-loop system. A methodology for the robust control design is proposed. We prove that a performance robustness problem with multiple objectives is equivalent to a stability robustness problem with extra repeated uncertainty blocks. We show how the equivalent transformation can be exploited to propose a method of controller designs. A numerical example illustrates how the design approach proposed deals with trade-off between two objectives effectively and finds satisfactory controllers achieving two performance objectives robustly.

Key Words: multiple objective control, robust performance, robust stability, structured uncertainty, robust control

1. Introduction

In recent years, $\mathcal{H}^\infty$ control design has got an unshakable position as one of effective and elegant design techniques¹⁾. Since $\mathcal{H}^\infty$ control design itself, however, can only deal with nominal performance problems, a lot of attention is also directed to robust performance problems. An interesting approach which is available now is so-called D-K iteration for μ -synthesis²⁾. These two design techniques have been proposed within the framework of the linear fractional transformation on a generalized plant by a controller. In such a feedback arrangement, both $\mathcal{H}^\infty$ design and μ -synthesis minimize H^∞ norm of the transfer function from an exogenous input to an exogenous output. That is, they deal with a single performance objective. In practice, we often require several performance objectives for a single control system. Recently, the subject of controller design with multiple objectives has received considerable attention. For example, multiple $\mathcal{H}^2$ performance criteria has been dealt with in Khargonekar and

Rotea³⁾ and $\mathcal{H}^2/\mathcal{H}^\infty$ control problem has been investigated by Bernstein and Haddad⁴⁾. Another approach was developed by Khargonekar et al.⁵⁾, which showed how to combine individually designed controllers. Those papers consider nominal performance. On the other hand, this paper considers control problems with multiple $\mathcal{H}^\infty$ performance criteria, and we tackle the robustness problem of the performance for structured uncertainty.

Let Ψ_i (for $i=1, 2$) be some specified performance objective as a transfer function associated with a closed-loop system. Let $\|\cdot\|$ denote some norm on systems. If we could combine the two objectives as $\Psi_c = \text{diag}[\Psi_1, \Psi_2]$, we could get $\|\Psi_i\| \leq \gamma$ for $i=1, 2$ by making $\|\Psi_c\| \leq \gamma$. Of course, in general it is impossible to express Ψ_c as a linear fractional transformation on a generalized plant and a controller, since the whole function Ψ_c is not a transfer function of a single closed-loop system any more. This implies that standard H^∞ control and μ -synthesis cannot handle Ψ_c . If Ψ_1 and Ψ_2 share the same input in the system, we may use a sufficient condition

$$\left\| \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix} \right\| \leq \gamma \quad (1)$$

to achieve $\|\Psi_i\| \leq \gamma$ for $i=1, 2$, and carry out standard $\mathcal{H}^\infty$ or μ synthesis. However, this is not always the case. In general Ψ_1 and Ψ_2 cannot share the same

[†] Presented 16th Dynamical System Theory Symposium (1993-12)

* Faculty of Science and Technology, Keio University, 3-14-1 Hiyoshi, Kohoku-ku, Yokohama (Received July 5, 1994)
(Revised November 15, 1994)

input and the same output as well. A typical example is shown in **Fig. 1**. Let T_{ur} denote the map from r to u , and T_{yd_1} denotes the map from d_1 to y . To attenuate input disturbance, we want to make $\|T_{yd_1}\|$ small. Meanwhile, since a large plant input can drive the plant out of the operating region for which the linear model description with specified uncertainty is valid, $\|T_{ur}\|$ should be less than a suitable bound. The inputs and outputs of these two maps are completely different. In addition, we often see such a kind of multi-objective problems when we design multi-input multi-output systems. It should be also emphasized that we cannot get tight bound for each individual objective by (1) even if the objectives can formally share the same input in the form of (1).

For $\mathcal{H}^\infty$ design with multiple objectives, an approach was proposed by Hung and Pokrud⁶. Since they consider nominal $\mathcal{H}^\infty$ performance and stability which are directly assessable, it is straightforward to come up with a synthesis technique. This is not the case where robustness is required. In this paper, first, we deal with analysis problem of robust performance with two objectives, and the idea established for the analysis is utilized to propose an approach to control design.

This paper is organized as follows: In Section 2, a framework of analysis for robust performance with two objectives is developed. A key theorem is what we call the main loop theorem with multiple objectives. In Section 3, an iterative solution to the multiobjective control problems is proposed, where the main loop theorem plays an important role. Finally, a numerical example is provided to illustrate the algorithm in Section 4. Conclusions are stated in Section 5.

In the sequel, $\|\cdot\|_\infty$ denotes $\mathcal{H}^\infty$ -norm. $\mathcal{F}_l(\cdot, \cdot)$ denotes linear fractional transformation defined by

$$\mathcal{F}_l(M, \Delta) := M_{11} + M_{12}\Delta(I - M_{22}\Delta)^{-1}M_{21},$$

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}.$$

For complex matrices M and Δ , the structured singular value with underlying structure Δ is defined as

Fig. 1 Feedback control system

$$\mu_\Delta(M) := \{\min\{\bar{\sigma}(\Delta) : \Delta \in \Delta, \det(I - M\Delta) = 0\}\}^{-1}$$

unless no $\Delta \in \Delta$ makes $\det(I - M\Delta) = 0$, in which case $\mu_\Delta(M) := 0$ ⁷. $\mathbf{R}_+$ denotes the set of nonnegative real numbers, i. e., $[0, \infty)$.

2. Single Test for Two Objectives

In this section, we deal with μ -analysis for multiple performance objectives. Consider the feedback configuration shown in **Fig. 2** consisting of complex matrices M and Δ . Suppose that Δ has block diagonal structure as

$$\Delta = \text{diag}[\Delta_1, \Delta_2, \dots, \Delta_n]. \quad (2)$$

Let Δ denote the above block diagonal structure. Here, Ψ_1 represents the transfer function from d_1 to e_1 and Ψ_2 denotes that from d_2 to e_2 :

$$\Psi_1 := M_{11} + M_{13}\Delta(I - M_{33}\Delta)^{-1}M_{31}$$

$$\Psi_2 := M_{22} + M_{23}\Delta(I - M_{33}\Delta)^{-1}M_{32},$$

where M_{ij} s denote portions of M partitioned as

$$\begin{bmatrix} e_1 \\ e_2 \\ v \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ w \end{bmatrix}. \quad (3)$$

For a single performance objective, Doyle et al.⁸ proved a property called the main loop theorem. We extend the main loop theorem to include multiple objectives by proposing the following.

Theorem 1 : Define a matrix M_f as

$$M_f := \begin{bmatrix} M_{11} & 0 & M_{13} & 0 & -M_{13} \\ 0 & M_{22} & M_{23} & -M_{23} & 0 \\ M_{31} & M_{32} & M_{33} & 0 & 0 \\ M_{31} & 0 & 0 & M_{33} & 0 \\ 0 & M_{32} & 0 & 0 & M_{33} \end{bmatrix} \quad (4)$$

Let $\mathbf{B}\Delta := \{\Delta \in \Delta : \bar{\sigma}(\Delta) < 1\}$, and define a block structure Δ_f as

$$\Delta_f := \{\text{diag}[\Delta_p, \Delta_r] : \Delta_p \in \Delta_p, \Delta_r \in \Delta_r\},$$

where

$$\Delta_r := \{\text{diag}[\Delta, \Delta, \Delta] : \Delta \in \Delta\}$$

$$\Delta_p := \{\text{diag}[\Delta_{\Psi_1}, \Delta_{\Psi_2}] : \Delta_{\Psi_i} \in \Delta_{\Psi_i}\}$$

$$\Delta_{\Psi_i} : \text{fictitious full block for } \Psi_i$$

Then the following are equivalent :

Fig. 2 Robust performance problem

$$\mu_{\mathcal{A}f}(M_f) \leq 1 \Leftrightarrow \begin{cases} \bar{\sigma}(\Psi_1) \leq 1 \text{ for all } \Delta \in \mathcal{B}\mathcal{A}, \\ \bar{\sigma}(\Psi_2) \leq 1 \text{ for all } \Delta \in \mathcal{B}\mathcal{A} \\ \text{and} \\ \mu_{\mathcal{A}}(M_{33}) \leq 1. \end{cases} \quad \blacksquare$$

Proof: From the definition of M_f , we have

$$\begin{bmatrix} \Psi_1 & 0 \\ 0 & \Psi_2 \end{bmatrix} = \mathcal{F}_l(M_f, \Delta_r) \\ = \mathcal{F}_l(M, \Delta) - \begin{bmatrix} 0 & \mathcal{F}_{12}(M, \Delta) \\ \mathcal{F}_{21}(M, \Delta) & 0 \end{bmatrix}$$

where $\mathcal{F}_{12}(M, \Delta) := M_{12} + M_{13}\Delta(I - M_{33}\Delta)^{-1}M_{32}$ and $\mathcal{F}_{21}(M, \Delta) := M_{21} + M_{23}\Delta(I - M_{33}\Delta)^{-1}M_{31}$. Then, we obtain

$$\mu_{\mathcal{A}p}(\mathcal{F}_l(M_f, \Delta_r)) \leq 1, \forall \Delta_r \in \mathcal{B}\mathcal{A}_r \\ \Leftrightarrow \begin{cases} \bar{\sigma}(\Psi_1) \leq 1, \forall \Delta \in \mathcal{B}\mathcal{A} \\ \text{and} \\ \bar{\sigma}(\Psi_2) \leq 1, \forall \Delta \in \mathcal{B}\mathcal{A} \end{cases} \quad (5)$$

where $\mathcal{B}\mathcal{A}_r := \{\Delta_r \in \mathcal{A}_r : \bar{\sigma}(\Delta_r) < 1\}$. Let $M_{f22} := \text{diag}[M_{33}, M_{33}, M_{33}]$. The following equivalence is clear:

$$\mu_{\mathcal{A}f}(M_{f22}) \leq 1 \Leftrightarrow \mu_{\mathcal{A}}(M_{33}) \leq 1. \quad (6)$$

Applying the Schur formula to $I - M_f\Delta_f$, we have

$$\det(I - M_f\Delta_f) = \det(I - M_{f22}\Delta_r)\det(I - \mathcal{F}_l(M_f, \Delta_r)\Delta_p). \quad (7)$$

This implies that the left hand sides of the equivalence (5) and (6) hold if and only if $\mu_{\mathcal{A}f}(M_f) \leq 1$ is satisfied. This completes the proof. $\blacksquare$

The theorem provides an equivalence allowing the treatment of performance robustness of the interconnection shown in Fig. 2 and stability robustness of the interconnection shown in Fig. 3. Now if we define

$$M_{f1} := \begin{bmatrix} M_{11} & M_{13} \\ M_{31} & M_{33} \end{bmatrix}, \quad M_{f2} := \begin{bmatrix} M_{22} & M_{23} \\ M_{32} & M_{33} \end{bmatrix} \quad (8)$$

and corresponding block diagonal perturbations:

$$\mathcal{A}_{f1} := \{\text{diag}[\Delta_{\Psi_1}, \Delta] : \Delta_{\Psi_1} \in \mathcal{A}_{\Psi_1}, \Delta \in \mathcal{A}\}$$

$$\mathcal{A}_{f2} := \{\text{diag}[\Delta_{\Psi_2}, \Delta] : \Delta_{\Psi_2} \in \mathcal{A}_{\Psi_2}, \Delta \in \mathcal{A}\},$$

then, the next corollary holds.

Corollary 1:

$$\mu_{\mathcal{A}f}(M_f) = \max\{\mu_{\mathcal{A}f1}(M_{f1}), \mu_{\mathcal{A}f2}(M_{f2})\}. \quad \blacksquare$$

The μ test in Theorem 1 is essentially identical to series of two μ tests, one test for the robustness of

Fig. 3 Robust stability problem

$\bar{\sigma}(\Psi_1) \leq 1$ and the other for the robustness of $\bar{\sigma}(\Psi_2) \leq 1$. In Theorem 1, two criteria for robust performance are expressed by a single μ test.

Consider the feedback control system in Fig. 4, where G , C and $\mathcal{A}$ are finite-dimensional, linear, time-invariant (FDLTI) and causal. Here G is the nominal plant, C is the controller, and $\mathcal{A}$ is the plant perturbation. The perturbation Δ belongs to the set $\mathcal{D}(n)$ defined as

$$\mathcal{D}(n) := \{\Delta(s) \in \mathcal{A} : \text{proper, stable and} \\ \text{real-rational function, } \|\Delta\|_{\infty} < 1\}.$$

Here, Laplace transform of a linear system T is indicated by $T(s)$. Define $M(s)$ by

$$M = \mathcal{F}_l(G, C), \quad (9)$$

which describes the nominal closed-loop system. Let $\Psi_1(s)$ and $\Psi_2(s)$ denote transfer functions from d_1 to e_1 and from d_2 to e_2 respectively. The nominal feedback system shown in Fig. 4 with $\Delta=0$ is called internally stable if the state of the unforced nominal feedback system is asymptotically stable. On the assumption that G is stabilizable and detectable from u and y respectively, internal stability implies stability of the transfer function from $[d_1^T, d_2^T, w^T, u_1^T, u_2^T]^T$ to $[e_1^T, e_2^T, v^T, y_1^T, y_2^T]^T$ and vice versa. The feedback system shown in Fig. 4 is said to be robustly stable for $\Delta \in \mathcal{D}(n)$ if the nominal system is internally stable and the transfer function from $[d_1^T, d_2^T, u_1^T, u_2^T, u_3^T, u_4^T]^T$ to $[e_1^T, e_2^T, y_1^T, y_2^T, y_3^T, y_4^T]^T$ is stable for all $\Delta \in \mathcal{D}(n)$. The next corollary follows directly from Theorem 1.

Corollary 2: The system in Fig. 4 is robustly stable for $\Delta \in \mathcal{D}(n)$ and both $\|\Psi_1\|_{\infty} \leq 1$ and $\|\Psi_2\|_{\infty} \leq 1$ are satisfied for all $\Delta \in \mathcal{D}(n)$ if and only if the nominal system is internally stable and $\sup_{\omega \in R+\mu_{\mathcal{A}f}(M_f(j\omega))} \leq 1$. $\blacksquare$

It is straightforward to extend the framework shown in this section to the general case of n objectives.

Fig. 4 Feedback control system with multiple objectives

Moreover, in Ito et al.⁹⁾, the equivalence theorem between a performance robustness problem with multiple objectives and a stability problem with extra repeated uncertainty block is extended to various control systems.

3. Multiobjective Controller Synthesis

In this section we will develop an iterative solution to the robust performances problem with two objectives. The outline of this section is similar to Hung and Pokrud⁶⁾ proposed for nominal performance. This section, however, tackles robust performance problems. Consider the feedback system shown in Fig. 4 again. The control problem to be considered is to find stabilizing controllers C with which the system in Fig. 4 achieves $\|\Psi_1\|_\infty \leq 1$ and $\|\Psi_2\|_\infty \leq 1$ for all $\Delta \in \mathcal{D}(n)$. As described in Section 2, we can achieve required robust performance by making $\mu_{\Delta_f}(M_f)$ less than or equal to 1. Therefore we consider an alternative problem as follows:

Problem 1 : Find controllers C which stabilize G and minimize $\sup_{\omega \in R} \mu_{\Delta_f}(M_f(j\omega))$. ■

The difficulty in solving the above is that there exists no interconnection matrix P which satisfies $M_f = \mathcal{F}_l(P, C)$. We can overcome this difficulty by solving the problem for C iteratively. This is the basis of the following algorithm.

Algorithm 1 :

Step 1) Let $i=1$ and

$$P_1 = \begin{bmatrix} G_{11} & G_{12} & G_{13} & 0 & 0 & G_{14} \\ G_{21} & G_{22} & G_{23} & 0 & 0 & G_{24} \\ G_{31} & G_{32} & G_{33} & 0 & 0 & G_{34} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ G_{41} & G_{42} & G_{43} & 0 & 0 & G_{44} \end{bmatrix}. \quad (10)$$

Step 2) Solve the minimization problem :

$$\gamma_i = \min_{C_i} \sup_{\omega \in R_+} \mu_{\Delta_f}(\bar{M}_f^{(i)}(j\omega)) \quad (11)$$

subject to the condition that C_i stabilizes P_i , where $\bar{M}_f^{(i)} := \mathcal{F}_l(P_i, C_i)$.

Step 3) If γ_i is satisfactorily small, then exit to Step 4), else define

$$P_{i+1} = P_i + \begin{bmatrix} 0 & -M_{12}^{(i)} & 0 & -M_{13}^{(i)} & 0 \\ -M_{21}^{(i)} & 0 & 0 & -M_{23}^{(i)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ M_{31}^{(i)} & 0 & 0 & M_{33}^{(i)} & 0 & 0 \\ 0 & M_{32}^{(i)} & 0 & 0 & M_{33}^{(i)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad (12)$$

where $M^{(i)} := \mathcal{F}_l(G, C_i)$. Increment i and return to Step 2).

Step 4) Put $\gamma = \gamma_i$, $C = C_i$ and terminate. ■

With the controller C obtained by using the above algorithm, the system is robustly stable for $\|\Delta\|_\infty < 1/\gamma$ and achieves corresponding robust performance in terms of $\|\Psi_1\|_\infty \leq \gamma$ and $\|\Psi_2\|_\infty \leq \gamma$. The next theorem provides the necessary justification of the proposed algorithm.

Theorem 2 : On the assumption that G is stabilizable by the feedback arrangement of Fig. 4, the following holds.

- i) For all $i \geq 1$, P_i is stabilizable, and C_i stabilizes P_i if and only if C_i stabilizes G .
- ii) The sequence $\{\gamma_i\}$ is monotonic nonincreasing and convergent.
- iii) The controller C obtained by using Algorithm 1 stabilizes the nominal feedback system of Fig. 4 and satisfies $\sup_{\omega \in R} \mu_{\Delta_f}(M_f(j\omega)) \leq \gamma$. ■

To prove the theorem, we need the following lemma.

Lemma 1 : Suppose that a matrix

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

is compatible in size with $\Delta_A = \text{diag}[\Delta_1, \Delta_2, \Delta_3]$. Then

$$\mu_{\Delta_A}(A) \geq \max\{\mu_{\Delta_1}(A_{11}), \mu_{\Delta_2}(A_{22}), \mu_{\Delta_3}(A_{33})\} \\ = \mu_{\Delta_A}(\text{diag}[A_{11}, A_{22}, A_{33}]). \quad \blacksquare$$

The above is immediate from the definition of the $\mu(\cdot)$. The following is known as the main loop theorem⁸⁾ which is a special case of Theorem 1.

Lemma 2 : Suppose that a matrix $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ is

compatible in size with $\Delta_A = \text{diag}[\Delta_1, \Delta_2]$. Then

$$\mu_{\Delta_A}(A) \leq \beta \Leftrightarrow \begin{cases} \mu_{\Delta_2}(A_{22}) \leq \beta \text{ and} \\ \sup_{\Delta_1 \in \mathcal{D}_1, \sigma(\Delta_1) < 1/\beta} \mu_{\Delta_1}(\mathcal{F}_l(A, \Delta_1)) \leq \beta \end{cases} \quad \blacksquare$$

Proof of Theorem 2 : i) : We prove this part i) by induction. For $i=1$, the statement is clearly true by

the construction of P_i . As an induction assumption, we suppose that P_i is stabilizable by some C_i which also stabilizes G . By this induction assumption, $M^{(i)}$ is stable. From the construction of P_{i+1} it follows that the stability of the closed-loop system consisting of P_i and C_{i+1} is equivalent to the stability of the closed-loop system consisting of P_{i+1} and C_{i+1} . Hence, P_{i+1} is stabilizable by some C_{i+1} which also stabilizes G .

ii) : Suppose that C_i is a stabilizing controller for P_i . By part i), C_i is a stabilizing controller for G and therefore it also stabilizes P_{i+1} . Now define $M_f^{(i)}$ by

$$M_f^{(i)} := \begin{bmatrix} M_{11}^{(i)} & 0 & M_{13}^{(i)} & 0 & -M_{13}^{(i)} \\ 0 & M_{22}^{(i)} & M_{23}^{(i)} - M_{23}^{(i-1)} & 0 & 0 \\ M_{31}^{(i)} & M_{32}^{(i)} & M_{33}^{(i)} & 0 & 0 \\ M_{31}^{(i)} & 0 & 0 & M_{33}^{(i)} & 0 \\ 0 & M_{32}^{(i)} & 0 & 0 & M_{33}^{(i)} \end{bmatrix}. \quad (13)$$

Then the following follows from the construction of P_{i+1} .

$$\mathcal{F}_i(P_{i+1}, C_i) = M_f^{(i)}. \quad (14)$$

From the definition of P_i we have

$$\bar{M}^{(i)} = \mathcal{F}_i(P_i, C_i) = \begin{bmatrix} M_{11}^{(i)} & M_{12}^{(i)} - M_{12}^{(i-1)} & M_{13}^{(i)} & 0 \\ M_{21}^{(i)} - M_{21}^{(i-1)} & M_{22}^{(i)} & M_{23}^{(i)} - M_{23}^{(i-1)} & 0 \\ M_{31}^{(i)} & M_{32}^{(i)} & M_{33}^{(i)} & 0 \\ M_{31}^{(i-1)} & 0 & 0 & M_{33}^{(i-1)} \\ 0 & M_{32}^{(i-1)} & 0 & 0 \\ -M_{13}^{(i-1)} \\ 0 \\ 0 \\ 0 \\ M_{33}^{(i-1)} \end{bmatrix}. \quad (15)$$

From Equation (13) and (15), it is shown that

$$\begin{aligned} \mathcal{F}_i(\bar{M}^{(i)}, \Delta_r) &= \mathcal{F}_i(M^{(i)}, \Delta) + \begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix} \\ &= \mathcal{F}_i(M_f^{(i)}, \Delta_r) + \begin{bmatrix} 0 & *' \\ *' & 0 \end{bmatrix}, \end{aligned} \quad (16)$$

where $\Delta_r = \text{diag}[\Delta, \Delta, \Delta]$. Here note $\mathcal{F}_i(M_f^{(i)}, \Delta_r) = \text{diag}[\Psi_1(C_i), \Psi_2(C_i)]$. According to Lemma 1,

$$\begin{aligned} &\mu_{\Delta_r} \left(\begin{bmatrix} M_{33}^{(i)} & 0 & 0 \\ 0 & M_{33}^{(i)} & 0 \\ 0 & 0 & M_{33}^{(i)} \end{bmatrix} \right) \\ &\leq \mu_{\Delta_r} \left(\begin{bmatrix} M_{33}^{(i)} & 0 & 0 \\ 0 & M_{33}^{(i-1)} & 0 \\ 0 & 0 & M_{33}^{(i-1)} \end{bmatrix} \right) \end{aligned} \quad (17)$$

holds for all $\omega \in \mathbf{R}_+$. In the above inequality, $j\omega$ is omitted. By applying Lemma 1 to (16) again, we have

$$\mu_{\Delta_r}(\mathcal{F}_i(M_f^{(i)}, \Delta_r)) \leq \mu_{\Delta_r}(\mathcal{F}_i(\bar{M}^{(i)}, \Delta_r)) \quad (18)$$

for all ω , where $\Delta_r := \text{diag}[\Delta_{\Psi_1}, \Delta_{\Psi_2}]$. Using Lemma 2 and (15), it is proved that

$$\mu_{\Delta_r}(\bar{M}^{(i)}) \leq \beta \quad (19)$$

if and only if

$$\begin{cases} \mu_{\Delta_r} \left(\begin{bmatrix} M_{33}^{(i)} & 0 & 0 \\ 0 & M_{33}^{(i-1)} & 0 \\ 0 & 0 & M_{33}^{(i-1)} \end{bmatrix} \right) \leq \beta \\ \sup_{\Delta_r \in \mathbf{A}_r, \sigma(\Delta_r) < 1/\beta} \mu_{\Delta_r}(\mathcal{F}_i(\bar{M}^{(i)}, \Delta_r)) \leq \beta. \end{cases} \quad (20)$$

Hence, (17), (18) and (13) deduce the following.

$$\begin{aligned} \mu_{\Delta_r}(\mathcal{F}_i(P_{i+1}, C_i)) &= \mu_{\Delta_r}(M_f^{(i)}) \\ &\leq \mu_{\Delta_r}(\bar{M}^{(i)}) \quad \forall \omega. \end{aligned} \quad (21)$$

Since obviously

$$\begin{aligned} \gamma_{i+1} &= \min_{C_{i+1}} \sup_{\omega \in \mathbf{R}_+} \mu_{\Delta_r}(\bar{M}^{(i+1)}(j\omega)) \\ &\leq \sup_{\omega \in \mathbf{R}_+} \mu_{\Delta_r}(\mathcal{F}_i(P_{i+1}, C_i)(j\omega)) \end{aligned} \quad (22)$$

is true, $\gamma_i = \sup_{\omega \in \mathbf{R}_+} \mu_{\Delta_r}(\bar{M}^{(i)}(j\omega))$ implies that $\gamma_{i+1} \leq \gamma_i$ holds. Clearly, the sequence $\{\gamma_i\}$ is bounded by zero, and it is convergent.

iii) : This part follows directly from part i) and part ii). ■

Remark 1 : The algorithm stands on the minimization problem (11), which is justified by (21). Theorem 2 holds even if C_i is a local minimum of the problem (11) for all i . Therefore, we can use the D-K iteration approach for solving (11). It should be, however, noted that γ_i is not guaranteed to converge to the global minimum of Problem 1. ■

Remark 2 : The generalized plants P_i and P_{i+1} can be realized easily in the state-space. Suppose the state-space representations of G and C_i are given. Then P_i and P_{i+1} are obtained by

$$\begin{aligned} P_i &= \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} G \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix}, \quad (23) \\ P_{i+1} &= P_i + \end{aligned}$$

$$Q_i = \begin{bmatrix} 0 & 0 & I & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix} \quad (24)$$

$$Q_i = \text{diag}[F_{1i}\mathcal{F}_i(G, C_i)F_{1r}, F_{2i}\mathcal{F}_i(G, C_i)F_{2r}]$$

$$F_{1i} = \begin{bmatrix} 0 & -I & 0 \\ 0 & 0 & I \end{bmatrix}, \quad F_{2i} = \begin{bmatrix} -I & 0 & 0 \\ 0 & 0 & I \end{bmatrix}$$

$$F_{1r} = \begin{bmatrix} I & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix}, \quad F_{2r} = \begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{bmatrix}$$

Here, $+$ and $\mathcal{F}_i(\cdot, \cdot)$ can be computed easily in the state-space. ■

Remark 3 : Let the McMillan degree of P_i be denoted by $\delta(P_i)$. From Remark 2, we have

$$\delta(P_i) \leq n \quad (25)$$

$$\delta(P_{i+1}) \leq 3n + 2\delta(C_i) - 2r_i, \quad i \geq 1 \quad (26)$$

where $n = \delta(G)$, and r_i denotes the number of pole-zero cancellations in Q_i . It is known that the degree of C_i solving (11) satisfies $\delta(C_i) \leq \delta(P_i) + s$, where s_i is interpreted as {the degree of D -scales}. Hence, for $i \geq 1$, we obtain

$$\delta(C_i) \leq (2^{i+1} - 3)n - \sum_{j=1}^{i-1} 2^j r_{i-j} + \sum_{j=0}^{i-1} 2^j s_{i-j}. \quad (27)$$

This inequality of McMillan degree is the same as that for the nominal performance design⁹⁾ except the last term consisting of s_{i-j} . To prevent the degree of controller from increasing rapidly with iteration i , we can perform model reduction of C_i obtained by the D-K synthesis at each i , which, in effect, makes s_i as small as possible in (27). We will answer how to introduce model reduction techniques by a design example shown in the next section. ■

Remark 4 : Unfortunately, no technique to construct D -scales for repeated blocks as real-rational functions has not developed yet. We can, however, treat them as independent blocks in the D-K iteration since the D -scales also give an upper bound of μ . ■

4. A Numerical Example

Consider a feedback system shown in Fig. 5. Here P_o denotes the nominal plant given by

$$P_o(s) = \frac{4-s}{(80s+1)(4+s)}.$$

The uncertainty Δ is a stable and causal system

Fig. 5 FDLTI feedback control system

which satisfies

$$|\Delta(j\omega)| < |W_f(j\omega)| \quad \forall \omega \in [0, \infty),$$

$$W_f = \frac{5(s+0.1)}{s+2}.$$

The set to which actual plants belong is

$$\mathcal{P} = \{P_o(1+\Delta) : \forall \Delta\}.$$

The signal d is the disturbance acting on the input at the plant $\mathcal{P}$ and its characteristic in the frequency domain is described by

$$W_d(s) = \frac{s+6}{100(s+0.01)}.$$

The exogenous input r denotes the reference signal and its bandwidth is represented by

$$W_r(s) = \left(\frac{10}{(s+10)} \right)^2.$$

It is desired to design a feedback controller $C(s)$ which internally stabilizes every member of $\mathcal{P}$, and attenuates the influence of the disturbance d on the output signal y , and prevents the control input u from being large. Let the transfer function from d to e be denoted by T_{ed} , and that from r to u be denoted by T_{ur} . The desired performance is summarized as follows:

S1) The closed-loop is robustly stable for $\mathcal{P}$;

S2) $\|W_e T_{ed} W_d\|_\infty \leq 1$ holds for $\mathcal{P}$;

S3) $\|W_u T_{ur} W_r\|_\infty \leq 1$ holds for $\mathcal{P}$,

where weights in the cost functions are

$$W_e(s) = \frac{100}{3}, \quad W_u = \frac{1}{5} \left(\frac{(s/10+1)}{(s/1000+1)} \right)^2.$$

The robust stability S1 must be achieved, and we want to make the costs of S2 and S3 as small as possible. Let M appeared in (3) be defined by

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} = \begin{bmatrix} W_e & 0 & 0 \\ 0 & W_u & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} T_{ed} & T_{er} & T_{ef} \\ T_{ud} & T_{ur} & T_{uf} \\ T_{ud} & T_{ur} & T_{uf} \end{bmatrix} \begin{bmatrix} W_d & 0 & 0 \\ 0 & W_r & 0 \\ 0 & 0 & W_f \end{bmatrix} = \mathcal{F}_i(G, C). \quad (28)$$

Design 1) Note that both d and r are large at low

frequencies, and making S2 satisfied does not mean that the cost in S3 becomes small. To see this fact, first, we design a controller by skipping S3. In this case, the problem becomes robust performance design with a single objective. Using the D-K iteration method, a controller is found and closed-loop response of the resulting system is shown in **Fig. 6** (a). The peak values of μ over all frequencies are shown in **Table 1**. The figure and table show that the controller achieves satisfactory robust stability and robust disturbance attenuation. The input signal u , however, becomes very large at middle and high frequencies. How good disturbance attenuation can we achieve if the magnitude of u is constrained? This

(a) Design 1

(b) Design 2

(c) Design 3

Fig. 6 Frequency response of nominal closed-loop functions

question will be answered.

Design 2) Next, we combine the two performance costs S2 and S3 as in the form of

$$\left\| \begin{bmatrix} W_e & 0 \\ 0 & W_n \end{bmatrix} \begin{bmatrix} T_{ed} & T_{er} \\ T_{ud} & T_{ur} \end{bmatrix} \begin{bmatrix} W_d & 0 \\ 0 & W_r \end{bmatrix} \right\|_{\infty}. \quad (29)$$

Using D-K iteration approach again, we solve the robust performance problem defined with S1 and (29). The achieved response is plotted in **Fig. 6**(b), and peak values of μ are shown in **Table 1**. The obtained close-loop is far from what we required.

Design 3) Now, we tackle S1, S2 and S3 simultaneously by using Algorithm 1. Here, Step 2 of Algorithm 1 is replaced by the following procedure.

Step 2a) Perform the balanced truncation on P_i with tolerance set at $10^{-5} \times \{\text{the largest Hankel singular value}\}$. And, set $P'_i = \{\text{the reduced-order model}\}$.

Step 2b) Solve (11) by using D-K iteration, where P_i is replaced by P'_i .

Step 2c) By using balanced truncation, find the lowest order (McMillan degree) controller which internally stabilizes G and satisfies $\mu_{\Delta f}(M_f^{(i)}) < \gamma_i^*(1 + 10^{-3})$, where γ_i^* is $\mu_{\Delta f}(M_f^{(i)})$ achieved with the full-order controller.

We adopted Step 2a to correctly detect pole-zero cancellations of P_i computed by (24). Step 2c is used for preventing the McMillan degree of C_i from becoming large as i increases. Note that Theorem 2 is still valid. The results are shown in **Table 2**. Since the 4th iteration does not improve the peak value of μ , we take C_3 as a satisfactory solution. Frequency response associated with the controller is shown in

Table 1 Peak values of μ achieved

	Design 1	Design 2	Design 3
$\mu_{\Delta f1}(M_{f1})$	0.833	30.3	2.49
$\mu_{\Delta f2}(M_{f2})$	50.5	29.9	2.61
$\mu_{\Delta}(M_{33})$	0.803	0.723	0.396

Table 2 Design 3 using the proposed algorithm

	$i=1$	$i=2$	$i=3$
McMillan degree of P'_i	8	19	22
McMillan degree of C_i	7	8	12
$\mu_{\Delta f}(M_f^{(i)})$	3.12	2.77	2.61
(1st objective)	(3.12)	(2.57)	(2.49)
(2nd objective)	(3.06)	(2.77)	(2.61)

Fig. 6(c), and peak values of μ are shown in Table 1. The controller achieves the robust stability S1. Although S2 and S3 are not attained exactly, we can see trade-off between two objectives. The performance levels are balanced indeed and much smaller than those obtained by other designs. The resulting μ plot for the above three designs is shown in Fig. 7. Compared with the plot in Fig. 7(b), the μ curve for the first objective in Fig. 7(c) is shifted to the right and the curve for the second objective is shifted to the left, and this shift results in balanced and small peak value of μ . This effect cannot be attained by using the combined objective (29) because at the frequency where μ curves of two objectives cross, the shift

brings large value of μ of the combined objective. This illustrates the ability of the proposed algorithm to evaluate two performance objectives independently.

5. Conclusions

We have considered robust performance problems with two performance objectives. A μ test for assessing performance robustness of two objectives is proposed. We have shown how the μ test is exploited successfully to develop an algorithm to the controller synthesis problems.

References

- 1) B. A. Francis: A Course in H^∞ Control Theory, Springer-Verlag (1987)
- 2) J. Doyle: Structured Uncertainty in Control System Design, in Proc. the 24th IEEE Conf. on Decision and Contr., 260/265 (1985)
- 3) P. P. Khargonekar and M. A. Rotea: Multiple Objective Optimal Control of Linear Systems: The Quadratic Norm Case, IEEE Trans., **AC-36**, 14/24 (1991)
- 4) D. S. Bernstein and W. H. Haddad: LQG Control with an H^∞ Performance Bound: A Riccati Equation Approach, IEEE Trans., **AC-34**, 293/305 (1989)
- 5) P. P. Khargonekar, M. A. Rotea and N. Sivashankar: Exact and Approximate Solution to a Class of Multiobjective Controller Synthesis Problems, in Proc. IEEE American Contr. Conf., 1602/1606 (1993)
- 6) Y. S. Hung and B. Pokrud: An H^∞ Approach to Feedback Design with Two Objective Functions, IEEE Trans., **AC-37**, 820/824 (1992)
- 7) J. Doyle: Analysis of Feedback Systems with Structured Uncertainty, IEE. Proc., **129-D**, 242/250 (1982)
- 8) J. Doyle, A. Packard and K. Zhou: Review of LFTs, LMIs and μ , in Proc. the 30th IEEE Conf. on Decision and Contr., 1227/1232 (1991)
- 9) H. Ito, H. Ohmori and A. Sano: Analysis of Robust Performance with Multiple Objectives, Systems & Control Letters., (1995) to appear

(a) Design 1

(b) Design 2

(c) Design 3

Fig. 7 μ plot for two performance objectives and stability

Hiroshi Iro (Student Member)

He received the B. E. and M. E. degrees from Keio University, Japan in 1990 and 1992, respectively. He is currently working toward the Dr. Eng. degree in the Department of Electrical Engineering, Keio University, Japan. His research interests include robust control theory, decentralized control, multirate sampled-data control and nonlinear control theory. He is a member of IEEE. From April 1994, he is a Research Fellow of the Japan Society for the Promotion of Science (JSPS).

Hiromitsu OHMORI (Member)

He received the B. E., M. E. and Dr. Eng. Degrees in Electrical Engineering, all from Keio University, Yokohama, Japan, in 1983, 1985, 1988 respectively. Since 1985 he has been with the Faculty of Science and Technology, Keio University, where he is presently an Assistant Professor of Department of Electrical Engineering. His research interests are in adaptive control and robust control theory, and their applications.

Akira SANO (Member)

He received the B. E., M. E. and Ph. D. degrees in mathematical engineering and information physics from University of Tokyo, in 1966, 1968 and 1971 respectively. He joined the Department of Electrical Engineering, Keio University in 1971. He is currently a Professor. His research interests include, control system design, adaptive control, system identification, digital signal processing, and control applications to biomedical systems and industrial systems. He was awarded the Kelvin Premium by the Institute of Electrical Engineering in United Kingdom in 1986. He is a member of ISCIE, IEICE, IEE, Japan, and IEEE.

=====