A Class of State Feedback Controllers with Simpler Structures for Uncertain Nonlinear Dynamical Systems

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Robust stabilization for a class of uncertain nonlinear dynamical systems is investigated. Based on the stability of the nominal systems, a new approach to synthesizing a class of continuous state feedback controllers with simpler structures for uncertain nonlinear dynamical systems is proposed. The state feedback controllers developed by using our approach are computationally simpler and can guarantee uniform ultimate boundedness and uniform asymptotic stability of uncertain nonlinear dynamical systems by choosing different control gain functions. Furthermore, for dynamical system with a linear nominal part and uncertainties bounded by constants, the state feedback controller proposed in this paper will become linear in the state. Therefore, for some practical robust control problems, the control given here may more easily be implemented. Finally, several illustrative examples are given to demonstrate the utilization of the approach developed in this paper.

Key Words: uniform ultimate boundedness, uniform asymptotic stability, nonlinear dynamical systems, uncertainty, state feedback controller, Lyapunov function

1. Introduction

In recent years, robust stabilization of dynamical systems with significant uncertainties is widely studied in the control research, and a number of new approaches have been proposed for synthesizing state feedback controllers which lead to some desired performance, e.g. asymptotic stability, practical stability, ultimate boundedness, exponential stability, etc., of the state of an uncertain dynamical system (see, e.g. Refs. 1) and 2)). For nonlinear and time-varying dynamical systems with significant uncertainties described only in terms of bounds on their possible sizes, it seems that the so called Lyapunov minimax approach be one of the most effective approaches for stabilizing controller synthesis.

The Lyapunov minimax approach is generally based on the stabilizability of a nominal system (i.e. the system in the absence of uncertainty). Roughly speaking, a Lyapunov function of the stable nominal system is employed as a Lyapunov function candidate for the actual uncertain dynamical system, and a control law is then chosen such that the Lyapunov function decreases along every possible trajectory of uncertain dynamical systems.

Based on the Lyapunov minimax approach, some stabilizing state feedback controllers have been proposed for dynamical systems with the so called matching uncertainties. In Ref. 3), for example, a class of discontinuous minimax controls are proposed which can guarantee uniform asymptotic stability of uncertain dynamical systems. In Ref. 4), a class of continuous saturation-type controls are proposed which can guarantee uniform ultimate boundedness of uncertain dynamical systems. In Ref. 6)~13), some other types of continuous state feedback controllers are proposed which guarantee some different types of stability results for uncertain dynamical systems. In particular, in Ref. 13), a class of continuous saturation-type controls are proposed which are computationally simpler than the existing state feedback controllers and can guarantee different types of stability of uncertain dynamical systems. In addition, for dynamical systems with mismatching uncertainties, there also are some standard method to relax the matching conditions (see, e.g. Refs. 7), 14)~18)).

In this paper, we discuss the problem of robust control for a class of uncertain nonlinear dynamical

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systems. Based on the stability of the nominal systems, a new approach to synthesizing a class of continuous state feedback controllers with simpler structures for uncertain nonlinear dynamical systems is proposed. Similarly to the controller given in Ref. 13), the state feedback controller developed by the approach proposed in this paper is computationally simpler than the existing state feedback controllers and can guarantee uniform ultimate boundedness and uniform asymptotic stability of uncertain nonlinear dynamical systems by choosing the different gain functions of the control. Furthermore, the newly proposed state feedback controller can also be decoupled with respect to possible uncertainties. That is, each component in the control vector requires only the bounding function of the corresponding local uncertainty. In particular, for uncertain dynamical system with a linear nominal part, the newly proposed state feedback controller will become linear in the state if each component of the uncertain vector is bounded by a constant. Therefore, for some practical robust control problems, the state feedback controller proposed in this paper may more easily be implemented. Several numerical examples are also given to demonstrate the synthesis procedure of state feedback controllers in the light of the approach presented in this paper.

The paper consists of the following parts. In Section 2, the problem to be tackled in this paper is stated, several standard assumptions are introduced, and some mathematical preliminaries are given. In Section 3, a continuous feedback controller is proposed to stabilize nonlinear dynamical systems with uncertainties, and as a special case, the problem of robust stabilization of the systems whose nominal part is linear is discussed. In Section 4, two numerical examples are given. Finally, the paper is concluded in Section 5 with a brief discussion of the results.

2. Problem Formulation and Assumptions

2.1 Problem Formulation

Consider a class of uncertain nonlinear dynamical systems described by the following state equations.

\[
\frac{dx(t)}{dt} = F(x(t), u(t), E(x,t)) \quad (1a)
\]

\[x(t_0) = x_0 \quad (1b)
\]

where \( t \in \mathbb{R} \) is the time, \( x(t) \in \mathbb{R}^n \) is the state vector, \( u(t) \in \mathbb{R}^m \) is the control vector, and \( E(x,t) \) represents the system uncertainties. Generally, \( E(x,t) \) is assumed to be bounded in magnitude, usually in its Euclidean norm denoted by \(|\cdot|\). The corresponding system without uncertainty, called nominal system, is described by

\[
\frac{dx(t)}{dt} = F(x(t), u(t)) \quad (2a)
\]

\[x(t_0) = x_0 \quad (2b)
\]

where \( F(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \) and \( G(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{n \times n} \) are known. Moreover, the so called unforced nominal system of the system \((1)\) is defined by

\[
\frac{dx(t)}{dt} = F(x(t)), \quad x(t_0) = x_0 \quad (3)
\]

Provided that all states are available, the state feedback controller can be represented by the following nonlinear functions:

\[
u_i(t) = p_i(x(t), t) \quad (4a)
\]

or in a vector form

\[u(t) = p(x(t), t) \quad (4b)
\]

where \( p(x(t), t) = [p_1(x(t), t) \ldots p_m(x(t), t)]^T \), and \( p_i(x(t), t) : \mathbb{R}^n \to \mathbb{R}^m \) for all \( i = 1, \ldots, m \).

Now, the question is how to synthesize a continuous state feedback controller \( u(t) \) that can guarantee some types of stability for nonlinear dynamical system \((1)\) in the presence of uncertainties \( E(x,t) \).

Before giving our synthesis approach, we first introduce for nonlinear system \((1)\) the following standard assumptions, then state two lemmas to facilitate the discussion of the main results in this paper.

2.2 Assumptions

Assumption 2.1 The known functions \( F(\cdot, \cdot) \) and \( G(\cdot, \cdot) \), as well as the unknown function \( E(\cdot, \cdot) \), are continuous, uniformly bounded with respect to time, and locally uniformly bounded with respect to the state \( x \). \( E(\cdot, \cdot) \in \mathcal{E} \), where \( \mathcal{E} \) is a specified set.

Assumption 2.2 For each \( E(\cdot, \cdot) \in \mathcal{E} \), there exists a mapping \( \xi(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m \) satisfying

\[E(x(t), t) = G(x(t), t)\xi(x(t), t) \quad (5)\]

for all \( (x(t), t) \in \mathbb{R}^n \times \mathbb{R} \).

Assumption 2.3 The uncertain \( \xi(\cdot, \cdot) \) is bounded element by element by known functions, i.e. there exist known nonnegative continuous functions \( \rho_i(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^+ \), \( i = 1, \ldots, m \), such that

\[|\xi(x(t), t)| \leq \rho_i(x(t), t), \quad i = 1, \ldots, m \quad (6)\]

for all \( (x(t), t) \in \mathbb{R}^n \times \mathbb{R} \).

Without loss of generality, the functions \( \rho_i(\cdot, \cdot), i = 1, \ldots, m \), are assumed to be uniformly bounded with respect to time and locally uniformly bounded with respect to the state \( x \).

Assumption 2.4 The origin \( x = 0 \) is an uniformly
asymptotically stable equilibrium point in the large for the unforced nominal system described by (3). More specifically, there exist a C\(^1\) function \(V(x, t)\) for dynamical system (3). That is, there exists a positive definite continuous scalar function \(\gamma_i(x)\) for dynamical system (3), which satisfy

\[
\gamma_i(0) = 0, \quad i = 1, 2, 3
\]

such that for all \((x, t) \in R^+\)

\[
\gamma_i(||x||) \leq V(x, t) \leq \gamma_i(||x||)
\]

\[
\frac{\partial V(x, t)}{\partial t} + P^T(x, t) F(x, t) < -\gamma_3(||x||)
\]

Remark 2.1 It is obvious that Assumption 2.1 is a technical assumption for mathematical completeness.

Assumption 2.2 defines the matching condition about the uncertainties. Assumption 2.3 is similar to the one given in Ref. 13), and defines element by element the uncertainty bands (in general state dependent) for \(f(x, t)\). Assumption 2.4 shows that the nominal system must be internally stable in the sense that there exists a Lyapunov function. Indeed, in order to guarantee robust stability of uncertain dynamical systems, their nominal systems must be stabilizable. It is under such an assumption that we can discuss the robust stability of dynamical systems with uncertainties.

2.3 Mathematical Preliminaries

The following Lemma 2.1 is a compact form of the theorem proved in Ref. 4).

Lemma 2.1 Let \(V(x, t)\) be a Lyapunov function candidate for a given continuous dynamical system

\[
\frac{dx(t)}{dt} = f(x(t), t)
\]

with the following properties:

\[
\gamma_i(||x||) \leq V(x, t) \leq \gamma_i(||x||)
\]

\[
\frac{\partial V(x, t)}{\partial t} + P^T(x, t) F(x, t) < -\gamma_3(||x||) + 2\varepsilon
\]

where \(\varepsilon > 0\) is a constant, the functions \(\gamma_i\), \(i = 1, 2, 3\), are defined in Assumption 2.4. Then, if

\[
2\varepsilon < \lim_{r \to \infty} \inf \gamma_i(r) := \ell
\]

every solution \(x(t; t_0, x_0)\) of dynamical system (7) is both uniformly bounded and uniformly ultimately bounded. More precisely, one has the following results.

(i) Uniform boundedness: If \(x(t)\) is a solution of the system, then

\[
\|x(t)\| \leq \gamma_i(r) \quad \forall t \in [t_0, t_1]
\]

(ii) Uniform ultimate boundedness: If \(x(t)\) is a solution of the system, then

\[
\|x(t)\| \leq \gamma_i(r) \quad \forall t \geq t_0 + \tau(d, r)
\]

where

\[
\tau(d, r) = \frac{\gamma_i(r) - \gamma_i(R)}{\gamma_i(R) - 2\varepsilon}
\]

and

\[
R = \gamma_i^{-1}(2\varepsilon)
\]

Remark 2.2 Let \(V(x, t)\) be a Lyapunov function candidate for the given continuous dynamical system described by (7), and with the following properties:

\[
\gamma_i(||x||) \leq V(x, t) \leq \gamma_i(||x||)
\]

\[
\frac{\partial V(x, t)}{\partial t} + P^T(x, t) F(x, t) < -\gamma_3(||x||) + \phi(t)
\]

where the functions \(\gamma_i\), \(i = 1, 2, 3\), are defined in Assumption 2.4, and \(\phi(t)\) is a continuous function satisfying

\[
\lim_{t \to \infty} \int_{t_0}^{t} \phi(t) dt \leq \phi < \infty
\]

where \(\phi\) is any constant. Then, system (7) is uniformly asymptotically stable. That is, for any solution \(x(t)\) of (7),

\[
\lim_{t \to \infty} \|x(t)\| = 0
\]

Proof: By the continuity of dynamical system (7), it is obvious that the dynamical system described by (7) has a continuous solution \(x(t; t_0, x_0)\).

It follows from (9) that for any \(t \geq t_0\)

\[
0 \leq V(x(t)) \leq V(x(t_0)) + \int_{t_0}^{t} V(x(\tau), \tau) d\tau
\]

\[
\leq \gamma_i(||x(t)||) - \int_{t_0}^{t} \gamma_i(||x(\tau)||) d\tau + \int_{t_0}^{t} \phi(\tau) d\tau
\]

Therefore, from (11) we can obtain the following two results. First, taking the limit as \(t\) approaches infinity on both sides of inequality (11), we have
\[ 0 \leq y_\alpha(x^a) - \lim_{t \to t_0} \int_{t_0}^t \gamma(x(\tau)) d\tau + \bar{\gamma} \]
\[ + \lim_{t \to t_0} \phi(\tau) d\tau \quad (12) \]

It follows from (10) and (12) that
\[ 0 \leq y_\alpha(x^a) - \lim_{t \to t_0} \int_{t_0}^t \gamma(x(\tau)) d\tau + \bar{\gamma} \]
i.e.
\[ \lim_{t \to t_0} \int_{t_0}^t \gamma(x(\tau)) d\tau \leq y_\alpha(x^a) + \bar{\gamma} \quad (13) \]

On the other hand, from (11) we also have
\[ 0 \leq \gamma(\|x(t)\|) = y_\alpha(x^a) + \int_{t_0}^t \phi(\tau) d\tau \quad (14) \]

Since the function \( \phi(\cdot) \) is continuous and satisfies (10), we can define a constant as follows.
\[ \Psi := \sup_{t \in [t_0, t_1]} \left| \int_{t}^{\tau} \phi(\tau) d\tau \right| \quad (15) \]

It follows from (14) and (15) that
\[ 0 \leq \gamma(\|x(t)\|) \leq y_\alpha(x^a) + \Psi \quad (16) \]

which implies that \( x(\cdot) \) is uniformly bounded. Since \( x(\cdot) \) has been shown to be continuous, it follows from (7) that \( x(\cdot) \) is uniformly continuous. Applying the Barbalat lemma to inequality (13) yields that
\[ \lim_{t \to \infty} y_\alpha(x(t)) = 0 \quad (17) \]

Since \( y_\alpha(\cdot) \) is a positive definite scalar function, it is obvious from (17) that we can have
\[ \lim_{t \to \infty} \|x(t)\| = 0 \]

That is, dynamical system (7) is uniformly asymptotically stable.

**Remark 2.2** From the proof of Lemma 2.2, it is shown that one does not require that the value of limitation of
\[ \lim_{t \to t_0} \int_{t_0}^t \phi(\tau) d\tau \quad (18) \]
must exist. Acturally, one only requires that (18) is bounded by a constant.

**Remark 2.3** It is obvious that Lemma 2.2 includes the results of the lemma given in Ref. 13). That is, the lemma of Ref. 13) is a special case of the lemma developed here. For instance, it is obvious that the function \( \phi(\cdot) = \cos(\cdot) \) does not satisfy the condition \( \lim_{t \to \infty} \phi(\cdot) = 0 \) given in the lemma of Ref. 13), but from Remark 2.2 it satisfies condition (10). In addition, it is also worth pointing out that in Ref. 21), some improvement of the results given in Ref. 13) has been made.

### 3. Stabilizing State Feedback Controllers

In this section, we consider uncertain nonlinear dynamical system defined in (1). Based on the Lyapunov stability theory, we propose the following state feedback controller:
\[ u_i(t) = p_i(x, t) \]
\[ = -k_i(t)p_i(x, t)p_i(x, t), \quad i = 1, \ldots, m \quad (19a) \]
or in a vector form
\[ u(t) = p(x, t) \]
\[ = -K(t)p(x, t)p(x, t) \quad (19b) \]

where \( K(t) \) and \( p(x, t) \) are diagonal matrices defined by \( K(t) = \text{diag} \{ k_1(t), \ldots, k_m(t) \} \) and \( p(x, t) = [p_1(x, t), \ldots, p_m(x, t)] \), respectively; the scalar control gain functions \( k_i(t) > 0, i = 1, \ldots, m \), are chosen to be continuous; \( p_i(x, t) \), is the \( i \)-th element of the vector function \( p(x, t) \), and \( p(x, t) := G^T(x, t)P_xV(x, t) \in \mathbb{R}^n \).

**Remark 3.1** It is apparent that the state feedback controller proposed in (19) has many nice properties similar to those stated in Ref. 13), e.g. continuity, computational simplicity, resulting different stability results, and so on. However, the controller proposed in Ref. 13) is a saturation-type. Therefore, even if the nominal part of uncertain dynamical system is linear and with uncertainties bounded by some constants, the controller proposed in Ref. 13) is also nonlinear since it is naturally a nonlinear controller. The controller proposed in (19) has rather simpler structures than the existing controllers, and is more easier to implement in practical control problems. Since it is not constructed by fractional functions or saturation-type functions, no chattering will appear in implementation for the control. In addition, the function \( k_i(t) \), for each \( i \in \{1, \ldots, m\} \) may be called a control gain parameter and by appropriately selecting it, one may obtain better transient behavior for closed-loop dynamical system (1), (19) (see, e.g. Section 4). Therefore, the controller proposed in this paper may be considered as a candidate of state feedback controllers for some practical robust control problems.

Thus, we have the following theorem which shows some types of stability of uncertain dynamical system (1) under the state feedback controller (19).

**Theorem 3.1** Consider dynamical system (1) satisfying Assumptions 2.1-2.4. Then, for every \( E(\cdot, \cdot) \in \mathcal{E} \), the closed-loop dynamical system (1), (19) has the following properties:

1. If the gain function matrix \( K(t) \) is chosen such that \( \frac{1}{k_i(t)} \) is finite for all time and all \( i \), and
satisfies
\[ \sum_{i=1}^{m} \frac{1}{4k_i(t)} \leq 2e < l \] (20)
where \( e \) is a constant and \( l \) is also a constant defined as in (8), then the closed-loop dynamical system described by (1) and (19) is uniformly ultimately bounded.

(ii) If the gain function matrix \( K(t) \) is chosen such that
\[ \lim_{t \to \infty} \int_{t_0}^{t} \frac{1}{4k_i(t)} \, dt \leq \tilde{k} < \infty \] (21)
where \( \tilde{k} \) is any constant, then the closed-loop dynamical system described by (1) and (19) is uniformly asymptotically stable.

Proof: Let \( x(t) = x(t; t_0, x_0) \) be a solution of the closed-loop dynamical system described by (1) and (19). Furthermore, employing the same Lyapunov function as given in Assumption 2.4, we have
\[
\frac{dV(x, t)}{dt} = \frac{\partial V(x, t)}{\partial x} F(x, t) + \frac{\partial V(x, t)}{\partial x} G(x, t) p(x, t) + \frac{\partial V(x, t)}{\partial x} E(x, t) \\
+ \mu^T(x, t) \xi(x, t) - \gamma_0 \|x\|^2
\]
where \( \mu \) is any constant, \( \xi \) is the control deviation, and \( E(x, t) = B \mu(x, t) \) (24).

Therefore, from (22) we can obtain the following two analysis results:

(i) If \( k_i(t), i=1, \ldots, m, \) are selected such that the inequality (20) is satisfied, then it is obvious from Lemma 2.1 that the closed-loop dynamical system described by (1) and (19) is uniformly ultimately bounded.

(ii) If \( k_i(t), i=1, \ldots, m, \) are chosen such that (21) is satisfied, then it follows from Lemma 2.2 that the closed-loop dynamical system described by (1) and (19) is uniformly asymptotically stable.

Remark 3.2 It is worth pointing out that though we can theoretically guarantee asymptotic stability of system (1) by selecting the control gain functions \( k_i(t), i=1, \ldots, m, \) which satisfy (21), it may be impossible to select such gain functions for practical robust control problems since such gains \( k_i(t) \) must go to infinity as time \( t \) goes to infinity. But, in practical control problems, we may choose the gains \( k_i(t) \) satisfying (20) (e.g. select \( k_i(t) \) as some finite constants) such that system (1) is uniformly ultimately bounded. Or we may choose the gains \( k_i(t) \) satisfying (21) and stop the control in some finite time after the required performance of the system is achieved. In some practical problems, the control processes may be considered to be finished in this finite time. Moreover, if the processes continue after this finite time, an additive controller is needed to guarantee that the required performance of the system continue to hold. For instance, if the required performance of the system is achieved at the time \( t_i \), we may select the control gain \( k_i(t_i) \) which takes the place of the gain \( k_i(t) \) in (19a) to obtain such an additive controller. It is obvious from (22) that the required performance of the system can be maintained for any \( t \geq t_i \).

In the rest of this section, as a special case of the above results, we consider the problem of robust stabilization of uncertain dynamical systems with linear nominal part described by
\[
\frac{dx}{dt} = Ax(t) + Bu(t) + E(x, t) (23a) \\
x(t_0) = x_0 (23b)
\]
where \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{n \times m} \) are constant matrices.

Here, for systems (23) we make the following standard assumptions.

Assumption 3.1 The matrix pair \((A, B)\) defined in (23) are completely controllable.

Assumption 3.2 For each \( E(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{R} \) satisfying \( E(x, t) = B \xi(x, t) \) (24) for all \((x, t) \in \mathbb{R}^n \times \mathbb{R}.

It follows from Assumption 3.1 that for any positive definite matrix \( Q \in \mathbb{R}^{n \times n} \), the algebraic Riccati equation of the form
\[
A^T P + PA - PBB^T P = -Q
\]
(25)
has a solution \( P \in \mathbb{R}^{n \times n} \), which is the positive definite matrix.

Here, we propose the following feedback controller for the system (23).
\[
u(t) = p_1(x, t) + p_2(x, t) (26a)
\]
where
\[
p_1(x, t) = -\frac{1}{2} B^T P x(t) (26b)
\]
\[ p_2(x, t) = -K(t)p_2(x, t)B^TPx(t) \]  
(26c)

where \( P \) is a positive definite matrix satisfying (25); \( K(t) \) and \( p_2(x, t) \) are diagonal matrices defined by \( K(t) = \text{diag} \{ k_1(t), \ldots, k_m(t) \} \) and \( p_2(x, t) = \text{diag} \{ p_1(x, t), \ldots, p_m(x, t) \} \), respectively; the scalar control gain functions \( k_i(t) > 0, \) \( i = 1, \ldots, m, \) are chosen to be continuous.

**Remark 3.3** The control (26) consists of two parts, \( p_1(\cdot, \cdot) \) and \( p_2(\cdot, \cdot) \). Here, \( p_1(\cdot, \cdot) \) is a linear state feedback controller which is used to stabilize the nominal system, and \( p_2(\cdot, \cdot) \) is a continuous (nonlinear) state feedback controller which is used to compensate for the system uncertainties in the system (23) to produce some types of stability results.

**Remark 3.4** It is worth noting that even if system (23) has a linear nominal part, the state feedback controllers which have been reported so far in the literature (see, e.g. Refs. 3~8), 13) are nonlinear. However, if the system uncertainties are bounded by some constants, the controller proposed in (26) is not only linear in the state \( x \), but also can guarantee an uniform asymptotic stability results of the dynamical system (23) by selecting an appropriate gain function matrix \( K(t) \).

**Corollary 3.1** Consider dynamical system (23) satisfying Assumptions 3.1, 3.2, and Assumption 2.3. Then, for every \( E(\cdot, \cdot) \subseteq \Sigma_x \), the closed-loop dynamical system described by (23) and (26) has the following properties:

(i) If the gain function matrix \( K(t) \) is chosen such that \( \frac{1}{k_i(t)} \) is finite for all time and all \( i \), and satisfies (20), then the closed-loop dynamical system described by (23) and (26) is uniformly ultimately bounded.

(ii) If the gain function matrix \( K(t) \) is chosen such that the inequality (21) is satisfied, then the closed-loop dynamical system described by (23) and (26) is uniformly asymptotically stable.

**Proof:** Applying the state feedback controller (26) to (23) yields
\[
\frac{dx(t)}{dt} = \left( A - \frac{1}{2} BB^TP \right)x(t) + Bp_2(x(t), t) + E(x, t) 
\]  
(27)

For the nominal dynamical system
\[
\frac{dx(t)}{dt} = \left( A - \frac{1}{2} BB^TP \right)x(t) 
\]  
(28)

we define a scalar function as follows.
\[
V(x) = x^T(t)Px(t) 
\]

where \( P \) is the solution of the algebraic Riccati equation (25).

Since \( P \) is a symmetric positive definite matrix, by the Rayleigh principle, we have
\[
\lambda_{\min}(P)\|x\|^2 \leq V(x) \leq \lambda_{\max}(P)\|x\|^2 
\]  
(29)

where \( \lambda_{\min}(\cdot) \) and \( \lambda_{\max}(\cdot) \) denote the minimum and maximum eigenvalues of the matrix \( \cdot \), respectively.

Thus, for the nominal dynamical system described by (28), we have
\[
\frac{dV(x)}{dt} = -x^T(t)Qx(t) \leq -\lambda_{\max}(Q)\|x(t)\|^2 
\]  
(30)

It is clear from (29) and (30) that one can take the bounding functions \( \gamma_i(\cdot), \) \( i = 1, 2, 3 \), to be
\[
\gamma_1(r) := \lambda_{\min}(P)r^2 \quad \gamma_2(r) := \lambda_{\max}(P)r^2 \quad \gamma_3(r) := \lambda_{\max}(Q)r^2 
\]  
(31)

Here, notice that \( \lim_{r \to \infty} \inf \gamma_3(r) = \infty \). Therefore, by making use of the proof method similar to that of Theorem 3.1, from (31) we can obtain the results given in this corollary.

**4. Illustrative Examples**

To illustrate the utilization of our approach, we consider the following two numerical examples.

**Example 4.1** Here, consider uncertain nonlinear dynamical system described by
\[
\frac{dx(t)}{dt} = F(x) + G(x)[u(t) + E(x, t)] 
\]  
(32a)

where
\[
F(x) = \begin{bmatrix} -10x_1 - x_1x_2 + x_2 \\ 10 + x_2 \end{bmatrix} \quad G(x) = \begin{bmatrix} 1 \\ 10x_2(10 + x_2) \end{bmatrix} 
\]  
(32b, c)

Here, for illustrating our approach, we have employed a modified dynamical system whose original version is given in Ref. 9). Similarly to Ref. 9), we employ the following positive definite function as a Lyapunov function candidate for the nominal system of (32).
\[
V(x, t) = 2x_1^2 + 10x_1 + 3x_2 + \frac{x_2^2}{10 + x_2} 
\]  
(33)

From (33), we have
\[
V_x(x, t) = 2x_1 + \frac{x_2}{10 + x_2} + \frac{x_2^2}{(10 + x_2)^2} 
\]  
(34)

Therefore, from (32a) and (34) we have
\[
\frac{dV(x, t)}{dt} + \frac{dV(x, t)}{dt}F(x) = -4x_1^2 - 4\left( \frac{x_2}{10 + x_2} \right)^2 
\]  
(35)

It is obvious from (35) that the positive definite
function defined by (33) is a Lyapunov function for the nominal dynamical system described by (32).

Then, in the light of (19), the state feedback controller guaranteeing some types of stability of (32) can be represented by

\[ p_1(x,t) = -2k_1(t)\bar{\rho}(x,t) \left[ 2x_1 + \frac{x_3}{10+x_2} \right] \]  
\[ p_2(x,t) = -2k_2(t)\bar{\rho}(x,t) \left[ -\frac{10x_1}{(10+x_2)^2} + \frac{3x_2}{(10+x_2)^3} \right] \]

where \( \bar{\rho}(\cdot,\cdot), i=1,2, \) are known nonnegative functions satisfying the conditions

\[ |\xi_i(x,t)| \leq \rho_i(x,t) \]  
\[ |\xi_i(x,t)| \leq \rho_i(x,t) \]

and \( k_i(t), i=1,2, \) will be selected here such that the uniform asymptotic stability of (32) can be guaranteed.

For simulation, we give the following representations of uncertain function \( \xi(x,t) : \)

\[ \xi(x,t) = \begin{bmatrix} 0 \\ \frac{1}{2} x_1 x_2 \end{bmatrix} \]

It is obvious from (38) that

\[ p_1(x,t) = 0, \quad p_2(x,t) = \frac{1}{2}|x_1 x_2| \]

Since \( p_1(x,t) = 0, \) it is sufficient to only use the control (36 b). Here, we select

\[ k_2(t) = k_2 \exp{[\beta(t-t_0)]} \]

where \( k_2 \) and \( \beta \) are any positive constants.

It is obvious that \( k_2(t) \) defined in (40) satisfies condition (21). Therefore, the control composed of (36 b) and (40) stabilizes asymptotically the uncertain nonlinear dynamical system described by (32). The results of simulation of this example for \( k_2=0.5 \) and \( \beta=1.0,2.0,\cdots,8.0 \), are depicted in Fig. 1. In addition, the results of our simulation also show that the uncertain system (32) without the control (36) is unstable. It is shown from Fig. 1 that the closed-loop dynamical system described by (32) and (36) is indeed uniformly asymptotically stable. From Fig. 1 (a) and Fig. 1 (b), we can observe that for a larger \( \beta \), the system has a better convergence property, but as a trade-off, a larger control energy will be required (see Fig. 1 (c)).

Example 4.2 In this example, we consider an uncertain dynamical system with linear nominal part described by

\[ \frac{dx(t)}{dt} = \begin{bmatrix} 1.0 & 3.0 \\ 2.0 & 0.0 \end{bmatrix} x(t) \]

In the light of Corollary 3.1, we can develop a state feedback controller such that the asymptotic stability
and \( k_0(t) \) and \( k_0(t) \) will be selected as follows.

\[
k_0(t) = k_0(t) = 0.02 \exp \left( 0.5(t - k_0) \right)
\] (43d)

For simulation, we give the following representations of uncertain function \( \xi(x, t) \):

\[
\xi(x, t) = \begin{bmatrix} 4.0 \sin(x_2) \\ 0 \end{bmatrix}
\]

For the above representation, we have

\[
\rho(x, t) = 4.0, \quad \rho(x, t) = 0
\] (43e)

It is obvious from (43e) that the state feedback controller (43) is linear in the state \( x \). Therefore, the control described by (43) can rather easily be implemented. The results of simulation of this example are depicted in Fig. 2. It is known from Fig. 2 that the uncertain nonlinear system (41) is uniformly asymptotically stable under the linear state feedback controller (43).

5. Conclusion

The robust control problem of a class of uncertain nonlinear dynamical systems has been discussed. Based on the stabilizability of the nominal system, we have proposed an approach to synthesizing a class of continuous state feedback controllers with rather simpler structures which guarantee uniform ultimate boundedness and uniform asymptotic stability of uncertain nonlinear dynamical systems by choosing different gain functions of the control.

Moreover, the state feedback controller developed by our approach is computationally simpler, and can be decoupled with respect to possible uncertainties. For dynamical system with a linear nominal part and uncertainties bounded by constants, by the approach developed in this paper, we can also obtain an stabilizing linear state feedback controller. It is shown from two numerical examples that the approach developed in this paper is effective for a class of uncertain nonlinear dynamical systems, and can be expected to have some further applications to some practical robust control problems.

References


