A Synthesis for Robust Tracking Systems Based on H_∞ Control[†]

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In this paper, we consider a synthesis problem of a feedback compensator which achieves the following three design specifications: 1) closed-loop internal stability, 2) desired feedback characteristics such as robust stability, sensitivity reduction and disturbance attenuation, and 3) robust tracking. We show that the Glover-Doyle's solution on H_∞ control problem can be directly applied to the case where the integrated error is evaluated if we modify the problem appropriately. The properties of the Riccati equations to be solved and the order reduction of the resultant controller are also investigated. The proposed method is applied to the position control of a linear AC servo motor.

Key Words: robust tracking, H_∞ control

1. Introduction

In this paper, we consider a synthesis problem of a feedback compensator which achieves the following three design specifications: 1) closed-loop internal stability, 2) desired feedback characteristics such as robust stability, sensitivity reduction and disturbance attenuation, and 3) robust tracking.

The H_∞ control is a powerful tool to accomplish the first two specifications. It was shown that the third specification can be also expressed as a norm constraint. There are several approaches to design a servo controller based on the H_∞ control. Sugie et al.⁸⁾ developed a method using the free parameters to obtain a servo controller. The drawback of the method is its indirect specification for the low frequency characteristics. Augmented H_∞ control problems with integral type weighting functions have been

investigated in ^{4), 7), 9)}. They, however, are not solvable with usual technique such as Glover-Doyle's solution²⁾, since the related generalized plants are not detectable for the modes of reference inputs and it has a $j\omega$ -axis zeros caused by the integrator in the weighting function³⁾.

The purpose of this paper is to propose an approach to design an integral type controller based on the H_∞ control. To overcome the difficulty mentioned above, we transform the augmented problem to a modified problem with which we can apply the usual technique in Section 3. The method can be applied to any H_∞ problem and the usual package can be used without modification. In Section 4, we will show the properties of the Riccati equations to be solved and the order reduction of the resultant controller. We then apply the proposed method to the position control of a linear AC servo motor to verify the usefulness of the proposed method. The experimental results are shown in Section 5.

2. Problem Formulation

Consider a unity feedback control system shown in Fig. 1, where $P(s)$ and $K(s)$ denote the plant and controller, respectively. The purpose of the control system design is to find a feedback compensator

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Fig. 1 Feedback Control System

Fig. 2 Original Problem

$K(s)$ which satisfies the following three specifications: 1) closed-loop internal stability, 2) desired feedback characteristics such as robust stability, sensitivity reduction and disturbance attenuation, and 3) robust tracking. Here, the robust tracking means that the output of the plant $y(t)$ tracks for any step type reference command $r(t)$ without steady-state error, i.e., $\lim_{t \rightarrow \infty} e(t) = 0$, under the plant perturbation and/or step type disturbance inputs $d(t)$.

The integral type weighted sensitivity function $(I + \tilde{P}K)^{-1}/s$ must be stable for any perturbed plant $\tilde{P}$ for complete robust tracking, and it is well known that the integral type controller is required for the robust tracking. Also note that the smaller norm of the integral type weighted sensitivity function

$$\left\| \frac{W_1}{s} (I + PK)^{-1} W_2 \right\|_{\infty}$$

yields the better performance for the tracking, where $W_1(s)$ and $W_2(s)$ are appropriate unimodular weighting functions^{4),9)}.

It is well known that the $\mathcal{H}_{\infty}$ control is useful for the first two specifications with an appropriate generalized plant $G_o(s)$ having exogenous input w , controlled output z_o , control input u , and measured output y . Hence, we can formulate the synthesis problem

as an $\mathcal{H}_{\infty}$ control problem with an additional controlled variables $z_e = \frac{W_e(s)}{s} y$ as depicted in Fig. 2:

Robust Tracking $\mathcal{H}_{\infty}$ Control Problem:

Find an integral type controller $K(s)$ which satisfies

S1) $K(s)$ stabilizes $G_o(s)$.

S2)

$$\|T(s)\|_{\infty} = \left\| \begin{bmatrix} T_{wz_o}(s) \\ T_{wz_e}(s) \end{bmatrix} \right\|_{\infty} < \gamma$$

where $T_{wz_o}(s)$ denotes the transfer function from w to z_o and it is related to the second requirement $\|T_{wz_o}(s)\|_{\infty}$. The third specification, robust tracking, is reflected by $\|T_{wz_e}(s)\|_{\infty}$, where $T_{wz_e}(s)$ denotes the transfer function from w to z_e . The generalized plant $G_e(s)$ is expressed as

$$G_e(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ \frac{W_e(s)}{s} G_{21}(s) & \frac{W_e(s)}{s} G_{22}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \quad (1)$$

where

$$G_o(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \quad (2)$$

We, however, have a difficulty to solve the $\mathcal{H}_{\infty}$ control problem for G_e , since it does not satisfy the detectability from y , i.e., the state of $\frac{W_e(s)}{s}$ is not detectable. Thus, we can not directly apply the usual Glover-Doyle's solution²⁾ for the $\mathcal{H}_{\infty}$ control problem defined above.

3. Main Results

In order to overcome the difficulty in solving the $\mathcal{H}_{\infty}$ control problem for $G_e(s)$, we modify the original problem shown in Fig. 2 to an equivalent problem depicted in Fig. 3.

Proposition. There exists an integral type controller $K(s)$ satisfying S1 and S2, if and only if there exists a controller $\hat{K}(s)$ satisfying

S_m1) $\hat{K}(s)$ stabilizes $G_m(s)$.

S_m2)

$$\|T(s)\|_{\infty} = \left\| \begin{bmatrix} \hat{T}_{wz_o}(s) \\ \hat{T}_{wz_e}(s) \end{bmatrix} \right\|_{\infty} < \gamma$$

where

Fig. 3 Modified Problem

Fig. 4 Implementation of a Compensator

$$G_m(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ \frac{W_e(s)}{s} G_{21}(s) & \frac{W_e(s)}{s} G_{22}(s) \\ \frac{s+\alpha}{s} G_{21}(s) & \frac{s+\alpha}{s} G_{22}(s) \end{bmatrix} \quad (3)$$

and α is an arbitrary positive number. $\hat{T}_{wz_o}(s)$ and $T_{wz_e}(s)$ respectively denote the transfer function from w to z_o and z_e in Fig. 3.

Proof: It is readily seen from (1) and (3) that setting

$$K(s) = \frac{s+\alpha}{s} \hat{K}(s)$$

as shown in Fig. 4 yields the equivalence. Here, we note that any integral type controller $K(s)$ must have a form above and that any solution $\hat{K}(s)$ for $G_m(s)$ has no zeros at the origin for assuring the stability. ■

The greatest advantage for considering the modified problem instead of the original one is that we can directly apply the usual algorithms such as Glover-Doyle's solution²⁾ for any $\mathcal{H}_\infty$ problem including the

mixed sensitivity problem and LQG type formulation. In other words, no approximation technique such as $1/s \rightarrow 1/(s+\varepsilon)$, ($\varepsilon > 0$: small) and no specialized algorithm are required. In the rest of this section, we will show this property.

Let the state space realization of $G_o(s)$ be given by

$$G_o(s) = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{bmatrix} \quad (4)$$

where $A \in \mathbb{R}^{n \times n}$, $B_1 \in \mathbb{R}^{n \times m_1}$, $B_2 \in \mathbb{R}^{n \times m_2}$, $C_1 \in \mathbb{R}^{p_1 \times n}$, $C_2 \in \mathbb{R}^{p_2 \times n}$, $D_{12} \in \mathbb{R}^{p_1 \times m_2}$ and $D_{21} \in \mathbb{R}^{p_2 \times m_1}$. We make the following natural assumptions:

A1) The following six conditions in²⁾ are satisfied for the state-space model $G_o(s)$ given by (4):

C1) (A, B_2) : stabilizable

C2) $\text{rank}(D_{12}) = m_2$

C3) $\forall \omega \in \mathbb{R}$;

$$\text{rank} \left(\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix} \right) = n + m_2$$

O1) (A, C_2) : detectable

O2) $\text{rank}(D_{21}) = p_2$

O3) $\forall \omega \in \mathbb{R}$;

$$\text{rank} \left(\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix} \right) = n + p_2$$

A2) The plant $P(s)$ satisfies the robust tracking condition for the step type reference command, i.e.,

$$\text{rank} \left(\begin{bmatrix} A & B_2 \\ C_2 & 0 \end{bmatrix} \right) = n + p_2 \quad (5)$$

For simplicity, we also assume that $W_e(s)$ is a constant matrix of dimension $p_2 \times p_2$ with full rank and denoted by W_e . Then, $G_m(s)$ is realized as follows:

$$G_m(s) = \begin{bmatrix} \hat{A} & \hat{B}_1 & \hat{B}_2 \\ \hat{C}_1 & 0 & \hat{D}_{12} \\ \hat{C}_2 & \hat{D}_{21} & 0 \end{bmatrix} = \begin{bmatrix} A & 0 & B_1 & B_2 \\ C_2 & 0 & D_{21} & 0 \\ C_1 & 0 & 0 & D_{12} \\ 0 & W_e & 0 & 0 \\ C_2 & \alpha I_{p_2} & D_{21} & 0 \end{bmatrix} \quad (6)$$

The modified problem has the following property:

Property 1. Suppose that the assumptions **A1** and **A2** hold. Then the state-space model (6) for $G_m(s)$ satisfies the corresponding six assumptions in 2).

Proof:

Cm1) From **C1)**, for $\omega \neq 0$, we have

$$\text{rank} \left(\begin{bmatrix} A - j\omega I & 0 & B_2 \\ C_2 & -j\omega I & 0 \end{bmatrix} \right) = n + p_2$$

For $\omega = 0$, we see from (5) that

$$\begin{aligned} & \text{rank} \left(\begin{bmatrix} A - j\omega I & 0 & B_2 \\ C_2 & -j\omega I & 0 \end{bmatrix} \right) \\ &= \text{rank} \left(\begin{bmatrix} A & B_2 \\ C_2 & 0 \end{bmatrix} \right) = n + p_2 \end{aligned}$$

Hence, we conclude that $(\hat{A}, \hat{B}_2)$ is stabilizable.

Cm2)

$$\text{rank}(\hat{D}_{12}) = \text{rank}(D_{12}) = m_2$$

Cm3) $\forall \omega \in \mathbb{R}$;

$$\begin{aligned} & \text{rank} \left(\begin{bmatrix} A - j\omega I & 0 & B_2 \\ C_2 & -j\omega I & 0 \\ C_1 & 0 & D_{12} \\ 0 & W_e & 0 \end{bmatrix} \right) \\ & \geq \text{rank} \left(\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix} \right) + \text{rank}(W_e) \\ & = n + p_2 + m_2 \quad (\text{from C3}) \end{aligned}$$

Om1) We can readily see from **O1)** that

$$\text{rank} \left(\begin{bmatrix} A - sI & 0 \\ C_2 & -sI \\ C_2 & \alpha I \end{bmatrix} \right) = n + p_2$$

for $\text{Re}(s) \geq 0$. This clearly implies that $(\hat{A}, \hat{C}_2)$ is detectable.

Om2)

$$\text{rank}(\hat{D}_{21}) = \text{rank}(D_{21}) = p_2$$

Om3) $\forall \omega \in \mathbb{R}$

$$\begin{aligned} & \text{rank} \left(\begin{bmatrix} A - j\omega I & 0 & B_1 \\ C_2 & -j\omega I & D_{21} \\ C_2 & \alpha I_{p_2} & D_{21} \end{bmatrix} \right) \\ & \geq \text{rank} \left(\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix} \right) + \text{rank}(\alpha I_{p_2}) \\ & = n + 2p_2 \quad (\text{from O3}) \end{aligned}$$

4. Properties of the Modified Problem

In this section, we will discuss the properties of the corresponding Riccati equations to be solved and the resultant controllers.

Let us first focus on the properties, structure and α -independenceness, of the two Riccati equations, the control Riccati equation:

$$\begin{aligned} & \hat{A}^T \hat{X} + \hat{X} \hat{A} + \hat{C}_1^T \hat{C}_1 \\ & - (\hat{X} \hat{B} + \hat{C}_1^T \hat{D}_X) \hat{R}_X^{-1} (\hat{X} \hat{B} + \hat{C}_1 \hat{D}_X)^T = 0 \end{aligned} \quad (7)$$

and the estimation Riccati equation:

$$\begin{aligned} \hat{A}\hat{Y} + \hat{Y}\hat{A}^T + \hat{B}_1\hat{B}_1^T \\ - (\hat{Y}\hat{C}^T + \hat{B}_1\hat{D}_Y^T)\hat{R}_Y^{-1}(\hat{Y}\hat{C}^T + \hat{B}_1\hat{D}_Y^T)^T = 0 \end{aligned} \quad (8)$$

where

$$\begin{aligned} \hat{B} &= \begin{bmatrix} \hat{B}_1 & \hat{B}_2 \end{bmatrix}, \quad \hat{D}_X = \begin{bmatrix} 0 & \hat{D}_{12} \end{bmatrix} \\ \hat{R}_X &= \hat{D}_X^T \hat{D}_X - \begin{bmatrix} \gamma^2 I & 0 \\ 0 & 0 \end{bmatrix} \\ \hat{C}^T &= \begin{bmatrix} \hat{C}_1^T & \hat{C}_2^T \end{bmatrix}, \quad \hat{D}_Y^T = \begin{bmatrix} 0 & \hat{D}_{21}^T \end{bmatrix} \\ \hat{R}_Y &= \hat{D}_Y \hat{D}_Y^T - \begin{bmatrix} \gamma^2 I & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

Property 2. The stabilizing solutions $\hat{X}$ and $\hat{Y}$ of Riccati equations (7) and (8) are both independent of α , and $\hat{Y}$ is given by

$$\hat{Y} = \begin{bmatrix} Y & 0 \\ 0 & 0 \end{bmatrix} \quad (9)$$

where Y is the stabilizing solution of the following Riccati equation:

$$\begin{aligned} AY + YA^T + B_1 B_1^T \\ - (YC^T + B_1 D_Y^T)R_Y^{-1}(YC^T + B_1 D_Y^T)^T = 0 \end{aligned} \quad (10)$$

where

$$\begin{aligned} C^T &= \begin{bmatrix} C_1^T & C_2^T \end{bmatrix}, \quad D_Y^T = \begin{bmatrix} 0 & D_{21}^T \end{bmatrix} \\ R_Y &= D_Y D_Y^T - \begin{bmatrix} \gamma^2 I & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

Proof: It is trivial that $\hat{X}$ is independent of α , since all matrices in (7) are α -independent.

The eigenvalue problem related to the Riccati equations (8) and (10) are given by

$$\hat{H}_Y \begin{bmatrix} \hat{U} \\ \hat{V} \end{bmatrix} = \begin{bmatrix} \hat{U} \\ \hat{V} \end{bmatrix} \hat{\Lambda}, \quad H_Y \begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} U \\ V \end{bmatrix} \Lambda$$

respectively, where

$$\begin{aligned} \hat{H}_Y &:= \begin{bmatrix} \hat{A}^T & 0 \\ -\hat{B}_1 \hat{B}_1^T & -\hat{A} \end{bmatrix} - \begin{bmatrix} \hat{C}^T \\ -\hat{B}_1 \hat{D}_Y^T \end{bmatrix} \\ &\quad \times \hat{R}_Y^{-1} \begin{bmatrix} \hat{D}_Y \hat{B}_1^T & \hat{C} \end{bmatrix} \end{aligned}$$

$$\begin{aligned} H_Y &:= \begin{bmatrix} A^T & 0 \\ -B_1 B_1^T & -A \end{bmatrix} - \begin{bmatrix} C^T \\ -B_1 D_Y^T \end{bmatrix} \\ &\quad \times R_Y^{-1} \begin{bmatrix} D_Y B_1^T & C \end{bmatrix} \end{aligned} \quad (11)$$

and $\hat{\Lambda}$ and Λ are appropriate Jordan form matrices made up of the stable eigenvalues of $\hat{H}_Y$ and H_Y , respectively.

We can readily verify that $\hat{U}$, $\hat{V}$ and $\hat{\Lambda}$ have the following forms:

$$\begin{aligned} \hat{U} &= \begin{bmatrix} U & 0 \\ * & I \end{bmatrix}, \quad \hat{V} = \begin{bmatrix} V & 0 \\ 0 & 0 \end{bmatrix} \\ \hat{\Lambda} &= \begin{bmatrix} \Lambda & 0 \\ 0 & -\alpha I_{p_2} \end{bmatrix} \end{aligned}$$

Here, we note that $\alpha > 0$ guarantees the stability of $\hat{\Lambda}$. Hence, we have

$$\hat{Y} = \hat{V} \hat{U}^{-1} = \begin{bmatrix} VU^{-1} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} Y & 0 \\ 0 & 0 \end{bmatrix}$$

Note that the Riccati equation (10) is the same one appeared in the $\mathcal{H}_\infty$ problem for $G_o(s)$, and hence it is independent of α . ■

We next investigate the order of the resultant controller. Let $\hat{K}_c(s)$ denotes the central solution of the $\mathcal{H}_\infty$ control problem related to $G_m(s)$. Also, let $K_c(s)$ denotes the controller corresponding to $\hat{K}_c(s)$, i.e., $K_c(s) = \frac{s+\alpha}{s} \hat{K}_c(s)$. The order of $\hat{K}_c(s)$ is at most $n + p_2$, since the size of Riccati equations are $n + p_2$. Then, the order of $K_c(s)$ seems to be $n + 2p_2$ in general. Fortunately, we can show that the pole-zero cancelation at $s = -\alpha$ leads to the smaller order of the central controller $K_c(s)$. The order is reduced to $n + 2p_2 - p_2 = n + p_2$. The next property shows the stronger result that such an order reduction is possible for any $\mathcal{H}_\infty$ controller obtained from the modified problem.

Property 3. The zeros $-\alpha$ of $\frac{s+\alpha}{s} I$ are completely canceled by the poles of any $\hat{K}(s)$ in $\hat{\mathcal{K}}$, where $\hat{\mathcal{K}}$ denotes the class of $\mathcal{H}_\infty$ controllers for $G_m(s)$. Hence, there exists an $\mathcal{H}_\infty$ controller $K(s)$ satisfying S1) and S2) with order $\hat{n}$ if we have an $\mathcal{H}_\infty$ controller $\hat{K}(s) \in \hat{\mathcal{K}}$ with order $\hat{n}$.

Proof: We first note that $\hat{K}$ is expressed as

$\mathcal{F}_l(\hat{K}_k, U)^{(1)}$ with $\|U\|_\infty < \gamma$ and $\hat{K}_k(s)$ is given as follows, following to the standard Glover-Doyle's solution for the $\mathcal{H}_\infty$ control problem²⁾:

Let $\hat{X}$ and $\hat{Y}$ respectively denote the stabilizing solutions of the control and the estimation Riccati equations corresponding to $G_m(s)$. Property 2 states that $\hat{X}$ and $\hat{Y}$ are independent of α . Also $\hat{Y}$ is given by (9). We also partition $\hat{X}$ into

$$\hat{X} = \begin{bmatrix} \hat{X}_1 & \hat{X}_3 \\ \hat{X}_3^T & \hat{X}_2 \end{bmatrix}$$

conformably with the structure of $G_m(s)$.

Then $\hat{Z} := (I - \hat{Y}\hat{X})^{-1}$ has the structure

$$\hat{Z} = \begin{bmatrix} \hat{Z}_1 & \hat{Z}_2 \\ 0 & I \end{bmatrix}$$

$$\hat{Z}_1 := (I - Y\hat{X}_1)^{-1}, \quad \hat{Z}_2 := \hat{Z}_1 Y \hat{X}_3$$

Straightforward but lengthy calculations using the structure of $\hat{Z}$ lead to the following state-space form of $\hat{K}_k(s)$:

$$\hat{K}_k(s) = \left[\begin{array}{c|cc} \hat{A}_k & \hat{B}_{1k} & \hat{B}_{2k} \\ \hline \hat{C}_{1k} & 0 & R_{X2}^{-\frac{1}{2}} \\ \hat{C}_{2k} & R_{Y2}^{-\frac{1}{2}} & 0 \end{array} \right]$$

$$\hat{A}_k := \begin{bmatrix} A_{k11} & A_{k12} - \alpha B_{1k1} \\ 0 & -\alpha I_{p_2} \end{bmatrix}$$

$$\hat{B}_{1k} := \begin{bmatrix} B_{1k1} \\ I_{p_2} \end{bmatrix}, \quad \hat{B}_{2k} := \begin{bmatrix} B_{2k1} \\ 0 \end{bmatrix}$$

$$\hat{C}_{1k} := \begin{bmatrix} C_{1k1} & C_{1k2} \end{bmatrix}$$

$$\hat{C}_{2k} := \begin{bmatrix} C_{2k1} & C_{2k2} - \alpha R_{Y2}^{-\frac{1}{2}} \end{bmatrix}$$

$$R_{X2} := D_{12}^T D_{12}, \quad R_{Y2} := D_{21} D_{21}^T$$

where

$$\begin{aligned} A_{k11} &:= A + B_1 B_1^T \hat{X}_1 - B_2 R_{X2}^{-1} (B_2^T \hat{X}_1 + D_{21}^T C_1) \\ &\quad - \hat{Z}_1 (Y C_2^T + B_1 D_{21}^T) R_{Y2}^{-1} (D_{21} B_1^T \hat{X}_1 + C_2) \\ &\quad - \hat{Z}_1 Y (C_2^T \hat{X}_3^T + \hat{X}_1 B_1 D_{21}^T \hat{X}_3^T) \\ &\quad - \hat{Z}_2 (C_2 + D_{21} B_1^T \hat{X}_1 + R_{Y2} \hat{X}_3^T) \end{aligned}$$

(1) $\mathcal{F}_l(K, U)$ denotes $K_{11} + K_{12}U(I - K_{22}U)^{-1}K_{21}$.

$$\begin{aligned} \hat{A}_{k12} &:= (B_1 B_1^T - B_2 R_{X2}^{-1} B_2^T) \hat{X}_3 \\ &\quad - \hat{Z}_1 Y (D_{21} B_1^T \hat{X}_1 + C_2)^T \hat{X}_2 \\ &\quad - \hat{Z}_2 D_{21} (B_1^T \hat{X}_3^T + D_{21}^T \hat{X}_2) \\ &\quad - \hat{Z}_1 (Y C_2^T + B_1 D_{21}^T) R_{Y2}^{-1} D_{21} B_1^T \hat{X}_3^T \end{aligned}$$

$$B_{1k1} := (Y C_2^T + B_1 D_{21}^T) R_{Y2}^{-1} + \hat{Z}_2$$

$$B_{2k1} := \hat{Z}_1 (B_2^T + Y C_1^T D_{12}) R_{X2}^{-\frac{1}{2}}$$

$$C_{1k1} := -R_{X2}^{-1} (B_2^T \hat{X}_1 + D_{12}^T C_1)$$

$$C_{1k2} := -R_{X2}^{-1} B_2^T \hat{X}_3$$

$$C_{2k1} := -R_{Y2}^{-\frac{1}{2}} (D_{21} B_1^T \hat{X}_1 + C_2) - R_{Y2}^{\frac{1}{2}} \hat{X}_3^T$$

$$C_{2k2} := -R_{Y2}^{-\frac{1}{2}} D_{21} B_1^T \hat{X}_3^T - R_{Y2}^{\frac{1}{2}} \hat{X}_2$$

Note that all the matrices A_{k11} , A_{k12} , B_{1k1} , B_{2k1} , C_{1k1} , C_{1k2} , C_{2k1} and C_{2k2} are independent of α . Consequently, a class of $\mathcal{H}_\infty$ controller $K(s)$ for $G_e(s)$ is given by $\mathcal{F}_l(K_k, U)$, where $\|U\|_\infty < \gamma$ and $K_k(s)$ is written by

$$K_k(s) = \hat{K}_k(s) \begin{bmatrix} \frac{s+\alpha}{s} I_{p_2} & 0 \\ 0 & I_{m_2} \end{bmatrix}$$

Then, a realization of $K_k(s)$ has the form

$$K_k(s) = \left[\begin{array}{c|cc} A_k & B_{1k} & B_{2k} \\ \hline C_{1k} & 0 & R_{X2}^{-\frac{1}{2}} \\ C_{2k} & R_{Y2}^{-\frac{1}{2}} & 0 \end{array} \right]$$

$$A_k := \begin{bmatrix} A_{k11} & A_{k12} - \alpha B_{1k1} & \alpha B_{1k1} \\ 0 & -\alpha I_{p_2} & \alpha I_{p_2} \\ 0 & 0 & 0 \end{bmatrix}$$

$$B_{1k} := \begin{bmatrix} B_{1k1} \\ I_{p_2} \\ I_{p_2} \end{bmatrix}, \quad B_{2k} := \begin{bmatrix} B_{2k1} \\ 0 \\ 0 \end{bmatrix}$$

$$C_{1k} := \begin{bmatrix} C_{1k1} & C_{1k2} & 0 \end{bmatrix}$$

$$C_{2k} := \begin{bmatrix} C_{2k1} & C_{2k2} - \alpha R_{Y2}^{-\frac{1}{2}} & \alpha R_{Y2}^{-\frac{1}{2}} \end{bmatrix}$$

where again all the matrices in the right hand side are independent of α . It is easy to see that the realization is uncontrollable and that canceling out of p_2 modes at $s = -\alpha$ yields the reduced version of the realization

$$K_k(s) = \begin{bmatrix} A_{k11} & A_{k12} & B_{1k1} & B_{2k1} \\ 0 & 0 & I_{p_2} & 0 \\ \hline C_{1k1} & C_{1k2} & 0 & R_{X2}^{-\frac{1}{2}} \\ C_{2k1} & C_{2k2} & R_{Y2}^{-\frac{1}{2}} & 0 \end{bmatrix}$$

with order $n + p_2$. This completes the proof. ■

The above proof includes the following property on the independenceness with respect to α :

Property 4. The parameterization of all $\mathcal{H}_\infty$ controllers $K(s)$ for $G_e(s)$ is independent of the choice of α , which is given by $\mathcal{F}_l(K_k, U)$, where $\|U\|_\infty < \gamma$.

Remark 1. The results in Sections 3 and 4 ensure that we can find an integral type controller for the robust tracking $\mathcal{H}_\infty$ control problem according to the standard Glover-Doyle's solution, and hence we can obtain the controller by using any standard computational package.

Although the final controller after minimal realization does not depend on the value of α in theory, the following situations may happen due to the numerical computation: $\hat{Y}$ does not have the structure as in (9) and/or the order reduction of the resultant controller does not occur. Those situations occur for the fully extremely two cases: 1) α : too small or too large, 2) α : near to one of the eigenvalues of H_Y in (11). Hence, the reasonable choice of α giving the successful result is normally possible by avoiding the above two cases.

As an another choice, we may implement the α -independent parameterization of $K(s)$ found in Property 4.

5. Application to Linear AC Servo Motor

We apply the proposed design method for robust tracking to the position control of a linear AC servo motor^{1),6)}. The nominal model $P(s)$ of the linear AC servo motor is given by

$$P(s) = \frac{1}{s} \frac{K_p}{T_p s + 1} \quad (12)$$

Fig. 5 Parameter Perturbation with Covering Ellipse

or

$$P(s) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1/T_p & K_p/T_p \\ \hline 1 & 0 & 0 \end{bmatrix} \quad (13)$$

The parameters T_p and K_p are identified based on 28 step responses, and the data are given in¹⁾. Here, we consider the perturbation of $a = 1/T_p$ and $b = K_p/T_p$, and we estimate the uncertainty based on the data replotted in (a, b) parameter space (See Fig. 5). As an model of the parameter perturbation, we employ an elliptic region defined by

$$\begin{pmatrix} a & b \end{pmatrix} = \begin{pmatrix} a_0 & b_0 \end{pmatrix} + \Delta \begin{pmatrix} r_{aa} & r_{ab} \\ r_{ba} & r_{bb} \end{pmatrix}$$

where $(a_0 \ b_0)$ represents the center of the ellipse, and $\Delta \in \mathbb{R}^2$ is any row vector such that $\Delta \Delta^T \leq 1$ holds. If the ellipse covers all data, we can adopt the following generalized plant $G_o(s)$ for the robust stabilization against the parameter perturbation:

$$G_o(s) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -a_0 & 1 & b_0 \\ \hline 0 & r_{aa} & 0 & r_{ab} \\ 0 & r_{ba} & 0 & r_{bb} \\ \hline 1 & 0 & 0 & 0 \end{bmatrix} \quad (14)$$

In this paper, the above parameters are selected as

Fig. 6 Step Responses of Resultant Control System

$$\begin{aligned} a_0 &= 1.681 \times 10^2, & b_0 &= 8.040 \times 10^6 \\ r_{aa} &= -1.751 \times 10^1, & r_{ab} &= -3.884 \times 10^{-4} \\ r_{ba} &= -8.515 \times 10^1, & r_{bb} &= 3.839 \times 10^6 \end{aligned}$$

so that the corresponding ellipse covers all data. In addition, a constant weighting matrix W_e is used to tune the tracking performance.

The experimental results are illustrated in Fig. 6 with corresponding simulation results. We can confirm the efficiency of the weighting matrix W_e , i.e., the tracking error is converged to be zero more quickly with the larger W_e . Moreover, these results agree with the simulation results.

References

- 1) T. Asai and S. Hara, "Robust Servo System Design Based on Descriptor Form Representation with Structured Uncertainty," in Proc. of the 12th IFAC World Congress, pp. 43-46, 1993.
- 2) K. Glover and J.C. Doyle, "State-Space Formulae for All Stabilizing Controllers That Satisfy An H_∞ Norm Bound And Relations to Risk Sensitivity," Syst. Contr. Lett., vol. 11, pp. 167-172, 1988.
- 3) S. Hara, "Synthesis of Robust Servo System," in summer seminar '92 text, pp. 55-67, SICE, 1992. In Japanese.
- 4) S. Hosoe, F. Zhang, and M. Kono, "Synthesis of Servo Mechanism Problem via H_∞ Control," in Robust Control (Lecture Notes in Control and Information Sciences, vol. 183), pp. 154-161, Springer-Verlag, 1992.
- 5) J. Hozumi, H. Fujioka, and S. Hara, " H_∞ Control Synthesis for Robust Servo Systems," in Proc. of the 1st ASCC, vol. 2, pp. 33-36, 1994.
- 6) T. Kosugiyama, "Robust Servo System Design with H_2/H_∞ Criterion." Bachelor Thesis, Tokyo Institute of Technology, 1993. In Japanese.
- 7) K.Z. Liu and T. Mita, "Generalized H_∞ Control Theory," in Proc. Amer. Contr. Conf, pp. 2245-2249, 1992.
- 8) T. Sugie, M. Fujita, and S. Hara, " H_∞ -Suboptimal Controller Design of Robust Tracking Systems," in

Robust Control (Lecture Notes in Control and Information Sciences, vol. 183), pp. 162-169, Springer-Verlag, 1992.

- 9) F. Zhang, S. Hosoe, and M. Kono, "Synthesis of Robust Output Regulators via H_∞ -Control," in Proc. of the 13th SICE Symposium on Dynamical System Theory, pp. 5-10, 1991. In Japanese.

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