Bounds for the BMI Eigenvalue Problem[†]

— A Good Lower Bound and A Cheap Upper Bound —

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The Bilinear Matrix Inequality (BMI) eigenvalue problem is considered. Upper and Lower bounds of the BMI eigenvalue to combine with the BMI branch and bound algorithm are derived. The proposed lower bound is better in compare to existing one, and still computable via the LMI-optimization. While, the proposed upper bound is computationally cheap utilizing the lower bound optimizer. The worst case gap of the proposed bounds is characterized by the problem data. Numerical examples are also shown.

Key Words: BMI eigenvalue problem, branch and bound algorithm, global optimization

1. Introduction

Motivated by increasing computer power and sophisticated algorithms^{14)~16), 20)}, analysis and design of control systems via numerical optimization is getting widely spread^{1)~3), 10)}. Especially, it has been revealed that a number of control problems can be characterized as linear matrix inequalities (LMIs) over the past few years^{2), 4), 10)}. Since LMIs are now easy to handle with computer software^{4), 19)}, we can obtain a solution of LMI-reducible control problems. However, we sometimes encounter control problems those are at the moment irreducible to LMIs.

Safonov et al.¹⁸⁾ pointed out that the bilinear matrix inequality (BMI) is a simple but flexible framework for analysis and synthesis of control systems. It is clear that BMIs can capture a larger class of control problems in compare to LMIs, and hence is potentially a powerful tool. Several researchers have derived BMI-characterizations of their control problems those are not reduced to LMIs yet, and also proposed local^{5), 8), 9), 11)} and global^{23), 24)} algorithms for special classes of BMIs.

Unfortunately, checking the solvability of a BMI is NP-hard¹⁷⁾. Although this fact indicates that it is computationally difficult to solve BMIs, it does not eliminates the importance of BMIs at all. Goh et al.^{6), 7)} formulated a general class of BMIs as the BMI eigenvalue problem, and investigated its computational issues. Furthermore, a global algorithm⁶⁾ with upper and lower bounds was proposed based on the branch and bound algorithm.

In this paper, we will propose new bounds for the BMI branch and bound algorithm. The proposed lower bound is better in compare to existing one, and still computable via the LMI-optimization. While, the proposed upper bound utilizes the lower bound optimizer, and is computationally cheap. We will also show the worst case gap of proposed bounds. Several numerical examples will be given to show the effectiveness of the BMI branch and bound algorithm with proposed bounds for the BMI eigenvalue problem.

2. BMI Eigenvalue Problem⁶⁾

2.1 Problem Formulation

Let $F: \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \rightarrow \mathbb{R}^{m \times m}$ be a biaffine function:

$$F(x, y) := F_{00} + \sum_{i=1}^{n_x} x_i F_{i0} + \sum_{j=1}^{n_y} y_j F_{0j} + \sum_{i=1}^{n_x} \sum_{j=1}^{n_y} x_i y_j F_{ij}$$

where $F_{ij} = F_{ij}^T \in \mathbb{R}^{m \times m}$ are symmetric matrices. Let also $\mathcal{Q}_D \subset \mathbb{R}^{n_x} \times \mathbb{R}^{n_y}$ be a closed bounded hyperrectangle:

$$\mathcal{Q}_D := [B_{x_1}^L, B_{x_1}^U] \times \cdots \times [B_{x_{n_x}}^L, B_{x_{n_x}}^U] \\ \times [B_{y_1}^L, B_{y_1}^U] \times \cdots \times [B_{y_{n_y}}^L, B_{y_{n_y}}^U]$$

where $-\infty < B_{x_i}^L \leq B_{x_i}^U < \infty$, $-\infty < B_{y_j}^L \leq B_{y_j}^U < \infty$.

Now we are ready to formulate the problem:

Definition (BMI Eigenvalue Problem⁶⁾). For given F and $\mathcal{Q}_D$, compute

$$\Phi(\mathcal{Q}_D) := \min_{(x, y) \in \mathcal{Q}_D} \bar{\lambda}\{F(x, y)\} \quad (1)$$

where $\bar{\lambda}\{\cdot\}$ denotes the greatest eigenvalue.

2.2 BMI Branch and Bound Algorithm

Goh et al.⁶⁾ proposed a branch and bound algorithm for the BMI eigenvalue problem, and derived a condition on which it terminates in finite time:

Lemma 1.⁶⁾ Given F and $\mathcal{Q}_D$, suppose that two functions Φ_L and Φ_U satisfy the following two conditions:

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C1. $\forall \mathcal{Q} \subset \mathcal{Q}_D, \Phi_L(\mathcal{Q}) \leq \Phi(\mathcal{Q}) \leq \Phi_U(\mathcal{Q})$

C2. $\forall \epsilon > 0, \exists \delta > 0$ such that

$$\forall \mathcal{Q} \subset \mathcal{Q}_D, \text{Size}(\mathcal{Q}) < \delta \Rightarrow \Phi_U(\mathcal{Q}) - \Phi_L(\mathcal{Q}) < \epsilon$$

where $\text{Size}(\mathcal{Q})$ denotes the length of the longest side of the hyper-rectangle $\mathcal{Q}$.

Then, $\Phi(\mathcal{Q}_D)$ is obtained in finite time within a given tolerance $\epsilon > 0$ by applying the following algorithm.

Algorithm 1 (BMI Branch and Bound Algorithm⁶⁾).

Given $F, \mathcal{P}, \Phi_L, \Phi_U$, and $\epsilon > 0$.

I. Set $k \leftarrow 0, \mathcal{Q}_0 \leftarrow \mathcal{Q}_D, \mathcal{S}_0 \leftarrow \{\mathcal{Q}_0\}, L_0 \leftarrow \Phi_L(\mathcal{Q}_0),$

$U_0 \leftarrow \Phi_U(\mathcal{Q}_0).$

II. Repeat {

R1. Select $\bar{\mathcal{Q}}$ from $\mathcal{S}_k$ such that $L_k = \Phi_L(\bar{\mathcal{Q}}).$

$\mathcal{S}_{k+1} \leftarrow \mathcal{S}_k \setminus \{\bar{\mathcal{Q}}\}.$

R2. Split $\bar{\mathcal{Q}}$ along its longest edge into $\bar{\mathcal{Q}}_1$ and $\bar{\mathcal{Q}}_2.$

R3. For $i = 1, 2,$

If $\Phi_L(\bar{\mathcal{Q}}_i) \leq U_k, \mathcal{S}_{k+1} \leftarrow \mathcal{S}_{k+1} \cup \{\bar{\mathcal{Q}}_i\}.$

R4. $U_{k+1} \leftarrow \min_{\mathcal{Q} \in \mathcal{S}_{k+1}} \Phi_U(\mathcal{Q}).$

R5. Pruning: $\mathcal{S}_{k+1} \leftarrow \mathcal{S}_{k+1} \setminus \{\mathcal{Q} : \Phi_L(\mathcal{Q}) > U_{k+1}\}.$

R6. $L_{k+1} \leftarrow \min_{\mathcal{Q} \in \mathcal{S}_{k+1}} \Phi_L(\mathcal{Q}).$

R7. $k \leftarrow k + 1.$

} until $U_k - L_k < \epsilon.$

Goh et al.⁶⁾ also proposed functions those can be used as Φ_L and Φ_U . In the following sections, we will not modify Algorithm 1, but propose new functions those can be used as Φ_L and Φ_U . They will improve the performance of Algorithm 1 in compare to those in Reference 6).

3. Upper and Lower Bounds

3.1 Lower Bound

In this subsection, we will propose a new function which can be used as a Φ_L .

Let us define an affine function $F_L: \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_x \times n_y} \rightarrow \mathbb{R}^{m \times m}$ related to F by

$$F_L(x, y, W) := F_{00} + \sum_{i=1}^{n_x} x_i F_{i0} + \sum_{j=1}^{n_y} y_j F_{0j} + \sum_{i=1}^{n_x} \sum_{j=1}^{n_y} w_{ij} F_{ij}$$

and consider $\mathcal{Q} \subset \mathcal{Q}_D$ of the form

$$\mathcal{Q} := [L_{x_1}, U_{x_1}] \times \cdots \times [L_{x_{n_x}}, U_{x_{n_x}}] \\ \times [L_{y_1}, U_{y_1}] \times \cdots \times [L_{y_{n_y}}, U_{y_{n_y}}]$$

Goh et al.⁶⁾ proposed Φ_{LG} defined by

$$\Phi_{LG}(\mathcal{Q}) := \min_{(x, y, W) \in \mathcal{P}_G(\mathcal{Q})} \bar{\lambda} \{F_L(x, y, W)\} \quad (2)$$

to use as Φ_L , where

$$\mathcal{P}_G(\mathcal{Q}) := \mathcal{Q} \times \{W \in \mathbb{R}^{n_x \times n_y} : w_{ij} \in [L_{w_{ij}}, U_{w_{ij}}]\}$$

$$L_{w_{ij}} := \min \{L_{x_i} L_{y_j}, L_{x_i} U_{y_j}, U_{x_i} L_{y_j}, U_{x_i} U_{y_j}\}$$

Fig. 1 $\mathcal{P}_G(\mathcal{Q})$ and $w = xy$

Fig. 2 $\mathcal{P}_H(\mathcal{Q})$ and $w = xy$

$$U_{w_{ij}} := \max \{L_{x_i} L_{y_j}, L_{x_i} U_{y_j}, U_{x_i} L_{y_j}, U_{x_i} U_{y_j}\}$$

Since F_L is an affine function and $\mathcal{P}_G(\mathcal{Q})$ is an affine set, we can compute $\Phi_{LG}(\mathcal{Q})$ via the LMI-optimization.

Our idea for the reduction of the conservativeness in Φ_{LG} is fairly simple: just replace $\mathcal{P}_G(\mathcal{Q})$ by a *smaller* affine set $\mathcal{P}_H(\mathcal{Q}) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_x \times n_y}$ defined by

$$\mathcal{P}_H(\mathcal{Q}) := \{(x, y, W) : (x_i, y_j, w_{ij}) \in \text{hull}(\mathcal{V}_{ij})\} \quad (3)$$

where $\text{hull}(\cdot)$ denotes the convex hull and $\mathcal{V}_{ij} \subset \mathbb{R}^3$ is defined by

$$\mathcal{V}_{ij} := \{(L_{x_i}, L_{y_j}, L_{x_i} L_{y_j}), (L_{x_i}, U_{y_j}, L_{x_i} U_{y_j}), \\ (U_{x_i}, L_{y_j}, U_{x_i} L_{y_j}), (U_{x_i}, U_{y_j}, U_{x_i} U_{y_j})\}$$

Figs. 1, 2 respectively show $\mathcal{P}_G(\mathcal{Q})$ and $\mathcal{P}_H(\mathcal{Q})$ for the case $n_x = n_y = 1$ and $\mathcal{Q} = [-0.5, 0.5] \times [-0.5, 0.5]$. We see that the cuboid $\mathcal{P}_G(\mathcal{Q})$ is *larger* than the tetrahedron $\mathcal{P}_H(\mathcal{Q})$ although both of them contain the face of $w = xy$ for $(x, y) \in \mathcal{Q}$. Now we define Φ_{LH} by

$$\Phi_{LH}(\mathcal{Q}) := \min_{(x, y, W) \in \mathcal{P}_H(\mathcal{Q})} \bar{\lambda} \{F_L(x, y, W)\} \quad (4)$$

The following result is then obtained:

Property 1. One has

$$\Phi_{LG}(\mathcal{Q}) \leq \Phi_{LH}(\mathcal{Q}) \leq \Phi(\mathcal{Q}) \quad (5)$$

Proof. It is obvious that we have

$$\Phi(\mathcal{Q}) = \min_{(x, y, W) \in \mathcal{P}(\mathcal{Q})} \bar{\lambda} \{F_L(x, y, W)\}$$

where $\mathcal{P}(\mathcal{Q})$ is defined by

$$\mathcal{P}(\mathcal{Q}) := \{(x, y, W) : (x, y) \in \mathcal{Q}, W = xy^T\}$$

For a given $(x, y, W) \in \mathcal{P}(\mathcal{Q})$, we have

$$\forall i, j, \exists \alpha_x \in [0, 1], \alpha_y \in [0, 1], \text{ such that}$$

$$x_i = \alpha_x L_{x_i} + (1 - \alpha_x) U_{x_i}, y_j = \alpha_y L_{y_j} + (1 - \alpha_y) U_{y_j}$$

by definition of $\mathcal{P}(\mathcal{Q})$. Using α_x and α_y , we also have

$$w_{ij} = \alpha_x \alpha_y L_{x_i} L_{y_j} + \alpha_x (1 - \alpha_y) L_{x_i} U_{y_j} \\ + (1 - \alpha_x) \alpha_y U_{x_i} L_{y_j} + (1 - \alpha_x)(1 - \alpha_y) U_{x_i} U_{y_j}$$

Noting that $\alpha_x \alpha_y \in [0, 1]$, $\alpha_x (1 - \alpha_y) \in [0, 1]$, $(1 - \alpha_x) \alpha_y \in [0, 1]$, $(1 - \alpha_x)(1 - \alpha_y) \in [0, 1]$, and

$$\alpha_x \alpha_y + \alpha_x (1 - \alpha_y) + (1 - \alpha_x) \alpha_y + (1 - \alpha_x)(1 - \alpha_y) = 1$$

we have $\mathcal{P}(\mathcal{Q}) \subset \mathcal{P}_H(\mathcal{Q})$, and the right inequality follows. On the other hand, the left inequality follows the fact $\mathcal{P}_H(\mathcal{Q}) \subset \mathcal{P}_G(\mathcal{Q})$. ■

Remark 1. Since $\text{hull}(\mathcal{V}_{ij})$ is an affine set, $\Phi_{L_H}(\mathcal{Q})$ is computable via the LMI-optimization. Actually, $\mathcal{P}_H(\mathcal{Q})$ is characterized by $4n_x n_y$ (affine and scalar) inequalities, while $\mathcal{P}_G(\mathcal{Q})$ by $2(n_x + n_y + n_x n_y)$ inequalities.

Remark 2. Recently, Yada et al.²²⁾ also proposed an improved lower bound by replacing $\mathcal{P}_G(\mathcal{Q})$. $\mathcal{P}_H(\mathcal{Q})$ for the case of, e. g., $n_x = n_y = 1$ and $\mathcal{Q} = [0, 1] \times [0, 1]$ is tighter than their alternative set to $\mathcal{P}_G(\mathcal{Q})$. Hence, Φ_{L_H} provides a better lower bound in compare to their one.

Remark 3. The following lower bound of $\Phi(\mathcal{Q})$ is computable via the convex programming:

$$\Phi_{L_C}(\mathcal{Q}) := \min_{(x, y, W) \in \mathcal{P}_C(\mathcal{Q})} \bar{\lambda} \{F_L(x, y, W)\} \quad (6)$$

where $\mathcal{P}_C(\mathcal{Q})$ is defined by

$$\mathcal{P}_C(\mathcal{Q}) := \text{hull}(\{(x, y, w) : (x, y) \in \mathcal{Q}, W = xy^T\})$$

and we readily have

$$\Phi_{L_H}(\mathcal{Q}) \leq \Phi_{L_C}(\mathcal{Q}) \leq \Phi(\mathcal{Q})$$

However, algorithms applicable to the general convex programming, such as the ellipsoid method, do not achieve good performances in compare to the interior point method in the LMI-optimization.

3.2 Upper Bound

It is in fact practical to compute a good upper bound of $\Phi(\mathcal{Q}_D)$. Goh et al.⁷⁾ proposed the method of centers. Algorithms for special classes of BMIs are found in literature (e. g. 8), (9), (11)).

In this subsection, we also propose an upper bound, which may not be good but computationally cheap: Let us define a function Φ_{U_H} by

$$\Phi_{U_H}(\mathcal{Q}) := \bar{\lambda} \{F(x^*, y^*)\} \quad (7)$$

where

$$(x^*, y^*, W^*) := \arg \min_{(x, y, W) \in \mathcal{P}_H(\mathcal{Q})} \bar{\lambda} \{F_L(x, y, W)\}$$

$\Phi_{U_H}(\mathcal{Q})$ is obviously an upper bound of $\Phi(\mathcal{Q})$.

Remark 4. Once we compute $\Phi_{L_H}(\mathcal{Q})$, $\Phi_{U_H}(\mathcal{Q})$ is obtained as just the greatest eigenvalue of a fixed matrix $F(x^*, y^*)$. This implies that Φ_{U_H} is computationally cheap when combined with Φ_{L_H} .

4. Worst Case Analysis

In this section, we discuss the worst case gap of proposed bounds.

Let us begin with the following lemma:

Lemma 2. Define a set $\mathcal{V} \subset \mathbb{R}^3$ by

$$\mathcal{V} := \{v_1, v_2, v_3, v_4\}$$

$$v_1 := (x_1, y_1, x_1 y_1), \quad v_2 := (x_1, y_2, x_1 y_2)$$

$$v_3 := (x_2, y_1, x_2 y_1), \quad v_4 := (x_2, y_2, x_1 y_2)$$

One has

$$\max_{(x, y, w) \in \text{hull}(\mathcal{V})} |xy - w| \leq \frac{1}{4} |(x_1 - x_2)(y_1 - y_2)| \quad (8)$$

Furthermore, the equality holds when

$$x = \frac{x_1 + x_2}{2}, \quad y = \frac{y_1 + y_2}{2}$$

Proof. For a given $(x, y, w) \in \text{hull}(\mathcal{V})$, there exists a vector $\alpha \in \mathcal{A} \subset \mathbb{R}^4$ such that

$$(x, y, w) = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \alpha_4 v_4$$

where $\mathcal{A}$ is defined by

$$\mathcal{A} := \{\alpha : \alpha_i \in [0, 1], \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1\}$$

Then we have

$$xy - w = ((\alpha_1 + \alpha_2)(\alpha_2 + \alpha_3) - \alpha_1)x_1 y_1 \\ + ((\alpha_1 + \alpha_2)(\alpha_2 + \alpha_4) - \alpha_2)x_1 y_2 \\ + ((\alpha_3 + \alpha_4)(\alpha_1 + \alpha_3) - \alpha_3)x_2 y_1 \\ + ((\alpha_3 + \alpha_4)(\alpha_2 + \alpha_4) - \alpha_4)x_2 y_2 \quad (9)$$

From the proof of Property 1, $(x, y, xy) \in \text{hull}(\mathcal{V})$, and hence the maximum of $|xy - w|$ is achieved on the surface of $\text{hull}(\mathcal{V})$. Therefore, we substitute $\alpha_i = 0$ for each i to (9), and obtain the following:

Case 1: $\alpha_1 = 0$ or $\alpha_4 = 0$

$$|xy - w| \leq \max_{\substack{\alpha_2 \in [0, 1], \alpha_3 \in [0, 1] \\ \alpha_2 + \alpha_3 \in [0, 1]}} \alpha_2 \alpha_3 |(x_1 - x_2)(y_1 - y_2)|$$

Case 2: $\alpha_2 = 0$ or $\alpha_3 = 0$

$$|xy - w| \leq \max_{\substack{\alpha_1 \in [0, 1], \alpha_4 \in [0, 1] \\ \alpha_1 + \alpha_4 \in [0, 1]}} \alpha_1 \alpha_4 |(x_1 - x_2)(y_1 - y_2)|$$

Hence, the maximum is achieved when

$$\alpha = \left(0, \frac{1}{2}, \frac{1}{2}, 0\right) \quad \text{or} \quad \alpha = \left(\frac{1}{2}, 0, 0, \frac{1}{2}\right)$$

and (8) follows. This completes the proof. ■

Now we characterize the worst case gap of proposed bounds by problem data:

Theorem 1. Define $\Delta(\mathcal{Q})$ by

$$\Delta(\mathcal{Q}) := \Phi_{U_H}(\mathcal{Q}) - \Phi_{L_H}(\mathcal{Q})$$

Then one has

$$\Delta(\mathcal{Q}) \leq \frac{1}{4} \sum_{i=1}^{n_x} \sum_{j=1}^{n_y} \tilde{\lambda}_{ij} (U_{x_i} - L_{x_i})(U_{y_j} - L_{y_j}) \quad (10)$$

where

$$\tilde{\lambda}_{ij} := \max \{ \bar{\lambda} \{F_{ij}\}, \bar{\lambda} \{-F_{ij}\} \}$$

Proof. By definition, we have

$$\Delta(\mathcal{Q}) = \bar{\lambda} \{F(x^*, y^*)\} - \bar{\lambda} \{F_L(x^*, y^*, W^*)\}$$

Noting that $\bar{\lambda}(A+B) \leq \bar{\lambda}(A) + \bar{\lambda}(B)$ when both A and B are symmetric,

$$\Delta(\mathcal{Q}) \leq \bar{\lambda} \left\{ \sum_{i=1}^{n_x} \sum_{j=1}^{n_y} (x_i^* y_j^* - w_{ij}^*) F_{ij} \right\}$$

$$\Delta(\mathcal{Q}) \leq \sum_{i=1}^{n_x} \sum_{j=1}^{n_y} \bar{\lambda} \{ (x_i^* y_j^* - w_{ij}^*) F_{ij} \}$$

Invoking Lemma 2, (10) follows. ■

Remark 5. Theorem 1 implies that the upper bound of $\Delta(\mathcal{Q})$ (RHS of (10)) becomes its one quarter when

$$\forall i, j; \begin{cases} U_{x_i} - L_{x_i} \rightarrow \frac{1}{2}(U_{x_i} - L_{x_i}) \\ U_{y_j} - L_{y_j} \rightarrow \frac{1}{2}(U_{y_j} - L_{y_j}) \end{cases} \quad (11)$$

Hence, it is easy to grid the entire domain to achieve a given tolerance. However, gridding the entire domain may not be *practical*. Note that $\Delta(\mathcal{Q})$ depends on (x^*, y^*) and it may not become its one quarter even in the case of (11).

The next theorem guarantees that Algorithm 1 with Φ_{U_H} and Φ_{L_H} terminates in finite time:

Theorem 2. Φ_{U_H} and Φ_{L_H} fulfill conditions C1 and C2 in Lemma 1.

Proof. From Property 1, it is obvious that Φ_{U_H} and Φ_{L_H} satisfy condition C1. From Theorem 1 and the definition of $\text{Size}(\cdot)$, we have

$$\Delta(\mathcal{Q}) \leq \frac{1}{4} (\text{Size}(\mathcal{Q}))^2 \cdot \sum_{i=1}^{n_x} \sum_{j=1}^{n_y} \tilde{\lambda}_{i,j}$$

Hence $\Delta(\mathcal{Q}) < \varepsilon$ when

$$\text{Size}(\mathcal{Q}) < 2 \cdot \sqrt{\frac{\varepsilon}{\sum_{i=1}^{n_x} \sum_{j=1}^{n_y} \tilde{\lambda}_{i,j}}}$$

This completes the proof. ■

5. Numerical Examples

In this section, we demonstrate the effectiveness of the BMI branch and bound algorithm with proposed bounds. The algorithm is implemented on Xmath²¹⁾ combined with SP¹⁹⁾.

5.1 Demonstrative Example

In this subsection, we consider a BMI⁶⁾ defined by

$$F(x, y) = F_{00} + F_{10}x + F_{01}y + F_{11}xy$$

Fig. 3 Progress of Upper (U_k) and Lower (L_k) Bounds

Fig. 4 Comparisons of Performances of Bounds

$$\begin{bmatrix} F_{00} & F_{10} \\ F_{01} & F_{11} \end{bmatrix} = \begin{bmatrix} -10 & -0.5 & -2 & 9 & 0.5 & 0 \\ -0.5 & 4.5 & 0 & 0.5 & 0 & -3 \\ -2 & 0 & 0 & 0 & -3 & -1 \\ -1.8 & -0.1 & -0.4 & 0 & 0 & -2 \\ -0.1 & 1.2 & -1 & 0 & -5.5 & 3 \\ -0.4 & -1 & 0 & 2 & 3 & 0 \end{bmatrix}$$

and $\mathcal{Q}_D = [-0.5, 2] \times [-3, 7]$.

We apply the BMI branch and bound algorithm with proposed bounds and $\varepsilon = 10^{-5}$ to the problem. **Fig. 3** shows the progress of the algorithm with proposed bounds. After 14 iterations, $(\text{gap}) < 10^{-5}$ is satisfied. It should be noted that the pruning step in the algorithm works well for this problem. Actually, only 2 branches appear simultaneously, i. e., S_k has 2 elements at most, until the tolerance is satisfied.

To evaluate the performance of Φ_{U_H} , we apply Algorithm 1 replacing U_k 's by the optimal value. Lower bounds L_k 's are computed using Φ_{L_H} . The results are plotted in **Fig. 4**: The curves labeled by $U_k - L_k$ and (I) indicate the gap of bounds related to Φ_{U_H} and Φ_{L_H} , and that between lower bounds and the optimal value, respectively. We see that the same accuracy is obtained when $k \geq 7$, and hence, in this case, we may conclude that a cheap upper bound Φ_{U_H} is sufficiently good.

Next, let us compare the performances of the lower bounds related to Φ_{L_H} and Φ_{L_G} . For the purpose, we apply Algorithm 1 with $U_0 = (\text{the optimal value})$ and $\Phi_L = \Phi_{L_G}$. The result is plotted as (II) in Fig. 4. With lower bounds derived by Φ_{L_G} , the tolerance $\varepsilon = 10^{-3}$ is

Fig. 5 Error vs. No. of Branches

not satisfied after 15 iterations even if we employ the optimal value as an upper bound. Actually, in the case, we need more than 800 iterations and more than 600 branches to satisfy $\varepsilon = 10^{-5}$. Note that the difference of curves (I) and (II) comes from that of lower bounds. The results implies that Φ_{LH} fairly improves the performance of Algorithm 1.

5.2 Randomly Generated BMI Problems

In this subsection, we compare proposed bounds and existing ones⁶⁾ by solving randomly generated BMIs.

We have generated more than 100 BMIs for $m = 3, 4, 5$, $n_x < 10$, $n_y < 10$. With proposed bounds, $(\text{gap}) < 10^{-5}$ was achieved by applying Algorithm 1 until $k = 50$ for about 70% problems, while it was not achieved with existing bounds⁶⁾.

For the 30% harder problems, pairs of the gap and the number of branches at $k = 50$ are plotted in Fig. 5. We see that smaller gap is achieved with less number of branches with proposed bounds.

5.3 Stabilization of Double Integrator

Fixed order controller design is a typical application of the BMI framework. In this subsection, we consider the problem to find a first order controller which stabilizes a double integrator.

The problem is easily formulated as a BMI: Find $X \in \mathbb{R}^{3 \times 3}$, $X = X^T > 0$ and $K \in \mathbb{R}^{2 \times 3}$ satisfying

$$(A + BKC)^T X + X(A + BKC) < 0 \quad (12)$$

where A , B and C are defined by

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} := \begin{bmatrix} 0 & 1 & 0 & | & 0 & 0 \\ 0 & 0 & 0 & | & 1 & 0 \\ 0 & 0 & 0 & | & 0 & 1 \\ \hline 1 & 0 & 0 & | & 0 & 0 \\ 0 & 0 & 1 & | & 0 & 0 \end{bmatrix}$$

Iwasaki and Skelton pointed out that the XY-centering algorithm¹¹⁾, a type of local optimization algorithm, works well for the problem. Applying the XY-centering

Fig. 6 Progress of Gap Varying Domain

Fig. 7 Error vs. No. of Branches Varying Domain

algorithm, we actually obtain a solution of (12):

$$X = X_0 := \begin{bmatrix} 1.1458 & -0.6910 & -1.6325 \\ -0.6910 & 1.7966 & 2.0712 \\ -1.6325 & 2.0712 & 3.6253 \end{bmatrix} \quad (13)$$

$$K = K_0 := \begin{bmatrix} -2.5695 & 3.5950 \\ 1.2955 & -2.1061 \end{bmatrix} \quad (14)$$

where X_0 is found in 11). With a fixed $X = X_0$, K_0 is obtained by solving (12) as an LMI in K .

Now we solve the BMI (12) as a BMI eigenvalue problem, i. e., we will optimize the greatest eigenvalue of the left hand side of (12), where the domain of X and K will be set around (13) and (14). Note that the pair of (13) and (14) is not the optimizer although it is a feasible solution of (12).

The results are shown in Figs. 6, 7, where the domain is determined by, e. g., $X \in [0.9X_0, 1.1X_0]$ and $K \in [0.9K_0, 1.1K_0]$ (element wise) for the case of $\pm 10\%$. When the domain is the case of $\pm 10\%$, we need less than 100 steps and less than 15 branches to achieve $(\text{gap}) < 10^{-4}$. While, if the domain is the case of $\pm 50\%$, the gap is not less than 10^{-2} even at $k = 800$ having more than 200 branches.

Through this example, we see that the BMI problem becomes much harder when its domain gets larger.

6. Conclusions

We have proposed upper and lower bounds for the BMI eigenvalue problem. The proposed lower bound is computable via the LMI-optimization, and better than existing one⁶⁾ by considering the geometry of the problem. The proposed upper bound utilizes the lower bound optimizer, and is computationally cheap. We also have derived the worst case gap of proposed bounds.

Through numerical examples, we have compared proposed bounds and existing ones⁶⁾, and have shown the effectiveness of the BMI branch and bound algorithm combined with proposed bounds.

Although proposed bounds improve the performance of the BMI branch and bound algorithm, it is not sufficiently good especially when its domain is large, as pointed out in Section 5. Hence, further enhancement is still required. An approach with parallel computation¹³⁾ and a new branching scheme¹²⁾ have been reported very recently. Better and cheaper bounds may be found for a certain control problem.

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