Parametrization of Discrete-Time H_∞ Filters Based on Model Matching[†]

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This paper is concerned with a state-space solution to the H_∞ filtering problem for linear discrete-time systems. We first reduce the H_∞ filtering problem to a model matching problem (MMP), and then employ a (J, J') -spectral factorization technique for solving the MMP. By this approach, we obtain a complete parametrization of all H_∞ filters. Using the chain scattering representation, the structure of the H_∞ filtering problem is also studied. Furthermore, we present a solution to the H_∞ prediction problem as a special case of the filtering problem.

Key Words: discrete-time H_∞ filtering, parametrization, model matching problem, chain scattering representation, H_∞ prediction

1. Introduction

This paper considers the state-space solution to the H_∞ filtering problem for linear discrete-time systems. The state estimation with H_∞ criterion is appropriate when the disturbance has significant uncertainty in its spectral density. As well as the continuous-time case, the discrete-time H_∞ filtering problem has been considered from various points of view. Yaesh and Shaked derived an H_∞ filter and an H_∞ predictor based on the bounded real lemma (BRL) under the assumption that the state estimator has a so-called full-order observer structure¹⁾. Also, game theoretic approach was employed for solving the H_∞ filtering and prediction problems, whose cost functions are different from each other²⁾. It may be also noted that parametrizations of all H_∞ filters and H_∞ predictors have been derived based on the orthogonal projection in Krein space for the case where the process disturbance and the measurement noise are mutually independent³⁾.

In this paper, we derive a complete parametrization of all H_∞ filters based on a model matching approach. We reduce the H_∞ filtering problem to a model matching problem (MMP) using the parametrization of all stable unbiased filters⁴⁾, and then solve the MMP using the (J, J') -spectral factorization⁵⁾. In comparison with the preceding works, our approach provides a straightforward proof in the pure frequency domain and a clear understanding of the structure of the H_∞ filtering problem

even though the process disturbance and the measurement noise are correlated.

Furthermore, we consider the H_∞ prediction problem as a special case of the H_∞ filtering problem, whereas the prediction problem was solved in a different setting from the filtering problem^{1), 2)}.

The following standard notations are used throughout this paper. The transpose and complex conjugate transpose are denoted by $(\cdot)^T$ and $(\cdot)^H$, respectively. The Z-transformation operator is denoted by σ . For a transfer matrix $G(\sigma)$, we define $G^\sim(\sigma) = G^T(\sigma^{-1})$. A transfer matrix in terms of state-space data is denoted by

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] := C(\sigma I - A)^{-1}B + D$$

RL_∞ : the set of proper real rational transfer matrices bounded on the unit circle

RH_∞ : the set of proper real rational stable transfer matrices

BH_∞ : the set of transfer matrices in RH_∞ whose H_∞ norm is less than $\gamma > 0$

In order to introduce the (J, J') -spectral factorization result, we define $J_{pq} = \begin{bmatrix} I_p & 0 \\ 0 & -\gamma^2 I_q \end{bmatrix}$.

2. Problem Formulation

We consider a linear time-invariant system given by

$$x_{k+1} = Ax_k + Bd_k \quad (1)$$

$$y_k = Cx_k + Dd_k \quad (2)$$

where $x_k \in \mathbf{R}^n$, $y_k \in \mathbf{R}^q$ and $d_k \in \mathbf{R}^m$ are the state vector, the measurement and the unknown disturbance, respectively. Let $z_k \in \mathbf{R}^p$ be the linear combination of the state variables given by

$$z_k = Lx_k \quad (3)$$

[†] 第22回制御理論シンポジウムで発表 (1993・5)

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(Received October 25, 1995)

(Revised February 4, 1997)

The following conditions are assumed to hold.

(A1) (C, A) is detectable.

$$(A2) \quad \text{rank} \begin{bmatrix} A - e^{j\omega} I_n & B \\ C & D \end{bmatrix} = n + q, \quad \forall \omega \in \mathbf{R}$$

Let $\hat{z}_k$ be an estimate of z_k based on the measurement set $\{y_t | t \leq k\}$ and $T_f(\sigma)$ be the transfer matrix of the filter, namely,

$$\hat{z}(\sigma) = T_f(\sigma)y(\sigma) \quad (4)$$

We also define $e_k = z_k - \hat{z}_k$. Then, from (1)–(4), we see that

$$e(\sigma) = \{T_{zd}(\sigma) - T_f(\sigma)T_{yd}(\sigma)\}d(\sigma) \quad (5)$$

where the transfer matrices from d_k to z_k and y_k are given by

$$T_{zd} = \left[\begin{array}{c|c} A & B \\ \hline L & 0 \end{array} \right], \quad T_{yd} = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] \quad (6)$$

Hence the transfer matrix from d_k to e_k is given by

$$T_{ed}(\sigma) = T_{zd}(\sigma) - T_f(\sigma)T_{yd}(\sigma) \quad (7)$$

The purpose of this paper is to give a necessary and sufficient condition for the existence of a filter $T_f(\sigma)$ satisfying the following conditions.

$$(S1) \quad T_f(\sigma) \in \mathbf{RH}_\infty^{p \times q}$$

$$(S2) \quad T_{ed}(\sigma) \in \mathbf{RH}_\infty^{p \times m}$$

$$(S3) \quad \|T_{ed}\|_\infty < \gamma \text{ for a given } \gamma > 0$$

Furthermore, if such a filter exists, we wish to derive a parametrization of all filters satisfying (S1)–(S3).

The conditions (S1) and (S2) are basic for a filtering problem. The former is required to hold for the internal stability of the filter itself. The latter implies that the estimation error e_k converges to zero as $k \rightarrow \infty$ for any initial states in the absence of disturbance. The condition (S3) is the requirement for $\mathbf{H}_\infty$ sub-optimality. Since the $\mathbf{H}_\infty$ norm of a transfer matrix is the $\mathbf{L}_2$ induced norm, (S3) can be interpreted as the design specification considering the worst-case disturbance in terms of energy gain.

3. Preliminaries

In this section, we give some results on (J, J') -spectral factorization, model matching problem and (J, J') -lossless matrix. They are useful for solving $\mathbf{H}_\infty$ filtering problem.

The first lemma gives the relationship between a model matching problem and a (J, J') -spectral factorization⁵⁾.

Lemma 1.⁵⁾ We define

$$G(\sigma) = \begin{bmatrix} T_2 & 0 \\ T_1 & -I_p \end{bmatrix}$$

for given $T_1(\sigma) \in \mathbf{RL}_\infty^{p \times m}$ and $T_2(\sigma) \in \mathbf{RL}_\infty^{q \times m}$. Suppose that $G(\sigma)$ has a right inverse in $\mathbf{RL}_\infty$. Then the following statements are equivalent.

(i) There exists a $Q(\sigma) \in \mathbf{RH}_\infty^{p \times q}$ satisfying

$$\|T_1 - QT_2\|_\infty < \gamma$$

(ii) There exists a unimodular matrix $\Pi(\sigma)$ such that

$$GJ_{mp}G^\sim = \Pi J_{qp}\Pi^\sim, \quad \Pi_{11}^{-1}(\sigma) \in \mathbf{RH}_\infty^{q \times q}$$

where $\Pi_{11}(\sigma)$ is the $(1, 1)$ -block matrix of $\Pi(\sigma)$.

The following two lemmas are related to the state-space computation of (J, J') -spectral factorization.

Lemma 2. For a given $(q+p) \times (m+\ell)$ transfer matrix

$$G(\sigma) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] \text{ with } A \text{ stable, there exists a unimodular matrix } \Pi(\sigma) \in \mathbf{RH}_\infty^{(q+p) \times (q+p)} \text{ such that}$$

$$G(\sigma)J_{m\ell}G^\sim(\sigma) = \Pi(\sigma)J_{qp}\Pi^\sim(\sigma)$$

if and only if

(i) The following algebraic Riccati equation (ARE) has a unique stabilizing solution P .

$$P = APA^T + BJ_{m\ell}B^T - (APC^T + S)V^{-1}(APC^T + S)^T \quad (8a)$$

$$V = DJ_{m\ell}D^T + CPC^T, \quad S = BJ_{m\ell}D^T \quad (8b)$$

(ii) There exists a nonsingular matrix W satisfying

$$WJ_{qp}W^T = V \quad (9)$$

When the above conditions hold, such a $\Pi(\sigma)$ is given by

$$\Pi(\sigma) = \left[\begin{array}{c|c} A & M \\ \hline C & I_{q+p} \end{array} \right] W \quad (10)$$

$$M = (APC^T + BJ_{m\ell}D^T)V^{-1} \quad (11)$$

Proof. See Appendix A. $\square$

Lemma 3. For a given real symmetric matrix $V = \begin{bmatrix} V_{11} & V_{21}^T \\ V_{21} & V_{22} \end{bmatrix} \in \mathbf{R}^{(q+p) \times (q+p)}$, we assume that $V_{11} > 0$ holds. Then a necessary and sufficient condition for the existence of a nonsingular matrix W satisfying $WJ_{qp}W^T = V$ is

$$\hat{V} := V_{21}V_{11}^{-1}V_{21}^T - V_{22} > 0$$

Proof. The lemma is immediate from the inertia theorem⁶⁾. It may be noted that the sufficiency is proved by the fact that

$$W = \begin{bmatrix} V_{11}^{\frac{1}{2}} & 0 \\ V_{21}V_{11}^{-\frac{1}{2}} & \gamma^{-1}\hat{V}^{\frac{1}{2}} \end{bmatrix} \quad (12)$$

satisfies $V = WJ_{qp}W^T$. $\square$

The notion of a (J, J') -lossless matrix is also very important.

Definition. Suppose that

$$\Theta(\sigma)J_{mt}\Theta^{\sim}(\sigma) = J_{qp} \quad \forall \sigma \in \mathbb{C}$$

$$\Theta(\sigma)J_{mt}\Theta^H(\sigma) \leq J_{qp} \quad \forall \sigma \text{ s.t. } |\sigma| \geq 1$$

hold for a given $\Theta(\sigma) \in \mathbf{RL}_{\infty}^{(q+p) \times (m+l)}$. Then, $\Theta(\sigma)$ is called (J_{mt}, J_{qp}) -lossless.

A remarkable property of a (J, J') -lossless matrix is shown in the following lemma.

Lemma 4. Suppose that

$$\Theta(\sigma) = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} \in \mathbf{RL}_{\infty}^{(q+p) \times (m+l)}$$

is (J_{mp}, J_{qp}) -lossless, and define

$$\Phi(\sigma) = (U\Theta_{12} + \Theta_{22})^{-1}(U\Theta_{11} + \Theta_{21})$$

Then $\Phi(\sigma) \in \mathbf{BH}_{\infty}^{p \times m}$ if and only if $U(\sigma) \in \mathbf{BH}_{\infty}^{p \times q}$.

Proof. For the proof, see the reference 7). $\square$

4. Solution to H_{∞} Filtering Problem

In this section, we will give a solution to the H_{∞} filtering problem based on a model matching approach.

We first introduce the class of all filters satisfying (S1) and (S2) in order to reduce the H_{∞} filtering problem to an MMP. This is the filtering equivalent of the class of stabilizing controllers⁸⁾.

Lemma 5. Suppose that H is a constant matrix such that $A_H := A - HC$ is stable. Then, the set of all filters satisfying (S1) and (S2) is given by

$$T_f(\sigma) = T_{f1}(\sigma) - Q(\sigma)T_{f2}(\sigma) \quad (13a)$$

$$T_{f1}(\sigma) = \left[\begin{array}{c|c} A_H & H \\ \hline L & 0 \end{array} \right] \quad (13b)$$

$$T_{f2}(\sigma) = \left[\begin{array}{c|c} A_H & H \\ \hline C & -I_q \end{array} \right] \quad (13c)$$

where $Q(\sigma)$ is an arbitrary transfer matrix in $\mathbf{RH}_{\infty}^{p \times q}$.

Proof. See the reference 4). $\square$

The filter $T_f(\sigma)$ is strictly proper if and only if $Q(\sigma)$ is strictly proper. A strictly proper $T_f(\sigma)$ implies that the filter does not use the measurement y_k for the estimation at time k . Therefore, the above parametrization includes both filters and predictors.

We assume that $T_f(\sigma)$ is expressed by (13). Substituting (13) into (7) yields

$$T_{ed}(\sigma) = T_1(\sigma) - Q(\sigma)T_2(\sigma) \quad (14a)$$

$$T_1(\sigma) = \left[\begin{array}{c|c} A_H & B_H \\ \hline L & 0 \end{array} \right] \quad (14b)$$

$$T_2(\sigma) = \left[\begin{array}{c|c} A_H & B_H \\ \hline C & D \end{array} \right] \quad (14c)$$

where $B_H = B - HD$. Therefore, the H_{∞} filtering problem is equivalent to the unilateral MMP of finding a matrix $Q(\sigma) \in \mathbf{RH}_{\infty}^{p \times q}$ such that

$$\|T_1 - QT_2\|_{\infty} < \gamma \quad (15)$$

It is well-known that the full-information (FI) H_{∞} control problem is equivalent to a bilateral MMP (see e.g. the reference 9)). Hence, the duality between state estimation and FI control does not hold from the model matching viewpoint.

We now introduce the transfer matrix

$$G(\sigma) = \left[\begin{array}{cc} T_2 & 0 \\ T_1 & -I_p \end{array} \right] = \left[\begin{array}{c|cc} A_H & B_H & 0 \\ \hline C & D & 0 \\ L & 0 & -I_p \end{array} \right] \quad (16)$$

For the subsequent reference, we define

$$\hat{C} = \left[\begin{array}{c} C \\ L \end{array} \right], \quad \hat{D} = \left[\begin{array}{cc} D & 0 \\ 0 & -I_p \end{array} \right], \quad R = DD^T, \\ \hat{S} = \left[\begin{array}{cc} BD^T & 0 \end{array} \right], \quad \hat{S}_H = \left[\begin{array}{cc} B_H D^T & 0 \end{array} \right]$$

By the assumptions (A1) and (A2), $G(\sigma)$ has a right inverse in $\mathbf{RL}_{\infty}$. Thus, Lemma 1 is applicable to the MMP of (15). That is, there exists a filter $T_f(\sigma)$ satisfying (S1)–(S3) if and only if there exists a unimodular matrix $\Pi(\sigma)$ satisfying

$$G(\sigma)J_{mp}G^{\sim}(\sigma) = \Pi(\sigma)J_{qp}\Pi^{\sim}(\sigma) \quad (17)$$

with $\Pi_{11}^{-1}(\sigma) \in \mathbf{RH}_{\infty}^{q \times q}$ for $G(\sigma)$ defined by (16).

We first consider the (J, J') -spectral factorization of (17). It follows from Lemma 2 that there exists a unimodular matrix $\Pi(\sigma)$ satisfying (17) if and only if the ARE

$$P = A_H P A_H^T + B_H B_H^T - (A_H P \hat{C}^T + \hat{S}_H) V^{-1} (A_H P \hat{C}^T + \hat{S}_H)^T \quad (18)$$

$$V = \hat{D} J_{mp} \hat{D}^T + \hat{C} P \hat{C}^T \\ = \left[\begin{array}{cc} R + C P C^T & C P L^T \\ L P C^T & -(\gamma^2 I_p - L P L^T) \end{array} \right] \quad (19)$$

has a stabilizing solution P and there exists a nonsingular W satisfying

$$V = W J_{qp} W^T \quad (20)$$

By the identity

$$\begin{bmatrix} R + CPC^T & CPL^T \end{bmatrix} V^{-1} = \begin{bmatrix} I_q & 0 \end{bmatrix},$$

we easily see that (18) reduces to the following ARE independent of the matrix H .

$$P = APA^T + BB^T - (AP\hat{C}^T + \hat{S})V^{-1}(AP\hat{C}^T + \hat{S})^T \quad (21)$$

Also, the matrix M corresponding to (11) is given by

$$M = [K_1 - H \quad K_2] \quad (22)$$

where we define

$$K = [K_1 \quad K_2] = (AP\hat{C}^T + \hat{S})V^{-1} \quad (23)$$

$$A_K = A - K\hat{C} = A - K_1C - K_2L \quad (24)$$

It then follows from (10) that

$$\Pi(\sigma) = \left[\begin{array}{c|c} A_H & M \\ \hline \hat{C} & I_{q+p} \end{array} \right] W \quad (25)$$

We next consider the condition $\Pi_{11}^{-1}(\sigma) \in \mathbf{RH}_{\infty}^{q \times q}$. Suppose that there exists a stabilizing solution to the ARE (21). Let W satisfying (20) be appropriately partitioned as

$$W = [W_1 \quad W_2] = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}$$

We now assume that $\Pi_{11}^{-1}(\sigma) \in \mathbf{RH}_{\infty}^{q \times q}$ holds. Then, we see from (25) that

$$\begin{aligned} \Pi_{11}^{-1}(\sigma) &= \left[\begin{array}{c|c} A_H & MW_1 \\ \hline C & W_{11} \end{array} \right]^{-1} \\ &= \left[\begin{array}{c|c} A_{\infty} & -MW_1 \\ \hline W_{11}^{-1}C & I_q \end{array} \right] W_{11}^{-1} \end{aligned} \quad (26)$$

where $A_{\infty} := A - KW_1W_{11}^{-1}C$. Since this realization is stabilizable and detectable, $\Pi_{11}^{-1}(\sigma) \in \mathbf{RH}_{\infty}^{q \times q}$ implies that A_{∞} has all eigenvalues inside the open unit disk. Furthermore, it follows from (21) that

$$P = A_{\infty}PA_{\infty}^T + N_{\infty}N_{\infty}^T \quad (27)$$

where

$$N_{\infty} = [B - K_{\infty}D \quad \gamma(KW_2 - K_{\infty}W_{12})] \quad (28)$$

$$K_{\infty} = KW_1W_{11}^{-1} \quad (29)$$

Hence, we have $P \geq 0$ by Lyapunov's theorem. Together with $\det V \neq 0$, this implies that $V_{11} > 0$. It thus follows from Lemma 3 that the existence of W satisfying (20) implies

$$\hat{V} := V_{21}V_{11}^{-1}V_{21}^T - V_{22} > 0 \quad (30)$$

where V_{ij} ($i, j = 1, 2$) are the (i, j) -block matrices of V compatible with the partition in (19).

Conversely, we assume that the stabilizing solution of (21) is positive semi-definite and (30) holds. Since $V_{11} > 0$

holds, there exists a nonsingular matrix W satisfying (20) by Lemma 3. Since $W_{11}W_{11}^T = V_{11} + \gamma^2W_{12}W_{12}^T > 0$ holds from (19) and (20), W_{11} is invertible. Thus, we can express (21) as (27). The pair (A_{∞}, N_{∞}) is stabilizable since A_K is stable. It thus follows from Lyapunov's theorem that A_{∞} is stable. This implies $\Pi_{11}^{-1}(\sigma) \in \mathbf{RH}_{\infty}^{q \times q}$. Therefore, from Lemma 1, the MMP (15) is solvable.

Summarizing the above results, we obtain the following theorem.

Theorem 1. The H_{∞} filtering problem, equivalently the MMP of (15), is solvable if and only if the ARE (21) has a positive semi-definite stabilizing solution P and (30) holds.

Next we give a parametrization of all H_{∞} filters satisfying (S1)–(S3).

Theorem 2. Suppose that the H_{∞} filtering problem is solvable. Then the parametrization of all filters satisfying (S1)–(S3) is given by

$$T_f(\sigma) = -(U\Omega_{12} - \Omega_{22})^{-1}(U\Omega_{11} - \Omega_{21}) \quad (31)$$

$$\Omega(\sigma) = W^{-1} \left[\begin{array}{c|cc} A_K & -K_1 & -K_2 \\ \hline C & I_q & 0 \\ L & 0 & I_p \end{array} \right] \quad (32)$$

where $U(\sigma)$ is an arbitrary transfer matrix in $\mathbf{BH}_{\infty}^{p \times q}$, and where $K = [K_1 \quad K_2]$, A_K and W are defined by (23), (24) and (20), respectively.

Proof. We define

$$\begin{aligned} \Theta(\sigma) &= \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} = \Pi^{-1}(\sigma)G(\sigma) \\ &= W^{-1} \left[\begin{array}{c|cc} A_K & B - K_1D & K_2 \\ \hline C & D & 0 \\ L & 0 & -I_p \end{array} \right] \end{aligned} \quad (33)$$

Some algebraic manipulations using (8) and (17) yield

$$\Theta(\sigma)J_{mp}\Theta^{\sim}(\sigma) = J_{qp} \quad \forall \sigma \in \mathcal{C}$$

$$\Theta(\sigma)J_{mp}\Theta^H(\sigma) \leq J_{qp} \quad \forall \sigma \text{ s.t. } |\sigma| \geq 1$$

Thus $G(\sigma)$ has the (J_{mp}, J_{qp}) -lossless factorization $G(\sigma) = \Pi(\sigma)\Theta(\sigma)$ with a (J_{mp}, J_{qp}) -lossless matrix $\Theta(\sigma)$ and a unimodular matrix $\Pi(\sigma)$. Let $X_1(\sigma) \in \mathbf{RH}_{\infty}^{p \times q}$ and $X_2(\sigma) \in \mathbf{RH}_{\infty}^{p \times p}$ be defined by

$$\begin{bmatrix} X_1 & X_2 \end{bmatrix} = \begin{bmatrix} -Q & I_p \end{bmatrix} \Pi \quad (34)$$

Then we get from (14)

$$\begin{aligned} \begin{bmatrix} T_{ed} & -I_p \end{bmatrix} &= \begin{bmatrix} -Q & I_p \end{bmatrix} G \\ &= \begin{bmatrix} -Q & I_p \end{bmatrix} \Pi \Theta \\ &= \begin{bmatrix} X_1 & X_2 \end{bmatrix} \Theta \end{aligned} \quad (35)$$

Hence $T_{ed}(\sigma)$ is expressed as

$$\begin{aligned} T_{ed} &= -\{(-U)\Theta_{12} + \Theta_{22}\}^{-1}\{(-U)\Theta_{11} + \Theta_{21}\} \\ &= -(U\Theta_{12} - \Theta_{22})^{-1}(U\Theta_{11} - \Theta_{21}) \end{aligned} \quad (36)$$

where $U(\sigma) := -X_2^{-1}X_1$. Since $\Theta(\sigma)$ is (J_{mp}, J_{qp}) -lossless, $T_{ed}(\sigma) \in \mathbf{BH}_{\infty}^{p \times m}$ if and only if $U(\sigma) \in \mathbf{BH}_{\infty}^{p \times q}$ by Lemma 4. Furthermore, from (34), we see that

$$-Q(\sigma) = X_1\hat{\Pi}_{11} + X_2\hat{\Pi}_{21} \quad (37)$$

$$I_p = X_1\hat{\Pi}_{12} + X_2\hat{\Pi}_{22} \quad (38)$$

where $\Pi^{-1}(\sigma)$ is partitioned as

$$\Pi^{-1}(\sigma) = \begin{bmatrix} \hat{\Pi}_{11} & \hat{\Pi}_{12} \\ \hat{\Pi}_{21} & \hat{\Pi}_{22} \end{bmatrix}.$$

Hence we get

$$Q(\sigma) = -(U\hat{\Pi}_{12} - \hat{\Pi}_{22})^{-1}(U\hat{\Pi}_{11} - \hat{\Pi}_{21}) \quad (39)$$

Substituting this into (13) yields the parametrization of (31), where

$$\begin{aligned} \Omega(\sigma) &= \begin{bmatrix} \hat{\Pi}_{11} & \hat{\Pi}_{12} \\ \hat{\Pi}_{21} & \hat{\Pi}_{22} \end{bmatrix} \begin{bmatrix} -T_{f2} & 0 \\ -T_{f1} & I_p \end{bmatrix} \\ &= W^{-1} \left[\begin{array}{c|cc} A_K & -K_1 & -K_2 \\ \hline C & I_q & 0 \\ L & 0 & I_p \end{array} \right] \end{aligned}$$

□

Remark 1: It may be noted from Theorems 1 and 2 that the solution to the H_{∞} filtering problem is independent of the constant matrix H which is introduced in Lemma 5.

Remark 2: It is easily seen from the proof of Lemma 3 that if the H_{∞} filtering problem is solvable, we can take $W_{12} = 0$ as in (12) without loss of generality. Then, $\Omega(\sigma)$ of Theorem 2 is given by

$$\Omega(\sigma) = \left[\begin{array}{c|cc} A_K & -K_1 & -K_2 \\ \hline W_{11}^{-1}C & W_{11}^{-1} & 0 \\ L' & -W_{22}^{-1}W_{21}W_{11}^{-1} & W_{22}^{-1} \end{array} \right] \quad (40)$$

where $L' = -W_{22}^{-1}W_{21}W_{11}^{-1}C + W_{22}^{-1}L$. Thus, by taking $U(\sigma) = 0$, we obtain

$$\begin{aligned} T_f(\sigma) &= -\Omega_{22}^{-1}(\sigma)\Omega_{21}(\sigma) \\ &= \left[\begin{array}{c|c} A - K_{\infty}C & K_{\infty} \\ \hline L - M_{\infty}C & M_{\infty} \end{array} \right] \end{aligned} \quad (41)$$

where

$$K_{\infty} = (APC^T + BD^T)(R + CPC^T)^{-1} \quad (42)$$

$$M_{\infty} = W_{21}W_{11}^{-1} = LPC^T(R + CPC^T)^{-1} \quad (43)$$

As γ tends to infinity, the ARE (21) reduces to the Kalman filtering type ARE, and hence (41) reduces to Kalman filter. Thus, we refer to the H_{∞} filter of (41) as the central H_{∞} filter.

Fig. 1 Chain scattering representation

Fig. 2 Cascade connection of chain scattering matrices

Although the derivation of $Q(\sigma)$ solving the MMP (15) is not a new result (see e.g. 5),8,10)), the properties of the transfer matrices defined in the proof of Theorem 2 are very useful to see the structure of the H_{∞} filtering problem in the next section.

5. Structure of H_{∞} Filtering Problem

In this section, we will study the structure of the H_{∞} filtering problem using the chain scattering representation.

First, we briefly review some results on the chain scattering representation¹¹⁾. Consider a system described by

$$\begin{bmatrix} u_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ y_1 \end{bmatrix}$$

where (u_1, y_2) and (u_2, y_1) are the inputs and outputs of the system, respectively. The transfer matrix $\Sigma(\sigma) = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$ is called a "chain scattering matrix". This system can be illustrated as in Fig. 1.

Using the chain scattering matrices $\Sigma_1(\sigma)$ and $\Sigma_2(\sigma)$, the input-output relationship of the cascade connection of two systems in Fig. 2 (a) is given by

$$\begin{bmatrix} u_3 \\ y_3 \end{bmatrix} = \Sigma_2 \begin{bmatrix} u_2 \\ y_2 \end{bmatrix} = \Sigma_2 \Sigma_1 \begin{bmatrix} u_1 \\ y_1 \end{bmatrix}$$

This implies that the cascade connection of systems can be represented by using the product of the chain scattering matrices of each system (Fig. 2(b)).

Fig. 3 Chain scattering representation of a closed-loop system

Fig. 4 Structure of H_∞ filtering problem

We consider a closed-loop system shown in Fig. 3. The input-output relationship of this system is described by

$$\begin{aligned}\Phi(\sigma) &= -\{(-Q)\Sigma_{11} + \Sigma_{21}\}^{-1}\{(-Q)\Sigma_{12} + \Sigma_{22}\} \\ &= -(Q\Sigma_{11} - \Sigma_{21})^{-1}(Q\Sigma_{12} - \Sigma_{22})\end{aligned}$$

As shown in Lemma 4, if $\Sigma(\sigma) \in \mathbf{RL}_\infty^{(q+p) \times (m+p)}$ is (J_{mp}, J_{qp}) -lossless, then a necessary and sufficient condition for $\Phi(\sigma) \in \mathbf{BH}_\infty^{p \times m}$ is $Q(\sigma) \in \mathbf{BH}_\infty^{p \times q}$.

Noting the above properties of the chain scattering representation, the structure of the H_∞ filtering problem is illustrated in Fig. 4. In the figure, $\Sigma_{sys}(\sigma)$ and $\Sigma_f(\sigma)$ are the chain scattering matrices associated with the system (1)–(3) and the filter (13), respectively.

$$\Sigma_{sys} = \begin{bmatrix} T_{yd} & 0 \\ T_{zd} & -I_p \end{bmatrix}, \quad \Sigma_f = \begin{bmatrix} -T_{f2} & 0 \\ -T_{f1} & I_p \end{bmatrix}$$

We see from the figure that $\Omega(\sigma)$ is independent of the constant matrix H in (13) due to the pole-zero cancellation between $\Pi^{-1}(\sigma)$ and $\Sigma_f(\sigma)$.

It may be noted that when the H_∞ filtering problem is solvable, $\Sigma_{sys}(\sigma)$ has a coprime factorization

$$\Sigma_{sys}(\sigma) = \Omega^{-1}(\sigma)\Theta(\sigma) \quad (44)$$

where Θ and Ω are respectively given by (32), (33), and Θ is (J_{mp}, J_{qp}) -lossless. Conversely, suppose that $\Sigma_{sys}(\sigma)$ has a (J_{mp}, J_{qp}) -lossless coprime factorization (44) without assuming any particular realizations of Ω and Θ . It then follows from Lemma 4 and Fig. 4 that an H_∞ filter is given by (31) for some $U(\sigma) \in \mathbf{BH}_\infty^{p \times q}$. Thus, we have the following theorem.

Theorem 3. The H_∞ filtering problem is solvable if and only if $\Sigma_{sys}(\sigma)$ has a (J_{mp}, J_{qp}) -lossless coprime factorization of (44).

The structure of Fig. 4 is almost dual to the pole-zero cancellation structure of the H_∞ control problem¹²⁾. It is well-known that the H_∞ control problem is solvable if and only if the chain scattering matrix of the plant has a (J_{mp}, J_{qp}) -lossless factorization, where the denominator must be unimodular in order to guarantee the internal stability of the feedback system¹²⁾. However, since the internal stability is not required for the filtering problem, the denominator $\Omega(\sigma)$ in (44) need not be a unimodular matrix.

6. Solution to H_∞ Prediction Problem

From the discussion in Section 4, we see that the key properties of H_∞ filters lie in the central H_∞ filter of (41)–(43) which utilizes $\{y_t | t \leq k\}$ for the estimation at time k . Therefore, even if the H_∞ filtering problem is solvable, there may not exist an H_∞ predictor (strictly proper $T_f(\sigma)$) which uses $\{y_t | t \leq k-1\}$ rather than $\{y_t | t \leq k\}$. In this section, we consider the H_∞ prediction problem as a special case of the H_∞ filtering problem.

Theorem 4. Suppose that the H_∞ filtering problem is solvable. Then, a necessary and sufficient condition for the existence of an H_∞ predictor satisfying (S1)–(S3) is that $V_{22} < 0$ holds for a positive semi-definite stabilizing solution P of the ARE (21).

Proof. Necessity: Suppose that there exists an H_∞ predictor $T_f(\sigma)$ satisfying (S1)–(S3). From (31), $U\Omega_{11} + \Omega_{21}$ must be strictly proper for the strictly proper $T_f(\sigma)$. We now assume $W_{12} = 0$ without loss of generality, so that $\Omega(\sigma)$ is given by (40). Moreover, define $U(\sigma) = \begin{bmatrix} A_U & B_U \\ C_U & D_U \end{bmatrix}$. Then, simple state-space calculation shows that the direct feed-through term of $U\Omega_{11} + \Omega_{21}$ is $(D_U - W_{22}^{-1}W_{21})W_{11}^{-1}$. Thus we obtain $D_U = W_{22}^{-1}W_{21}$. Further, since $\|U\|_\infty < \gamma$ holds from Theorem 2, we get

$$\begin{aligned}\gamma^2 I_p - D_U D_U^T &= \gamma^2 I_p - W_{22}^{-1}W_{21}W_{21}^T W_{22}^{-T} \\ &= -W_{22}^{-1}V_{22}W_{22}^{-T} > 0\end{aligned}$$

This implies $V_{22} = -(\gamma^2 I_p - LPL^T) < 0$.

Sufficiency: Suppose that $V_{22} < 0$ holds. Then,

$$W = \begin{bmatrix} \tilde{V}^{\frac{1}{2}} & -\gamma^{-1}V_{21}^T(-V_{22})^{-\frac{1}{2}} \\ 0 & \gamma^{-1}(-V_{22})^{\frac{1}{2}} \end{bmatrix} \quad (45)$$

is nonsingular and satisfies $WJ_{qp}W^T = V$, where $\tilde{V} = V_{11} - V_{21}^T V_{22}^{-1} V_{21}$. By taking W as in (45), we get the

following parametrization from Theorem 2.

$$T_f(\sigma) = -(U\Omega_{12} + \Omega_{22})^{-1}(U\Omega_{11} + \Omega_{21}) \quad (46)$$

$$\Omega(\sigma) = \left[\begin{array}{c|cc} A_K & -K_1 & -K_2 \\ \hline C' & W_{11}^{-1} & -W_{11}^{-1}W_{12}W_{22}^{-1} \\ \hline W_{22}^{-1}L & 0 & W_{22}^{-1} \end{array} \right] \quad (47)$$

where $U(\sigma)$ is an arbitrary transfer matrix in $BH_\infty^{p \times q}$ and $C' = W_{11}^{-1}C - W_{11}^{-1}W_{12}W_{22}^{-1}L$. Taking $U(\sigma) = 0$ yields an H_∞ predictor

$$T_f(\sigma) = \left[\begin{array}{c|c} A - K_1C & K_1 \\ \hline L & 0 \end{array} \right] \quad (48)$$

This completes the proof. $\square$

It is easy to verify that

$$K_1 = (A\bar{P}C^T + BD^T)(R + C\bar{P}C^T)^{-1}$$

$$\bar{P} = P(I_n - \gamma^{-2}L^TLP)^{-1}$$

Thus, as γ tends to infinity, the H_∞ predictor (48) reduces to Kalman predictor.

Remark 3: Since $V_{11} > 0$ holds, we get

$$V_{21}^T V_{11}^{-1} V_{21} - V_{22} \geq -V_{22}$$

Thus we see from Theorems 1 and 4 that the existence condition of an H_∞ filter is more relaxed than that of an H_∞ predictor. This implies that an H_∞ filter can achieve the smaller H_∞ error bound than an H_∞ predictor.

7. Conclusion

In this paper, we have derived a complete parametrization of all solution to the H_∞ filtering problem based on the model matching approach, and we have investigated the structure of this problem. The free parameter of the parametrization can be used for achieving an additional design specification as well as the H_∞ error bound. An example of such multi-objective filter design is given in 13). Furthermore, we have given a solution to the H_∞ prediction problem as a special case of the H_∞ filtering problem. The present approach provides a unified solution to the H_∞ filtering and prediction problem.

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Appendix A. Proof of Lemma 2

Although the proof was given by Green¹⁴⁾ based on the continuous-time result and the bilinear transformation, we give a different proof.

Sufficiency: We assume that there exist a stabilizing solution P to the ARE of (8) and that a nonsingular matrix W satisfying (9) exists. We define $\Psi(\sigma) := G(\sigma)J_{ml}G^\sim(\sigma)$ and let $\Pi(\sigma)$ and M be defined by (10) and (11), namely

$$\Pi(\sigma) = \left[\begin{array}{c|c} A & M \\ \hline C & I_{q+p} \end{array} \right] W$$

$$M = (APC^T + BJ_{ml}D^T)V^{-1}$$

Since P is a stabilizing solution to the ARE of (8), it is straightforward to show that

$$\Pi^{-1}(\sigma) = W^{-1} \left[\begin{array}{c|c} A - MC & -M \\ \hline C & I_{q+p} \end{array} \right]$$

is stable. Simple but tedious state-space calculation using the ARE (8) yields

$$\Psi(\sigma) = \left[\begin{array}{c|c} A & M \\ \hline C & I_{q+p} \end{array} \right] V \left[\begin{array}{c|c} A & M \\ \hline C & I_{q+p} \end{array} \right]^\sim \quad (A.1)$$

Substituting (9) into (A.1) yields $\Psi(\sigma) = \Pi(\sigma)J_{qp}\Pi^\sim(\sigma)$.

This completes a proof of sufficiency.

Necessity : Hereafter, for simplicity of discussion, we assume that (C, A) is observable. The basic idea of the proof is due to the reference 15). Since A is stable, there exists a unique solution to the Lyapunov equation

$$P_1 = AP_1A^T + BJ_{m\ell}B^T \quad (A.2)$$

Then we get

$$\begin{aligned} BJ_{m\ell}B^T &= (\sigma I_n - A)P_1(\sigma^{-1}I_n - A^T) \\ &\quad + AP_1(\sigma^{-1}I_n - A^T) + (\sigma I_n - A)P_1A^T \end{aligned}$$

Pre-multiplying by $\Phi(\sigma) := C(\sigma I_n - A)^{-1}$ and post-multiplying by $\Phi^{\sim}(\sigma)$ yield

$$\Phi BJ_{m\ell}B^T \Phi^{\sim} = \Phi AP_1C^T + CP_1A^T \Phi^{\sim} + CP_1C^T$$

Hence, we get

$$\begin{aligned} \Psi(\sigma) &= DJ_{m\ell}D^T + \Phi BJ_{m\ell}B^T \Phi^{\sim} \\ &\quad + \Phi BJ_{m\ell}D^T + DJ_{m\ell}B^T \Phi^{\sim} \\ &= DJ_{m\ell}D^T + CP_1C^T + G' + G'^{\sim} \end{aligned} \quad (A.3)$$

where

$$G'(\sigma) = \left[\begin{array}{c|c} A & AP_1C^T + BJ_{m\ell}D^T \\ \hline C & 0 \end{array} \right] \quad (A.4)$$

We now assume that there exists a unimodular matrix $\Pi(\sigma)$ such that $\Psi = \Pi J_{qp} \Pi^{\sim}$. From (A.3), we easily see that if λ is a pole of $\Psi(\sigma)$ then $1/\lambda$ is also a pole of $\Psi(\sigma)$, and that if such a pole λ satisfies $|\lambda| < 1$ then it is an eigenvalue of A . Thus, we can take $\Pi(\sigma)$ as

$$\Pi(\sigma) = \left[\begin{array}{c|c} A & \Gamma \\ \hline C & W \end{array} \right], \quad W: \text{nonsingular}$$

Note that since (C, A) is observable, Γ is unique for C and A . Since W is nonsingular, $V := WJ_{qp}W^T$ is also nonsingular. Similarly to the derivation of (A.3) and (A.4), we obtain

$$\Psi(\sigma) = \Pi J_{qp} \Pi^{\sim} = V + CP_2C^T + \Pi' + \Pi'^{\sim} \quad (A.5)$$

$$\Pi'(\sigma) = \left[\begin{array}{c|c} A & AP_2C^T + \Gamma J_{qp}W^T \\ \hline C & 0 \end{array} \right] \quad (A.6)$$

where P_2 is a unique solution to the Lyapunov equation

$$P_2 = AP_2A^T + \Gamma J_{qp}\Gamma^T \quad (A.7)$$

Since both $G'(\sigma)$ and $\Pi'(\sigma)$ are in $\mathbf{RH}_{\infty}^{q \times q}$ and strictly proper, comparing (A.3) with (A.5) yields

$$G'(\sigma) = \Pi'(\sigma) \quad (A.8)$$

$$DJ_{m\ell}D^T + CP_1C^T = V + CP_2C^T \quad (A.9)$$

Moreover, since (C, A) is observable, from (A.4), (A.6) and (A.8), we get

$$AP_1C^T + S = AP_2C^T + \Gamma J_{qp}W^T \quad (A.10)$$

where $S = BJ_{m\ell}D^T$. Define $P := P_1 - P_2$. It then follows from (A.9), (A.10) and the invertibility of V that

$$V = DJ_{m\ell}D^T + CPC^T \quad (A.11)$$

$$\Gamma = MW \quad (A.12)$$

$$M = (APC^T + S)V^{-1} \quad (A.13)$$

Substituting above equations into (A.7) yields

$$P_2 = AP_2A^T + (APC^T + S)V^{-1}(CPA^T + S^T)$$

By subtracting this from (A.2), we obtain the ARE of (8). Note that P is unique since P_1 , P_2 and Γ are unique.

Moreover, Π^{-1} is given by

$$\begin{aligned} \Pi^{-1}(\sigma) &= \left[\begin{array}{c|c} A - \Gamma W^{-1}C & -W^{-1}\Gamma \\ \hline W^{-1}C & W^{-1} \end{array} \right] \\ &= \left[\begin{array}{c|c} A - MC & -M \\ \hline W^{-1}C & W^{-1} \end{array} \right] \end{aligned}$$

Since $\Pi(\sigma)$ is unimodular, $A - MC$ is stable, i.e. P is a stabilizing solution of (8). $\square$

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