Constraint on Steady State Output Imposed by Zeros at $s = 0$ and Servo Synthesis Using Unstable Weight

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When an MIMO plant has zeros at $s = 0$, it will be shown that this imposes a constraint on the steady state output of the plant. In the control design using unstable weight, the structure of unstable weight can not be selected freely in this case. The required structure is exposed in this note. We then propose a particular generalized plant structure for an optimal servo design such as the one using $\mathcal{H}_\infty$ control.

Key Words: plant with zero at $s = 0$, MIMO systems, steady state constraint, structural constraint on unstable weight

1. Introduction

When an MIMO plant has zeros at $s = 0$ such as in 5), 6), it is well known that this imposes a constraint on the structure of unstable dynamics of the controller. In this note, we revisit this issue from a new standpoint, i.e. to expose the constraint on the steady state output of the plant due to this unstable zero. This, in tracking problems, implies that the reference is not free and is subject to a structural constraint. This study is necessary in the servo design using unstable weight. It will be shown that the unstable weight must also satisfy a structural constraint in this case.

We also propose a structure of generalized plant for optimal design. The Extended $\mathcal{H}_\infty$ control^{3),4)} can be applied to design a controller for asymptotic rejection of step disturbance or reference tracking.

The notation G_{yx} denotes the transfer matrix $x \mapsto y$.

2. Plant zero and steady state output constraint

Let $P(s)$ be an n th order plant of dimension $n_u \times n_u$. Suppose a stabilizable and detectable realization is

$$P(s) = \left[\begin{array}{c|c} A_p & B_p \\ \hline C_p & 0 \end{array} \right]. \quad (1)$$

Without loss of generality, it can be assumed that B_p has full column rank and C_p has full row rank. The plant has n_z zeros at $s = 0$, $n_u > n_z \geq 1$. So there exist a full row rank matrix $V = \begin{bmatrix} V_1 & V_2 \end{bmatrix}$, $\text{rank}(V) = n_z$, that satisfies the equation below.

$$\begin{matrix} (n_z) \\ (n_u) \end{matrix} \left[\begin{array}{c|c} V_1 & V_2 \end{array} \right] \left[\begin{array}{c} -\frac{A_p}{C_p} - \frac{B_p}{0} \\ 0 \end{array} \right] = 0 \quad (2)$$

Due to the stabilizability of (A_p, B_p) , V_2 must be full row rank.

Assumption 1. The reference input and/or persistent disturbance are step signals. $\square$

Proposition 1. For the plant given above, the following statements hold.

(i) the steady state output $y(\infty)$ of the plant must satisfy the constraint

$$V_2 \cdot y(\infty) = 0 \quad (3)$$

when $P(s)$ is stabilized by feedback control.

(ii) The step reference input r can be tracked asymptotically only if $r \in \text{Ker } V_2$.

(iii) The step output disturbance d can be rejected asymptotically only if $d \in \text{Ker } V_2$. $\square$

Proof. (i) Owing to Assumption 1, it can be proved that $\dot{x}_p(\infty) = 0$ holds in steady state so that

$$\begin{cases} 0 = A_p x_p(\infty) + B_p u(\infty) \\ y(\infty) = C_p x_p(\infty). \end{cases}$$

By (2) we have

$$\begin{aligned} & \left[\begin{array}{c|c} V_1 & V_2 \end{array} \right] \left[\begin{array}{c} 0 \\ y(\infty)^T \end{array} \right]^T \\ &= \left[\begin{array}{c|c} V_1 & V_2 \end{array} \right] \left[\begin{array}{c} A_p \\ C_p \end{array} \right] \left[\begin{array}{c} x_p(\infty) \\ u(\infty) \end{array} \right] \\ &= 0. \end{aligned} \quad (4)$$

This leads to $V_2 \cdot y(\infty) = 0$.

(ii) Since $y(\infty) = r$, it follows from (3) that $V_2 r = 0$.

(iii) The disturbance is added to the output, so $y(t) = C_p x(t) + d$. Then, following the proof of (i), we obtain $V_2 y(\infty) = V_2 d$. Therefore, $y(\infty) = 0$ is possible only if $V_2 d = 0$. $\blacksquare$

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3. Servo design using unstable weight

Let us consider the asymptotic rejection of step output disturbance of $P(s)$. For reference tracking problem, just take the tracking error as the plant output and the rest is the same. A useful way of doing this in the framework of optimal control is to use unstable weight at output y , as W_S shown in Fig. 1.

W_S of (5) is the general form for a transfer matrix having integrator dynamics only, and T_W is a design parameter and is to be determined. If fine tuning is required, extra stable weight can be added behind W_S and the results of this note are not affected.

$$W_S = \frac{1}{s} \times T_W = \left[\begin{array}{c|c} 0_{n_1} & T_W \\ \hline I_{n_1} & 0 \end{array} \right]. \quad (5)$$

Assumption 2. T_W has full rank. $\square$

Note at this stage, it is not clear whether T_W should have full row rank or full column rank and how many rows T_W should have. This is to be determined so that a control system that rejects disturbance asymptotically can be synthesized by using the unstable weight.

Fig. 1 Generalized plant structure

4. Determination of T_W

4.1 Main result

The key idea of using unstable weight in the design is to ensure asymptotic disturbance rejection by comprehensive stability^{1),2)}. That is, to guarantee $y(\infty) = 0$ by $G_{z_2w}(s) \in \mathcal{RH}_\infty$. The following proposition gives the requirement on the unstable weight in order to realize this purpose.

Proposition 2. Let T_W be an $n_1 \times n_u$ constant matrix and suppose there exists a controller $K(s)$ stabilizing $P(s)$ internally. Let V_y be the matrix in equation (6). Further, let $y(t)$ be the response of $P(s)$ to the step output disturbance $d \in \text{Ker } V_2$. Then the statements (i) and (ii) below are valid.

(i) To guarantee $y(\infty) = 0$ by $G_{z_2w}(s) \in \mathcal{RH}_\infty$, V_y must be nonsingular.

(ii) If $G_{z_2w}(s) \in \mathcal{RH}_\infty$ and V_y is nonsingular, then $y(\infty) = 0$.

$$V_y = \left[\begin{array}{c|c} (n_u) & \\ \hline V_2 & (n_z) \\ T_W & (n_1) \end{array} \right] \quad (6)$$

Proof. In both (i) and (ii) it is assumed that $G_{z_2w}(s) \in \mathcal{RH}_\infty$. So, we investigate the condition for this. As the closed loop system (P, K) is internally stable, the generalized plant of P with respect to exogenous input w and controlled output z_2 (Fig. 1) must be comprehensively stabilizable^{1),2)}. One of the comprehensive stabilizability condition requires that the unstable poles of W_S , which are not observable from y , be controllable from u . The transfer matrix $u \mapsto z_2$ has a realization as below

$$W_S P(s) = \left[\begin{array}{cc|c} A_p & 0 & B_p \\ \hline T_W C_p & 0_{n_1} & 0 \\ 0 & I_{n_1} & 0 \end{array} \right]. \quad (7)$$

We need only to find the controllability condition of this realization at $s = 0$. This requires

$$\text{rank } J(0) = n + n_1 \quad (8)$$

where

$$J(0) := \left[\begin{array}{cc|c} A_p & 0 & B_p \\ \hline T_W C_p & 0 & 0 \end{array} \right]. \quad (9)$$

Suppose $\exists \eta = [\eta_1 \ \eta_2] \neq 0$ such that $\eta V_y = 0$, i.e.

$$\eta_1 V_2 = -\eta_2 T_W.$$

Obviously $\eta_2 \neq 0$ since otherwise $\eta_1 = 0$ would follow from $\eta_1 V_2 = 0$, which contradicts the assumption $\eta \neq 0$.

So we have

$$\begin{aligned} & \left[\begin{array}{cc} \eta_1 V_1 & -\eta_2 \end{array} \right] J(0) \\ &= \left[\begin{array}{cc} \eta_1 V_1 & \eta_1 V_2 \end{array} \right] \left[\begin{array}{ccc} A_p & 0 & B_p \\ \hline C_p & 0 & 0 \end{array} \right] \\ &= 0. \end{aligned} \quad (10)$$

This means that if V_y does not have full row rank, at least a pole of W_S at $s = 0$ is not controllable. Therefore, $G_{z_2w}(s) \in \mathcal{RH}_\infty$ is possible only if V_y has full row rank.

Now, suppose $G_{z_2w}(s) \in \mathcal{RH}_\infty$. Since

$$G_{z_2w}(s) = \frac{1}{s} T_W G_{yw} = T_W G_{yw} \times \frac{1}{s} I_{n_u},$$

the response $y(t)$ of any output step disturbance vector satisfies

$$T_W \cdot y(\infty) = 0 \quad (11)$$

in steady state. According to proposition 1, $V_2 y(\infty) = V_2 d = 0$. So,

$$\left[\begin{array}{c} V_2 \\ T_W \end{array} \right] \cdot y(\infty) = 0 \quad (12)$$

holds. Then, $y(\infty) = 0$ is guaranteed only if V_y has full column rank. This proves (i).

Suppose $G_{zw}(s) \in \mathcal{RH}_\infty$. Then, it has been already shown in the first part of proof that (12) holds. Since V_y is nonsingular, $y(\infty) = 0$ follows immediately. This proves (ii). ■

Remark 1. As V_y is nonsingular, it follows that the number of rows of T_W must be $n_1 = n_u - n_z$. □

Remark 2. In the controller design, a good starting point to tune T_W is to choose it satisfying the relation $T_W \perp V_z$. □

4.2 A generalized plant

If other specifications are required, one needs only to add extra exogenous input and controlled output to the plant P . An example of mixed sensitivity problem is shown in Fig.1, where the case of plant having additive perturbation is considered. The stable weight $W_T = \{A_T, B_T, C_T, D_T\}$ is used to model the bound of additive perturbations. The realization of the generalized plant of Fig.1 is given by

$$G = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ \hline C_2 & D_{21} & 0 \end{array} \right] = \left[\begin{array}{ccc|ccc} A_p & 0 & 0 & 0 & B_p & \\ 0 & A_T & 0 & 0 & B_T & \\ T_W C_p & 0 & 0_{n_1} & T_W & 0 & \\ \hline 0 & C_T & 0 & 0 & D_T & \\ 0 & 0 & I_{n_1} & 0 & 0 & \\ \hline C_p & 0 & 0 & I & 0 & \end{array} \right] \quad (13)$$

The comprehensive stabilizability of this generalized plant requires that ^{1), 2)}:

- A1)** There should exist no eigenvalues of A both uncontrollable from u and unobservable from y .
- A2)** Among the unstable eigenvalues of A , those uncontrollable from u (resp. unobservable from y) should be invariant zeros of G_{12} (resp. G_{21})
- A3)** $[B_2^T \ D_{12}^T]^T$ should be full column rank and $[C_2 \ D_{21}]$ should be full row rank.

This is verified below.

- A1)** The only unstable and unobservable modes of (C_2, A) are eigenvalues of $A_S = 0_{n_1}$, which are controllable in (A, B_2) if T_W is chosen as in Proposition 2.
- A2)** Since for $Y_1^T = [0 \ 0 \ -I]$, $Y_2^T = T_W$

$$\left[\begin{array}{cc} Y_1^T & Y_2^T \end{array} \right] \left[\begin{array}{cc} A & B_1 \\ \hline C_2 & D_{21} \end{array} \right] = A_S \left[\begin{array}{cc} Y_1^T & 0 \end{array} \right]$$

holds, the eigenvalues of A_S are invariant zeros of $G_{21} = \{A, B_1, C_2, D_{21}\}$.

A3) Since B_p has full column rank and C_p has full row rank, this is automatically satisfied.

5. Example

Consider the plant below.

$$P = \left[\begin{array}{ccc|cc} -\tau_1 & -k_{t1} & 0 & k_{t1} & 0 \\ k_1 & 0 & -k_2 & 0 & 0 \\ 0 & -k_{t2} & -\tau_2 & 0 & k_{t1} \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array} \right] \quad (14)$$

This plant has $n_z = 1$ zero at $s = 0$ and the zero vector is

$$V = \left[\begin{array}{cc} V_1 & V_2 \end{array} \right] = \left[\begin{array}{cc|cc} 0 & 1 & 0 & -k_1 & k_2 \end{array} \right] \quad (15)$$

Following Proposition 2 and Remark 2, we choose T_W as

$$T_W = \left[\begin{array}{cc} k_2 & k_1 \end{array} \right] \quad (16)$$

So, the output weighting function is given by

$$W_S = \left[\begin{array}{c|cc} 0 & k_2 & k_1 \\ \hline 1 & 0 & 0 \end{array} \right] \quad (17)$$

6. Conclusion

We have shown that the zeros at $s = 0$ imposes a constraint on the steady state output of the plant. Due to the existence of this zero, the output unstable weight used in the servo synthesis must also be subject to a structural constraint. The constraint on the output unstable weight is also exposed.

If the constraint on the output weighting function is satisfied, one can use Extended $\mathcal{H}_\infty$ control theory to design a controller achieving asymptotic tracking and/or disturbance rejection.

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