Fault-Tolerance of Control Systems with Dihedral Group Symmetry

Reiko TANAKA* and Kazuo MUROTA**

This paper deals with the fault-tolerance of systems that are symmetric with respect to the dihedral group $D_m$. The group $D_m$ generally represents the geometric symmetry of a regular $m$-gon. We reveal the underlying mathematical mechanism of the loss of controllability for $D_m$-symmetric systems induced by failures and derive a necessary and sufficient condition to retain the controllability despite a failure. Moreover, we show the minimum number of functioning modules needed to retain the controllability despite the failures. It serves for a quantitative evaluation of fault-tolerance.

Key Words: fault-tolerance, dihedral group, controllability

1. Introduction

Study on the fault-tolerance of symmetric control systems are mostly concerned with the graph-theoretic connectivity\(^1\)\(^2\)\(^3\)\(^4\) and only few researches with the control theoretic characteristics. This paper discusses the fault-tolerance of symmetric systems with respect to controllability, which is a fundamental characteristic of control systems. We consider the controllability of a system as a characteristic that should be retained in spite of failures in some control channels, and clarify those failures which enable the symmetric system to retain its controllability.

A first attempt in this direction is found in 7), where the fault-tolerance of some symmetric systems has been evaluated. The analysis has revealed the failure patterns that retain the controllability of the system. Whereas 7) has dealt with the restricted class of symmetric systems as ring-type homogeneous systems\(^7\) Fig. 1, for example), the present paper will be concerned with systems with dihedral group symmetry so as to systematize and generalize the results in 7) on the fault-tolerance with respect to the controllability.

An interesting relationship between the symmetry and the fault-tolerance has been observed in 7). That is, when some failures cause a symmetric system to be uncontrollable, the system after the failures has certain symmetry as well. Now, a question comes about if all the symmetric failures cause the symmetric systems to be uncontrollable or not. The examples in Fig. 1 illustrate the motivation of this paper: whereas the system shown in Fig. 1(a) is uncontrollable because of the symmetric failures, the system shown in Fig. 1(b) retains its controllability despite the symmetric failures. Note that the system shown in Fig. 1(a) is symmetric regarding $\pi$ rotations and so is the one in Fig. 1(b) with respect to $\frac{2\pi}{3}$ rotations.

Based on the group representation theory, 8) has revealed the underlying mathematical mechanism of the loss of controllability by some failures for group symmetric systems, and 9) has shown a lower bound for the number of functioning modules needed to keep the entire symmetric system controllable. The present paper refines the results in 8) and 9) by applying them to the dihedral group $D_m$. The results bring us two major consequences. Firstly, it enables us to analyze more complicated $D_m$-symmetric systems (see Fig. 2, for example) including the ring-type systems and to give a necessary and sufficient condition for a $D_m$-symmetric system to keep the controllability despite some failures. Secondly, it leads us to show

\* Faculty of Science and Technology, Keio University, Yokohama
\* Research Institute for Mathematical Sciences, Kyoto University, Kyoto; Faculty of Engineering, University of Tokyo, Tokyo
(Received January 12, 1999)
(Revised March 24, 1999)
the minimum number of functioning modules needed to retain the controllability of the entire \( D_m \)-symmetric system, by showing that the lower bound obtained in \( \alpha \) is tight for some \( D_m \)-symmetric systems. It serves for the quantitative evaluation of fault-tolerance. It should be noted here that we are interested in the characteristics of a system determined by its symmetric structure.

The outline of this paper is as follows. In Section 2, the formulation of the systems treated in this paper is described. In Section 3, we analyze the fault-tolerance of systems with the dihedral group-theoretic symmetry. Section 4 gives the number of the functioning modules needed to retain the controllability. It is based on the decomposition into the orbits with respect to the group action. Preliminaries from group representation theory are given in Appendix.

2. Symmetric failures in symmetric systems

In this section we formulate the notions of symmetric systems and symmetric failure patterns in precise terms.\(^8\)

Consider a linear time-invariant system \( S \) that consists of \( m \) control modules \( \{ S_1, S_2, \ldots, S_m \} \), each of which has its own control channel. The entire system \( S \) is then described by

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t), \\
&= \begin{bmatrix}
A_{11} & A_{12} & \cdots & A_{1m} \\
A_{21} & A_{22} & \cdots & A_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
A_{m1} & A_{m2} & \cdots & A_{mm}
\end{bmatrix} \\
& \quad + \\
& \begin{bmatrix}
B_{11} & B_{12} & \cdots & B_{1m} \\
B_{21} & B_{22} & \cdots & B_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
B_{m1} & B_{m2} & \cdots & B_{mm}
\end{bmatrix} \begin{bmatrix}
u_1(t) \\
u_2(t) \\
\vdots \\
u_m(t)
\end{bmatrix}
\end{align*}
\]

where \( x_i(t) \in \mathbb{R}^{n_i} \) and \( u_i(t) \in \mathbb{R}^{r_i} \) denote the state of \( S_i \) and the input from its control channel, respectively, with \( \mathbb{R}^n \) being the set of real vectors of the size \( n \). The diagonal elements \( A_{ii} \) and \( B_{ii} \) in (1) represent the effects of each module \( S_i \) on its own state \( x_i \). The off-diagonal elements \( A_{ij} \) and \( B_{ij} \) (\( i \neq j \)) imply the interactions among modules through the interconnections. Therefore, if two modules \( S_i \) and \( S_j \) (\( i \neq j \)) are not connected, the corresponding matrices \( A_{ij} \) and \( B_{ij} \) are zero. By denoting the state and the input of the entire system \( S \) as

\[
\begin{bmatrix}
x_1(t) \\
x_2(t) \\
\vdots \\
x_m(t)
\end{bmatrix} \in \mathbb{R}^n, \quad \begin{bmatrix}
u_1(t) \\
u_2(t) \\
\vdots \\
u_m(t)
\end{bmatrix} \in \mathbb{R}^r,
\]

respectively, the equation (1) can be given in the standard form of a state transition equation:

\[
\dot{x}(t) = Ax(t) + Bu(t). \quad (2)
\]

Among the systems described by (2), we are interested in ones with group-theoretic symmetry. We say that the system (2) is symmetric with respect to a finite group \( G \) if

\[
T(g)A = AT(g), \quad T(g)B = BS(g), \quad g \in G, \quad (3)
\]

where \( T \) and \( S \) are unitary representations of \( G \) on \( \mathbb{R}^n \) and \( \mathbb{R}^r \), respectively (see Appendix for terminology from group representation theory). The equation (3) often reflects the underlying geometric symmetry in the system structures. It should be noted here that we are interested in the characteristics of a system determined by its symmetric structure and not by the numerical information of the system matrices \( A \) and \( B \). More precisely, we are mainly interested in a generic system subject to the symmetry constraint defined in terms of the equation (3). Since \( (A, B) \) satisfies this symmetry constraint if and only if \( A \) and \( B \) are block diagonalized on the basis of symmetry (see \( \alpha \), \( \beta \) for details), the genericity under the symmetry condition (3) is equivalent to the genericity of the distinct diagonal blocks in the sense that all of their entries are independent parameters.

**Example.** The formulation above is illustrated for a ring-type homogeneous system, as shown in Fig. 1, consisting of six identical modules \( (m = 6) \) with \( n_1 = n_0 \) and \( r_1 = r_0 \) (\( 1 \leq i \leq 6 \)). The matrices \( A \) and \( B \) in (2) are given as

\[
A = \begin{bmatrix}
P & 0 & 0 & 0 & 0 & 0 \\
Q & 0 & 0 & 0 & 0 & 0 \\
0 & Q & 0 & 0 & 0 & 0 \\
0 & 0 & Q & 0 & 0 & 0 \\
0 & 0 & 0 & Q & 0 & 0 \\
0 & 0 & 0 & 0 & Q & 0 \\
\end{bmatrix}, \quad B = \begin{bmatrix}
K & 0 & 0 & 0 & 0 & 0 \\
0 & K & 0 & 0 & 0 & 0 \\
0 & 0 & K & 0 & 0 & 0 \\
0 & 0 & 0 & K & 0 & 0 \\
0 & 0 & 0 & 0 & K & 0 \\
0 & 0 & 0 & 0 & 0 & K \\
\end{bmatrix},
\]

(1)

where the modules \( \{ S_i \} \) (\( 1 \leq i \leq 6 \)) are indexed clockwise from an arbitrary module. The matrices \( P \) and \( Q \) in (4) are \( n_0 \times n_0 \) and \( K \) is \( n_0 \times r_0 \). The system is therefore symmetric with respect to the dihedral group \( D_6 \). The dihedral group \( D_6 \), of order 12, is defined by

\[
D_6 = \{ e, \rho, \rho^2, \ldots, \rho^5; \sigma, \sigma \rho, \ldots, \sigma \rho^5 \},
\]

with \( \rho^6 = \sigma^2 = (\sigma \rho)^2 = e \) (\( e \) is the identity element). The group \( D_6 \) generally represents the geometric symmetry of a regular hexagon. The representations \( T(\rho) \) and \( T(\sigma) \) of \( G \) are given by

\[
T(\rho) = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}, \quad T(\sigma) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix},
\]

(6)

where \( I \) denotes the unit matrix of order \( n_0 \).

Let \( T(g) \) and \( S(g) \) be decomposed into a direct sum of irreducible representations as

\[
T = \sum_{\mu \in R(G)} a_\mu^* \mu, \quad S = \sum_{\mu \in R(G)} b_\mu^* \mu
\]

(7)

with \( R(G) \) being the index set of all the irreducible representations of \( G \), and \( a_\mu^* \) and \( b_\mu^* \) being the multiplicities of \( \mu \) in the representations \( T \) and \( S \) of \( G \).

In order to discuss the fault-tolerance of the symmetric systems defined above, we restrict the failure here to that of the control channels. Generally, if the control channel
of the module, say \( S_i \), is in the outage or replacement, the control input \( u_i(t) \) has no correlation with the state \( x(t) \). The input from the control channel in the outage has no influence on the state of the entire system. This situation is described in the mathematical model (1) by

\[
  u(t) = 0 \quad (S_i \text{ is in the outage}),
\]

that is, \( S_i \) is supposed to have no control input from its own control channel (1).

According to the failure defined in (8), let \( M \) and \( N \) denote the index sets of the functioning modules and of the modules in the outage, respectively. The failure pattern of the system is thus described by the pair of \( M \) and \( N \).

In addition, we introduce the failure matrix \( F \) of order \( r \) in such a way that the matrix \( BF \) has zero column blocks that correspond to the control channels in the outage. The system after the failures is thus denoted as \( (A, BF) \).

Such a matrix \( F \) is given by \( F = \bigoplus_{i=1}^{n} F_i \) with

\[
  F_i = \begin{cases} 
  I_i, & (i \in M), \\
  O_i, & (i \in N), 
  \end{cases}
\]

where the matrices \( I_k \) and \( O_k \) denote, respectively, the unit matrix and the zero matrix of order \( k \) in general.

This means

\[
  F = F_M \oplus F_N = I_f \oplus O_{r-f}
\]

with \( F_M = \bigoplus_{i \in M} F_i = I_f \) and \( F_N = \bigoplus_{i \in N} F_i = O_{r-f} \), where \( f = \sum_{i \in M} r_i \) by an appropriate permuation of the indices of the modules. Note that any failure pattern can be given in the form of (9).

The symmetry of a failure pattern can then be formulated similarly to (3) for the matrix \( F \) in (9). A failure pattern \( F \) is said to be symmetric with respect to a subgroup \( H \) of \( G \) if

\[
  S(h)F = FS(h), \quad h \in H.
\]

Note that the symmetry of failure patterns is defined with respect to a subgroup \( H \) of \( G \). Given a subgroup \( H \) of \( G \), we have a set of failure patterns \( F \) that satisfy (10). Conversely, for a given \( F \),

\[
  H(F) = \{ g \in G \mid S(g)F = FS(g) \}
\]

is a subgroup of \( G \) and can be chosen as the subgroup \( H \) in (10).

**Remark 1.** From the equations (3) and (10), the system after the failures \( (A, BF) \) with \( BF = BF \) is also symmetric with respect to \( H \) in the sense of (3), since

\[
  T(h)A = AT(h), \quad T(h)BF = BF S(h), \quad h \in H.
\]

(1) In the study of reliable control (2), it is customary to define a failure of a system as the output of the actuators being zero by a switching condition.

Therefore, the symmetry of the original system and that of the failure patterns yield a partial symmetry of the system after the failures.

From the assumption above, the form of the unitary representation matrices \( S(h) \) satisfying (10) is restricted to

\[
  S(h) = S_M(h) \oplus S_N(h), \quad h \in H,
\]

where \( S_M(h) \) is of order \( f \) and \( S_N(h) \) is of order \( r-f \), corresponding to the blocks of \( F \) in (9). Namely, the representation matrices \( S(h) \) in (10) splits into diagonal blocks for all \( h \in H \). Let \( S_M(h) \) and \( S_N(h) \) be decomposed into a direct sum of irreducible representations as

\[
  S_M = \sum_{\nu \in R(H)} b_M^\nu \nu, \quad S_N = \sum_{\nu \in R(H)} b_N^\nu \nu
\]

with \( R(H) \) being the index set of all the irreducible representations of \( H \), and \( b_M^\nu \) and \( b_N^\nu \) being the multiplicities. Similarly, let \( \alpha_M^\nu \) denote the multiplicity of \( \nu \) in the restriction of \( \mu \) to \( H \), i.e.,

\[
  \mu \downarrow H = \sum_{\nu \in R(H)} \alpha_M^\nu \nu.
\]

### 3. Fault-Tolerance of \( D_m \)-symmetric systems

This section shows the group-theoretic nature of loss of controllability by some failures for \( D_m \)-symmetric systems through a refinement of the general results obtained in 8). The dihedral group \( D_m \) generally represents the geometric symmetry of a regular \( m \)-gon. We will show a necessary and sufficient condition for generic \( D_m \)-symmetric systems to retain their controllability despite failures.

The dihedral group \( D_m \), of order \( 2m \), is defined as

\[
  D_m = \{ e, \rho, \cdots, \rho^{m-1}, \sigma, \sigma \rho, \cdots, \sigma \rho^{m-1} \},
\]

where \( \rho^m = e = \rho^2 \). The index set of all the irreducible representations of \( D_m \), denoted as \( R(D_m) \), is given by

\[
  R(D_m) = \{ \{ A_1, A_2, A_3 \} \mid m \text{ is even} \}, \quad \{ \{ A_1, A_2, A_3 \} \mid m \text{ is odd} \},
\]

where \( A_1, A_2, A_3 \) and \( B_2 \) are one-dimensional irreducible matrix representations given as

\[
  T^{A_1}(g) = \begin{cases} 
  1 & (g \in D_m), \\
  -1 & (g \in \{ \rho^i \}),
  \end{cases}
\]

with \( 0 \leq i \leq m-1 \),

\[
  T^{B_1}(g) = \begin{cases} 
  1 & (g \in \{ \rho^{2 j}, 2^{j+1} \}), \\
  -1 & (g \in \{ \rho^{2 j + 1}, 2^{j+1} \}),
  \end{cases}
\]

with \( 0 \leq j \leq \frac{m}{2} - 1 \),

\[
  T^{B_2}(g) = \begin{cases} 
  1 & (g \in \{ \rho^{2 j + 1}, 2^{j+1} \}), \\
  -1 & (g \in \{ \rho^{2 j}, 2^{j+1} \}),
  \end{cases}
\]

with \( 0 \leq j \leq \frac{m}{2} - 1 \), and \( E_k (k = 1, 2, \ldots) \) are two-dimensional ones given as
Theorem 1. A $D_m$-symmetric system $(A, B)$ retains its controllability in spite of the $D_p$-symmetric (p is a divisor of m) failure $F$ only if there exists no pair of irreducible representations $\rho$ of $D_m$ and $\nu$ of $D_p$ such that

$$a_\rho \neq 0, \quad a_\nu \neq 0, \quad b_M^\rho = 0,$$

where $a_\rho, a_\nu$ and $b_M^\rho$ are defined by (7), (14) and (13), respectively.

Example. Consider a system shown in Fig. 2(a), where $n_i = r_i = 1$ for $i = 1, \ldots, 6$. It is symmetric with respect to $D_6$. The index set of all the irreducible representations is given as $R(D_6) = \{A_1, A_2, B_1, B_2, E_1, E_2\}$ by (16) for $m = 6$.

The symmetry condition (3) holds for $T$ and $S$ defined naturally as representations of permutations. The irreducible representation decomposition of $S$ of $D_6$ is described as $S = 3A_1(+)2B_1(+)B_2(+)3E_1(+)3E_2$. (18)

The system shown in Fig. 2 (b) has the $D_3$-symmetric failure pattern $M = \{2, 4, 6, 7, 11, 15\}$. The restrictions of the irreducible representations of $D_6$ to $D_3$ are given as

$$A_1 \downarrow D_3 = A_1, \quad B_1 \downarrow D_3 = A_1, \quad E_1 \downarrow D_3 = E_1,$$

$$B_2 \downarrow D_3 = A_2, \quad E_2 \downarrow D_3 = E_1.$$

The representations $S_M$ and $S_N$ of $D_3$ are decomposed as $S_M = 2A_1 \oplus 2E_1, \quad S_N = 3A_1 \oplus 3E_1 \oplus 3E_2$.

Since the condition (17) holds for $\mu = B_2 \in R(D_6)$ and $\nu = A_2 \in R(D_3)$, Theorem 1 reveals that the system is uncontrollable. It is worth mentioning that $D_3$-symmetric failures in $D_6$-symmetric ring-type homogeneous system do not cause the system to be uncontrollable, as has been shown in 7) (see Fig. 1(b)).

For the $D_2$-symmetric failure pattern with $M = \{1, 4, 7, 10, 13, 16\}$, as shown in Fig. 2(c), the representations $S_M$ and $S_N$ of $D_2$ are decomposed as $S_M = 3A_1 \oplus 2B_1 \oplus B_2, \quad S_N = 3A_1 \oplus 3A_2 \oplus 3B_1 \oplus 3B_2$,

$$A_1 \downarrow D_2 = A_1, \quad A_2 \downarrow D_2 = A_2, \quad B_1 \downarrow D_2 = B_1, \quad B_2 \downarrow D_2 = B_2, \quad E_1 \downarrow D_2 = B_1 + B_2, E_2 \downarrow D_2 = A_1 + A_2.$$

Since the condition (17) holds for $\mu = E_2 \in R(D_6)$ and $\nu = A_2 \in R(D_2)$, Theorem 1 clarifies that the system is uncontrollable.

It should be mentioned that the condition (17) often turns out to be sufficient. Therefore, as a rule of thumb, we may hopefully expect that the system retains its controllability if the condition (17) is not satisfied by any $\mu \in R(D_m)$ and $\nu \in R(D_p)$. In particular, for the case $D_p = D_m$ in Theorem 1, the condition (17) is necessary and sufficient, as has been shown in Theorem 4 of 8).

Example. In a $D_8$-symmetric system shown in Fig. 2(a), a failure with $M = \{7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 18\}$ is $D_8$-symmetric. The representations $S_M$ and $S_N$ of $D_8$ are decomposed as $S_M = 2A_1 \oplus B_1 \oplus B_2 \oplus 2E_1 \oplus 2E_2, \quad S_N = A_1 \oplus B_1 \oplus E_1 \oplus E_2$.

Since the condition (17) does not hold for any $\mu(=\nu) \in R(D_8)$, the system retains its controllability.

We will show, in the following, a necessary and sufficient condition for the controllability of the system $(A, BF)$ for a generic $D_m$-symmetric system $(A, B)$. The condition will be given with reference to a unitary matrix $W$ of order $r$, where $W = (W^\mu | \mu \in R(G))$ consists of $r \times b^\nu$ submatrices $W^\nu$ representing the basis of the invariant subspace corresponding to $D^\nu$. Moreover, let $W_M^\rho$ denote the first $f$ rows of $W^\rho$ as $W^\rho = \left[ W_M^\rho, \, W_N^\rho \right]$, corresponding to the failure pattern, with $W_M^\rho = (W_{jk}^\rho | 1 \leq k \leq N^\rho), \quad W_{M_1}^{\rho j} = (w_{ij}^j | 1 \leq k \leq b^\nu) \in C^{f \times b^\nu}$ and $(W_{jk}^\rho)^* = (w_{ij}^j)^* = (w_{ij}^j^*)^* \quad 1 \leq i \leq f)$. Note that the matrix $W$ for the dihedral group $D_m$ can be obtained in an explicit form.
A necessary and sufficient condition for the controllability is given as follows.

**Theorem 2.** A generic \( D_m \)-symmetric system \((A, B)\) retains its controllability despite a failure \( F \) if and only if both of the following conditions are satisfied:

1. For each \( \mu \in R(D_m) \) such that \( N^\mu = 1 \) and \( a^\mu \neq 0 \), there exists an integer \( j^\mu \) (\( 1 \leq j^\mu \leq b^\mu \)) such that \( \text{rank} \left[ w_{j^\mu j^\mu} \right] = 1 \).
2. For each \( \mu \in R(D_m) \) such that \( N^\mu = 2 \) and \( a^\mu \neq 0 \), either (a) or (b) is satisfied:
   a. There exists an integer \( j^\mu \) (\( 1 \leq j^\mu \leq b^\mu \)) such that \( \text{rank} \left[ w_{j^\mu j^\mu} \right] = 2 \).
   b. There exist two pairs of indices \((i_1^\mu, i_2^\mu)\) (\( 1 \leq i_1^\mu, i_2^\mu \leq f \)) and \((j_1^\mu, j_2^\mu)\) (\( 1 \leq j_1^\mu, j_2^\mu \leq b^\mu \)) such that 
   \[
   \det \begin{bmatrix}
   w_{i_1^\mu j_1^\mu} & w_{i_2^\mu j_1^\mu} \\
   w_{i_1^\mu j_2^\mu} & w_{i_2^\mu j_2^\mu}
   \end{bmatrix} + \det \begin{bmatrix}
   w_{i_1^\mu i_1^\mu} & w_{i_2^\mu i_1^\mu} & w_{i_1^\mu i_2^\mu} & w_{i_2^\mu i_2^\mu}
   \end{bmatrix} \neq 0.
   \]  

\( \square \)

(Proof) Proof is shown in Appendix.

The theorem above implies the following necessary condition for the controllability.

**Corollary 1.** If a \( D_m \)-symmetric system \((A, B)\) becomes controllable because of a failure \( F \), then both of the following conditions are satisfied:

1. Condition (1) of Theorem 2.
2. For each \( \mu \in R(D_m) \) such that \( N^\mu = 2 \) and \( a^\mu \neq 0 \), there exists a pair of indices \( \{j_1^\mu, j_2^\mu\} \) (\( 1 \leq j_1^\mu, j_2^\mu \leq b^\mu \)) such that 
   \[
   \text{rank} \left[ w_{j_1^\mu j_1^\mu} \right] \text{ or } \left[ w_{j_2^\mu j_2^\mu} \right] = 2.
   \]

\( \square \)

(Proof) If the system \((A, BF)\) is controllable, it is generically controllable. Note that if (2)(b) of Theorem 2 is satisfied, either the first term or the second one of (21) is nonzero. Therefore, the condition (2) follows, which also contains the case (2)(a) of Theorem 2.

With the help of Theorem 2, the controllability of the system \((A, BF)\) is systematically determined in the following three steps.

**Step 1:** If (1) of Theorem 2 is not satisfied, then the system is uncontrollable. Otherwise go to Step 2.

**Step 2:** If (2)(a) of Theorem 2 is satisfied, then the system is controllable. Otherwise go to Step 3.

**Step 3:** The procedure to check for (2)(b) of Theorem 2 is summarized as follows.

1. Transform the matrix \( W_{M'} \) as \( U(W_{M'}^\mu)^*V \) with arbitrary nonsingular matrices \( U \) and \( V \), in such a way that the resulting submatrix \( W_{M_1}^\mu \) takes the rank normal form. From the assumption, the transformed matrix \( W_{M_3}^\mu = (W_{M_1}^\mu, W_{M_2}^\mu) \) is given as

\[
W_{M_1}^\mu = \begin{bmatrix}
1 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0
\end{bmatrix},
\]

\[
W_{M_2}^\mu = \begin{bmatrix}
\begin{array}{cccccc}
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{array}
\end{bmatrix},
\]

where \( l = \text{rank} \ W_{M_1}^\mu \).

2. We may judge the condition (2)(b) of Theorem 2 for the transformed matrix. The condition is satisfied in the following two cases:
   a. \( 1 \leq i_1 = j_1 \leq l, 1 \leq i_2 = j_2 \leq l, i_1 \neq i_2, \) and \( k_{j_1} \neq k_{j_2} \).
   b. \( 1 \leq i_1 = j_1 \leq l, l+1 \leq j_2 \leq b^\mu, i_1 \neq i_2, \) and \( w_{i_2 j_2} \neq 0 \).

4. **Quantitative Evaluation of Fault-Tolerance**

This section aims to clarify the minimum number of the functioning modules needed to keep the entire \( D_m \)-symmetric system controllable in spite of the failures, and show the failure patterns to attain the controllability by the minimum number of functioning modules. The discussion will be based on the orbit decomposition of the system. For simplicity of discussion, we deal with the cases where \( n_i = r_i = 1 (1 \leq i \leq m) \), i.e., each module has a one-dimensional state and input. However, a similar argument is possible for systems with higher-dimensional modules. In the following, we assume the representations \( T \) and \( S \) of \( D_m \) as permutations.

We denote by \( P \) the index set of the elements in the original system. Let

\[
P = \bigcup_{l=1}^{L} P_l
\]

be the decomposition of \( P \) into \( L \) disjoint orbits with respect to the action of a finite group \( G \). Similarly to (22), the index set \( M \) of the functioning modules are decomposed into \( L \) disjoint orbits as

\[
M = \bigcup_{l=1}^{L} M_l
\]

where \( M_l = M \cap P_l (1 \leq l \leq L) \). Hence \( |M| = \sum_{l=1}^{L} |M_l| \).
The permutation representations \( T \) and \( S \) in (3) are decomposed accordingly as
\[
T(g) = \bigoplus_{i=1}^{L} T_i(g), \quad S(g) = \bigoplus_{i=1}^{L} S_i(g), \quad g \in G. \tag{24}
\]

In this paper, we assume that the representations \( T_i \) and \( S_i \) \((1 \leq i \leq L)\) are faithful.

In (9), we derived the covering condition (25) which gives a lower bound for the number of functioning modules in each type of orbit, needed to keep the controllability of the systems with general symmetry.

**Theorem 3.** If a \( G \)-symmetric system \((A, B)\) retains its controllability in spite of a failure \( F \) with the functioning modules indexed by \( M = \bigcup_{l=1}^{L} M_l \), then
\[
\sum_{l \in M} |M_l| \geq N^\sigma \tag{25}
\]
holds for all \( \mu \in R(G) \) with \( \sigma \neq 0 \), where \( \sum_{l \in M} |M_l| \) denotes the sum of \( |M_l| \) over all orbits \( l \) such that \( T_l \) defined in (24) contains the irreducible representation \( \sigma \).

A tight lower bound for \( |M| \) of \( D_m \)-symmetric systems is then derived as follows.

**Theorem 4.** If a \( D_m \)-symmetric \((m \geq 3)\) system \((A, B)\) retains its controllability in spite of the failure \( F \) with the functioning modules indexed by \( M \), then
\[
|M| \geq 2. \tag{26}
\]
For a generic \( D_m \)-symmetric system with a faithful representation in each orbit, this bound can be attained. (Proof) From (16), \( N^\sigma \leq 2 \) for all \( \mu \in R(D_m) \). Then, (26) is immediate from Theorem 3. Concerning the attainment of the bound for generic systems, we present the basic idea of the proof below, instead of a formal proof.

The orbital decomposition of \( D_m \)-symmetric systems in general is induced from the geometric symmetry, and hence we may assume \([P_l] \in \{m, 2m\}^6\) for faithful permutation representations \( T_l \) and \( S_l \). The orbits are classified into three types, namely, type I, type II and type III, where \([P_l] = m\) for type I and type II, and \([P_l] = 2m\) for type III. The representation \( S \) for each type is decomposed into a direct sum of irreducible representations as follows:

**type I:** \( S = A_1 \otimes B_1 \oplus E_{2m-1} \) \((m: \text{even})\),
\[ A_1 \oplus B_1 \oplus \cdots \oplus E_{2m-1} \) \((m: \text{odd})\),

**type II:** \( S = A_1 \otimes B_2 \oplus E_{2m-1} \) \((m: \text{even})\),
\[ A_1 \oplus B_1 \oplus \cdots \oplus E_{2m-1} \) \((m: \text{odd})\),

**type III:** \( S = A_2 \oplus B_1 \oplus B_2 \oplus 2E_1 \oplus \cdots \oplus 2E_{m-1} \) \((m: \text{even})\),
\[ A_1 \otimes A_2 \oplus B_1 \oplus B_2 \oplus 2E_1 \oplus \cdots \oplus 2E_{m-1} \) \((m: \text{odd})\).

It should be noted that \( D_m \)-symmetric systems in general are obtained by combinations of these three types of orbits. For example, the \( D_6 \)-symmetric system in Fig. 3 consists of three orbits with \( P_l \) \((l = 1, 2, 3)\) given as
\[
P_1 = \{1, 2, 3, 4, 5\}, \quad P_2 = \{7, 8, 9, 10, 11, 12\}, \quad P_3 = \{13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24\},
\]
where \( P_1, P_2 \) and \( P_3 \) are type I, type II and type III, respectively, if \( \sigma \) represents the reflection which leaves the modules \([1, 4]\) invariant.

From the formulation of (25), the orbits of the same type are not distinguished in the covering condition. Consequently, it is enough to consider the systems which contain at most one orbit for each type (Fig. 3, for example). Then, the covering condition (25) for \( D_m \)-symmetric systems reads
\[
A_1: |M_1| + |M_2| + |M_3| \geq 1,
A_2: |M_3| \geq 1,
B_1: |M_1| + |M_3| \geq 1, \quad (m \text{ is even})
B_2: |M_2| + |M_3| \geq 1, \quad (m \text{ is even})
E_k: |M_1| + |M_2| + |M_3| \geq 2,
\]
where \( 1 \leq k \leq \frac{m}{2} - 1 \) for even \( m \) and \( 1 \leq k \leq \frac{m}{2} - 1 \) for odd \( m \). This system of inequalities is satisfied for \( M \) with \( |M| = 2 \) according to the three cases as follows:

1. If \( P_3 = \emptyset \) and \( P_1 \neq \emptyset \) or \( P_2 \neq \emptyset \), \( |M_2| = 2 \) or \( |M_1| = 2 \), respectively.
2. If \( P_3 = \emptyset, P_1 \neq \emptyset \) and \( P_2 \neq \emptyset \), (a) or (b) as below:
   (a) \( |M_1| = 2 \),
   (b) \( |M_1| = 2 \) or \( |M_2| = 2 \) (if \( m \) is odd).
3. If \( P_3 \neq \emptyset, (a) \) or (b) as below:
   (a) \( |M_3| = 2 \),
   (b) \( |M_3| = 1 \) and \( |M_1| = 1 \) or \( |M_2| = 1 \).

Corresponding to (1), (2)(a), (3)(a) and (3)(b), Fig. 4(a), (b), (c) and (d) provide examples of the failure patterns, respectively. The controllability of each system is verified by Theorem 2.

Moreover, some failure patterns corresponding to (1)–(3) above may cause the system to be uncontrollable. The examples of such uncontrollable systems obtained by Theorem 2 are shown in Fig. 1(a) for (1), Fig. 5(a) for (2)(a)
and Fig. 5(b) for (3)(a).

5. Conclusion

In this paper, we have revealed the underlying mathematical mechanism of the loss of controllability for regular polygonal symmetric systems induced by failures. According to the results, we have analyzed more complicated systems with the dihedral group symmetry including the ring-type systems. Moreover, we have shown the minimum number of functioning modules needed to retain the controllability of the entire symmetric system in spite of the failures.

Acknowledgement

This work is supported partly by a Grant-in-Aid of the Ministry of Education, Science, Sports and Culture of Japan and by the Sumitomo Foundation.

References


Fig. 4 Examples of $D_6$-symmetric systems with $|M| = 2$, which retain the controllability despite the failures.

Fig. 5 Examples of the failures with $|M| = 2$ in $D_6$-symmetric systems, which cause the systems to be uncontrollable.

Appendix A. Preliminaries on Group Representation

We describe the notations about group representations (see, for example 5)) utilized in the discussion above.

Let $G$ be a finite group. A representation of $G$ on representation space $V$ is a homomorphism $\tau$. An $n$-dimensional matrix representation of $G$ is denoted as $T$.

We denote by $\{\tau^\mu \mid \mu \in R(G)\}$ a complete list of nonequivalent irreducible representations of $G$, where $R(G)$ denotes an index set for the irreducible representations of $G$. A complete list of nonequivalent irreducible matrix representations of $G$ is denoted by $\{D^\mu \mid \mu \in R(G)\}$. We denote the dimension of $D^\mu$ by $N^\mu$. The matrix representation $T$ can be put into a block-diagonal form

$$T(g) = \bigoplus_{\mu \in R(G)} a^\mu D^\mu(g), \quad g \in G$$

which we write as $T = \sum_{\mu \in R(G)} a^\mu \mu$.

A subgroup $H$ of $G$ is a subset which is itself a group. If $T$ is a representation of $G$ on $V$, we can obtain a representation $T_H$ of any subgroup $H$ of $G$ by restricting $T$,

$$T_H(h) = T(h), \quad h \in H,$n$ and we write $T_H = T \downarrow H$. Similarly, $\mu \downarrow H$ means the restriction to $H$ of the irreducible representation $\mu$ of $G$.

A permutation of a nonempty set $X$ is a 1-1 mapping of $X$ onto itself. The set $S_X$ of all permutations of $X$ forms a group, the full symmetric group on $X$. Elements of $S_X$ are said to act or operate on elements of $X$. Two elements $x$ and $y$ $(x, y \in X)$ are said to belong to the same orbit if and only if $y = \tau(g)(x)$ for some $g \in G$. The orbit containing $x$ is the set $\{\tau(g)x \mid g \in G\}$. If there is only one orbit in $X$ we say $G$ is transitive.

Appendix B. Proof of Theorem 2

By the argument of 8), it suffices to show the controllability of subsystem $(\bigoplus_{k=1}^{N^\mu} A^\mu, (\bigoplus_{k=1}^{N^\mu} B^\mu)(W^\mu)^* F)$ for $\mu$
with $a^n \neq 0$, where $A^n \in \mathbb{C}^{a^n \times a^n}$ and $B^n \in \mathbb{C}^{a^n \times b^n}$ are generic matrices. Note that $N^n = 1$ or 2 for all $n \in \mathbb{R}(D_m)$.

**Lemma 1.** Consider a system $(\bar{A}, \bar{B}W^*)$ with the matrices $\bar{A}$ and $\bar{B}$ given as

$$
\bar{A} = \bigoplus_{i=1}^{N} A_i, \quad \bar{B} = \bigoplus_{i=1}^{N} B_i,
$$

where $A_i \in \mathbb{C}^{a_i \times a_i}$, $B_i \in \mathbb{C}^{a_i \times b}$, and $W = (W_k | 1 \leq k \leq N)$, $W_k = (w_{ij} | 1 \leq j \leq b)$ $\in \mathbb{C}^{i \times b}$ with $(w_{ij})^* = (w_{kj}^*) | 1 \leq i \leq j \leq f)$. Suppose that $\bar{A}$ and $\bar{B}$ are fully dense generic matrices. Then, the system $(\bar{A}, \bar{B}W^*)$ is controllable if and only if

1. For $N = 1$, there exists an integer $j (1 \leq j \leq b)$ such that $\text{rank} \ [w_1j] = 1$.
2. For $N = 2$, either (a) or (b) is satisfied:
   (a) there exists an index $j$ such that $\text{rank} \ [w_1j \ w_{2j}] = 2$.
   (b) there exist two pairs of indices $(i_1, i_2)$ and $(j_1, j_2)$ such that

$$
\det \begin{bmatrix} w_{i_1j_1} & w_{i_2j_1} \\ w_{i_1j_2} & w_{i_2j_2} \end{bmatrix} + \det \begin{bmatrix} w_{i_2j_1} & w_{i_2j_2} \\ w_{i_1j_1} & w_{i_1j_2} \end{bmatrix} \neq 0. \tag{B.2}
$$

**Proof.** By the genericity of the matrix $A$, $A$ can be diagonalized by a unitary matrix, say $U$, in the form

$$
\bar{A} = U^{-1}AU = \text{diag} (\alpha_1, \ldots, \alpha_b), \quad \text{with } \alpha_l \neq \alpha_l' \text{ for } l \neq l'.
$$

We now consider the transformed pair of matrices $\bar{A} = \bigoplus_{i=1}^{N} \tilde{A}_i$ and $\bar{B} = \bigoplus_{i=1}^{N} \tilde{B}_i$, where $\tilde{B} = U^{-1}B$ is a fully dense generic matrix by the genericity of $B$.

Then the system $(\bar{A},\bar{B}W^*)$ is controllable if and only if the modal controllability matrix $(\bar{A} - \alpha I | \bar{B}W^*)$ is of full rank for all $\alpha \in \mathbb{C}$, hence for all $\alpha = \alpha_l (1 \leq l \leq a)$. If $N = 1$, it is satisfied if and only if the rank of the $l$-th row of $\bar{B}W^*$ is one. If $N = 2$, it is satisfied if and only if the $l$-th rows of $\bar{B}_W^*_1$ and $\bar{B}_W^*_2$ are independent. The $l$-th rows of $\bar{B}_W^*$ and $\bar{B}_W^*$ are described as

$$
v_1^* = (\beta_1 \cdots \beta_b)W_1^* = \sum_{j=1}^{b} \beta_j (w_1^j)^* = (\sum_{j=1}^{b} \beta_j w_{1j} | 1 \leq i \leq f),
$$

$$
v_2^* = (\beta_1 \cdots \beta_b)W_2^* = \sum_{j=1}^{b} \beta_j (w_2^j)^* = (\sum_{j=1}^{b} \beta_j w_{2j} | 1 \leq i \leq f),
$$

where the vector $(\beta_1 \cdots \beta_b)$ is generic, from the genericity of $\tilde{B}$.

Therefore, for $N = 1$, the condition is equivalent to rank $v_1 = 1$, hence (1). For $N = 2$, the two vectors $v_1$ and $v_2$ are mutually independent if and only if there exists a pair of indices $(i_1, i_2)$ such that

$$
\det \begin{bmatrix} \sum_{j=1}^{b} \beta_j w_{i_1j} \sum_{j=1}^{b} \beta_j w_{i_2j} \\ \sum_{j=1}^{b} \beta_j w_{i_1j} \sum_{j=1}^{b} \beta_j w_{i_2j} \end{bmatrix} \neq 0. \tag{B.3}
$$

The determinant in (B.3) is a quadratic expression in $\{\beta_j \mid 1 \leq j \leq b\}$, with the coefficient of $\beta_j^2$ being

$$
\begin{bmatrix} w_{i_1j_1} & w_{i_2j_1} \\ w_{i_1j_2} & w_{i_2j_2} \end{bmatrix}
$$

and that of $\beta_i \beta_j (i \neq j)$ being

$$
\begin{bmatrix} w_{i_1j_1} & w_{i_2j_1} \\ w_{i_1j_2} & w_{i_2j_2} \end{bmatrix} + \begin{bmatrix} w_{i_2j_1} & w_{i_2j_2} \\ w_{i_1j_1} & w_{i_1j_2} \end{bmatrix}. \tag{B.5}
$$

Then the inequality (B.3) is satisfied if and only if (B.4) or (B.5) is nonzero, hence (2) follows.

**Theorem 2** is then proved by applying Lemma 1 to each subsystem $((\bigoplus_{i=1}^{N} \bar{A}_i),((\bigoplus_{i=1}^{N} \bar{B}_i)(W_1^*)^*)$ corresponding $\mu \in \mathbb{R}(D_m)$ with $a^n \neq 0$.

---

**Reiko Tanaka** (Member)

She received Bachelor, Master and Doctor Degrees of Engineering from the University of Tokyo in 1993, 1995 and 1998, respectively. She is now an instructor at Faculty of Science and Technology, Keio University. She received SICE paper award in 1998.

**Kazuo Murota** (Member)

He received Bachelor, Master and Doctor Degrees of Engineering from the University of Tokyo in 1978, 1980 and 1983, respectively. He is now a professor at Research Institute for Mathematical Sciences, Kyoto University, and is also a visiting professor at Faculty of Engineering, University of Tokyo. His research interest ranges from discrete mathematics to group-theoretic methods in engineering.