Interpolation of Non-Uniformly Sampled Sequences and Reconstruction of Sampled Sequences with Any Uniform Interval Based on Non-Uniform Sampling Theorem†

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In this paper we present a new non-uniform sampling theorem for band-limited signals, together with an associated interpolation formula of non-uniformly sampled sequences, and propose a practical algorithm for reconstructing sampled sequences of any uniform interval from a non-uniform one by using the frequency sampling technique and FFT algorithm. After a brief review of the ordinary uniform sampling theorem, the non-uniform sampling theorem is given, together with an aliasing free condition being made clear. Based on the fundamentals, an interpolation formula of non-uniformly sampled sequences is derived and a practical implementation algorithm to reconstruct uniformly sampled sequences of any desired interval from a non-uniformly sampled version is proposed by way of the frequency sampling technique and FFT algorithm. The principle as well as the effectiveness of the proposed algorithm are ascertained through numerical experiments with sinusoidal and FM signals of finite bandwidths.

Key Words: non-uniform sampling theorem, aliasing free condition, interpolation formula, frequency sampling technique, fast Fourier transformation

1. Introduction

Since the invention of the sampling theorem for band-limited signals by C. E. Shannon1) and I. Someya2), digital signal processing techniques and systems have made a remarkable progress, resulting in the very high precise and flexible techniques and/or systems using such techniques and high speed microprocessors of general use as multi-rate signal processing3) and DSP (digital signal processor), respectively, being easily available. Most of the digital techniques and/or systems are, however, based on the ordinary sampling theorem by uniform sampling of band-limited signals.

For the non-uniform sampling of band-limited signals, there have been only a little reports, that is, spectral estimation for random sampling using interpolation4), polynomial interpolation and prediction of continuous time processes5) by using random point process-based sampling, and some mathematical developments such as Paley-Wiener non-uniform sampling theorem6), frames theory6), and pseudo-biorthogonal bases7) in Hilbert spaces, rigorously representing and interpolating continuous signals with polynomials of the same order as number of sampled points or reproducing kernels in the spaces, and also estimating predesignated sampled values from those at different non-uniform points in the best approximation in the meaning of defined norm. Nevertheless the rigorous mathematical representations and interpolation formulae of considerable complexity are given, however, no concrete algorithm of engineering importance has been reported except for interpolation of a non-uniformly sampled sequence of band-limited signals and reconstruction of uniformly sampled values of any desired interval by the frequency sampling and FFT algorithm proposed by the authors8).

On the other hand, the image reconstruction from limited angle and/or restricted-region projections or non-uniform sampling of projection data for speedy data acquisition are required7), in the applications such as 2D- or 3D-image reconstruction in computed tomography. The reconstruction problem from such degraded projection data may be refered to as the degraded projection problem. In the current systems, pre- and/or post-processing such as spline and Lagrange interpolation or restoration method are used to compensate some of these degradations9),10).

In this paper, we review a new practical non-uniform

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sampling theorem for band-limited signals, together with an associated interpolation formula of non-uniformly sampled sequences, and the proposed simple implementation algorithm to reconstruct uniformly sampled sequences of any desired interval from a given non-uniformly sampled version by way of the frequency sampling technique and FFT algorithm\(^8\), and present the more thorough numerical experimental validation and the special features of the proposed algorithm.

After a brief review of the the ordinary sampling theorem for band-limited signals, the non-uniform sampling theorem is given, together with the condition of no aliasing error in section 2. In section 3, we propose a practical implementation algorithm for reconstructing uniformly sampled sequences of any desired interval from a given non-uniformly sampled version. In section 4, results of numerical experiments to ascertain the principle as well as the effectiveness of the proposed algorithm are presented, together with some illustrations, revealing the special features of the algorithm. Finally, in section 5, we summarize the obtained results and suggest the associated future applications.

2. Brief Reviews of Sampling Theorems for Band-Limited Signals

2.1 The Sampling Theorem Based on Uniform Sampling

For simplicity of discussions followed, let \( x(t) \) be a real-valued signal squared integrable on a real line, i.e., belonging to \( L^2(\mathbb{R}) \)-space. Then, the Fourier transform and the inverse one exist and are respectively given by

\[
X(f) = \mathcal{F}[x(t)] = \int_{-\infty}^{\infty} x(t) \exp(-j2\pi ft)dt, \quad (1)
\]

\[
x(t) = \mathcal{F}^{-1}[X(f)] = \int_{-\infty}^{\infty} X(f) \exp(j2\pi ft)df, \quad (2)
\]

where \( \mathcal{F}[\cdot] \) and \( \mathcal{F}^{-1}[\cdot] \) denote the Fourier and inverse Fourier transform operators, respectively. In what follows, the Fourier spectrum \( X(f) \) of \( x(t) \) is assumed to be limited within the band \([-f_{\text{max}}, f_{\text{max}}]\). That is,

\[
X(f) \equiv 0, \quad |f| > f_{\text{max}}. \quad (3)
\]

For the signal \( x(t) \), the next theorem is valid\(^1\),\(^2\).

**Theorem 1 (by Uniform Sampling):**

1. No information on \( x(t) \) is lost when it is sampled with equi-spaced interval \( T \) less than or equal to \( 1/(2f_{\text{max}}) \).
2. In actual, \( x(t) \) at any time \( t \) is completely reconstructed from a sampled sequence \( x(nT)(n = -\infty, \cdots, 0, \cdots, \infty) \) by the following interpolation formula by taking \( F_0 \equiv 1/(2T) \):

\[
x(t) = \sum_{n=-\infty}^{\infty} x(nT) \frac{\sin 2\pi F_0(t - nT)}{2\pi F_0(t - nT)}. \quad (4)
\]

2.2 The Sampling Theorem Based on Non-Uniform Sampling

Let us briefly review the non-uniform sampling theorem for band-limited signals proposed previously\(^8\). Let a non-uniformly sampled sequence be given as \( x(t_n)(n = -\infty, \cdots, 0, \cdots, \infty) \) with the maximum \( \Delta t_{\text{max}} \) of sampling interval \( \Delta t_n = t_{n+1} - t_n(n = -\infty, \cdots, 0, \cdots, \infty) \) satisfying the condition

\[
\Delta t_{\text{max}} \leq 1/(2f_{\text{max}}). \quad (5)
\]

Then, by defining \( \Delta_n \equiv (\Delta t_n + \Delta t_{n-1})/2 \), the next theorem is valid for the signal \( x(t) \)\(^8\):

**Theorem 2 (by Non-Uniform Sampling):**

1. No information on \( x(t) \) is lost when it is sampled with non-uniform sampling under the condition given by inequality (5).
2. The continuous signal \( x(t) \) is completely reconstructed from a non-uniformly sampled sequence \( x(t_n)(n = -\infty, \cdots, 0, \cdots, \infty) \) by the following interpolation formula by taking \( F_0 \equiv 1/(2\Delta t_{\text{max}}) \):

\[
x(t) = \sum_{n=-\infty}^{\infty} x(t_n) \Delta_n \frac{\sin 2\pi F_0(t - t_n)\pi F_0(t - t_n)}{2\pi F_0(t - t_n)}, \quad (6)
\]

where \( \Delta_n \equiv \Delta_n/\Delta t_{\text{max}} \) is an amplitude coefficient for a sample \( x(t_n) \) introduced for preserving integral norm by the Fourier and inverse Fourier transformations.

Proof of the **Theorem 2 (by Non-Uniform Sampling)** will be given in an Appendix A. One of drawbacks of the non-uniform sampling theorem is the fact that the sampled values \( x(t_n) \) become dependent one another as in the oversampling of uniform sampling\(^6\), in contrast to the ordinary shown in Theorem 1. Another is the redundant sampling, i.e., oversampling given by the inequality (5), compared with the ordinary, which makes it unimportant at a first glance. Only a simple extension of non-uniform sampling shown in Theorem 2, however, has the great possibilities to make it useful in many applications as suggested previously\(^8\), and the special features that any interpolation techniques using spline functions, Lagrange formula and so on can never provide. Some examples of them will be shown in the later section.
3. Reconstruction of Uniformly Sampled Sequences from A Non-Uniformly Sampled Version

In the previous section, the non-uniform sampling theorem for band-limited signals is given with interpolation formula of infinite non-uniformly sampled sequences. In this section, by utilizing the result, we propose a practical implementation algorithm to reconstruct uniformly sampled sequences of any desired interval from a non-uniformly sampled version of a band-limited signal with non-zero values only on a finite interval, from practical viewpoint.

Without loss of generality, let a non-uniformly sampled N point series \( x(t_n) (n = 0, 1, \cdots, N - 1) \) be given to satisfy the inequality in Eq. (5), and assume \( t_0 = 0 \). Then, by taking \( \Delta_0 = \Delta t_0, \Delta_{N-1} = \Delta t_{N-2} \), and data length \( T_0 = t_{N-1} + (\Delta_0 + \Delta_{N-1})/2 \), we propose the next practical algorithm for reconstructing uniformly sampled sequences of any desired interval from a non-uniformly sampled version of a band-limited signal with non-zero values only on a finite interval, from practical viewpoint.

Algorithm 1:
- Reconstruction of Uniformly Sampled Sequences of Any Desired Interval from A Non-Uniformly Sampled Sequence

\(<\text{Step 1}\>) \text{ (Initial Setting):} \\
According to a designated reconstruction interval \( T \), select an integer \( M \) as the minimum of powers of 2 greater than \( T_0/T \), and the upper limit of reconstruction frequency \( W \) as the minimum of \( F_0 \) and \( 1/(2T) \).

\(<\text{Step 2}\>) \text{ (Frequency Sampling):} \\
Sample the Fourier spectrum of \( \{x(t_n)\} \) by every \( 1/(MT) \) to obtain \( \{\hat{X}_s(k)\} \) as follows:

\[
\hat{X}_s(k) = \sum_{n=0}^{N-1} x(t_n)\Delta_n \exp[-j2\pi kn/(MT)],
\]

\[ k = 0, 1, \cdots, [MTW], \] (7)

\[
\hat{X}_s(k) = 0, \\
kd\] (8)

\[
\hat{X}_s(M - k) = \hat{X}_s(k)^*, \\
kd\] (9)

where \([A]\) denotes a Gauss notation, i.e., the maximum integer not greater than \( A \), and asterisk does a complex conjugate operation.

\(<\text{Step 3}\>) \text{ (Inverse Discrete Fourier Transform):} \\
Obtain the reconstructed sampled sequence \( \{\tilde{x}(nT)\} \) by inverse discrete Fourier transforming the frequency sampled values of the spectrum given by Eqs. (7) to (9) via the FFT algorithm as follows:

\[
\tilde{x}(nT) = \frac{1}{M} \sum_{k=0}^{M-1} \hat{X}_s(k) \exp(j2\pi kn/M),
\]

\[ n = 0, 1, \cdots, M - 1. \] (10)

In the above algorithm, the reconstructed data length is \( MT \), and the values outside of an interval \([0, T_0]\), over which the original signal \( x(t) \) is defined, are assumed to be zero. If we want to reconstruct the sequence over the exact length \( T_0 \), we are required to take \( M = \lfloor T_0/T \rfloor \) and carry out exactly the inverse discrete Fourier transform calculation given by Eq. (10) without use of FFT algorithm.

Moreover, in case of reconstructing uniformly sampled sequences of the lower sampling rate, i.e., \( F_0 > 1/(2T) \), spectral components of frequency higher than the Nyquist one of the reconstructed sequences exist in the original non-uniformly sampled sequence. So, we choose \( W = 1/(2T) \) in Step 1 and exclude these spectral components in Step 2, to avoid the aliasing due to the change of sampling rate. On the contrary, in case of \( F_0 < 1/(2T) \), there is no possibility of such an aliasing, and so we only choose \( W = F_0 \) in Step 1 and directly calculate all the spectral contents in the original sequence in Step 2.

As shown above, the essential part of the algorithm is the direct use of the well known frequency sampling technique in Step 2 to sample with uniform frequency interval the Fourier spectrum of a given non-uniformly sampled sequence. The representation of the spectrum of the non-uniformly sampled sequence is the key point of the newly considered non-uniform sampling theorem stated above. (See Eqs. (7) and (A.2)). With this sampled spectrum under the initial setting in Step 1, desired uniformly sampled sequences of any interval may be reconstructed only once by this simple algorithm. This is one of the special features of the proposed algorithm, and different from the conventional multi-rate signal processing. That is, in the latter, only a rational rate transformation is possible by rather a complicated algorithm based on combination of up and down samplings.

4. Results of Numerical Experiments

To ascertain the principle and effectiveness of the algorithm proposed in the previous section, numerical ex-
Experiments are carried out by assuming sinusoidal signals of various frequencies or FM signals of various bandwidths of amplitude being normalized to 1. In the experiments, only the reconstruction of uniformly sampled sequences of sampling rate higher than the maximum rate of the original sequence is examined, since the authors’ main application points are in the precise information retrieval from given non-uniformly sampled sequences such as precise and simple interpolation of the measured discrete data as suggested previously. However, it may be said that the possibility of reconstructing the lower rate sequences by the proposed algorithm is clear from the principle given in the previous section.

In what follows, signals are assumed to have non-zero values only on an interval \([0,T_0]=[0,1\text{sec}]\), and sampling frequency \(f_s\) is taken constant at 512Hz for uniformly sampled given sequences and changed linearly with a time from 414Hz to 612Hz for non-uniformly sampled versions. Reconstructed uniform interval \(T\) is changed inversely with powers of 2 from \(1/1024\text{sec}\) to \(1/8192\text{sec}\). The instantaneous frequency of FM signals is changed as follows:

\[
x(t) = \sin \left[ 2\pi f_0 t + \frac{\alpha}{\beta} \sin(2\pi \beta t) \right], \quad 0 \leq t \leq 1, \quad (11)
\]

where \(\beta\) is fixed as 0.5Hz, and \(\alpha\) is varied from 5Hz to 100Hz. As the result, the instantaneous frequency is parametrically changed with bandwidth 10Hz to 200Hz.

### 4.1 Preliminary Results of the Reconstruction from Uniformly Sampled Sequences

To test the validity of the proposed algorithm, reconstructions of uniformly sampled sequences of any desired interval from a uniformly sampled version are carried out. Two examples of the reconstructed values are shown in Figs. 1 and 2. In Fig.1, a truncated portion of a sinusoidal signal of frequency 250Hz is discretely reconstructed with sampling interval \(T=1/2048\text{sec}\) from a sampled version of \(f_s=512\text{Hz}\), and the 1st 0.05sec portion is displayed, while in Fig.2, the 1st 0.05sec portion of the reconstructed sequence of an FM signal of instantaneous frequencies 50Hz to 250Hz with the same sampling interval from the uniformly sampled version of \(f_s=512\text{Hz}\) is displayed. In these figures, solid lines denote the theoretically assumed signals, and symbols □ and \(\times\) are the given sampled values and reconstructed ones, respectively.

Since sampled sequences of uniform interval are reconstructed from the uniformly sampled versions in these results, the reconstruction is essentially equivalent to the conventional technique of multi-rate signal processing, i.e., rate transformation, except for the used algorithm and no extra constraint on the sampling rates, the ratio of which should be related rationally each other in the latter. In Fig. 1, the reconstructed values completely coincide with the theoretical ones within the precision of calculation, as the result of precise extraction of the Fourier spectrum according to the frequency sampling exactly every inverse of data length, which is the very multiple of the sinusoidal period. Relative RMS error of the reconstructed values in Fig.2, on the other hand, is about -23.7dB, since in this case the continuous Fourier spectrum of the FM signal is periodically sampled by uniform frequency interval, i.e., equivalently, the interpolation filter of infinite response is truncated for use with a rectangular or do-nothing window in time domain.

More precisely, however, it is clearly seen from the results that values of sinusoidal or pseudo-sinusoidal components are well reconstructed even from given small sam-

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**Figure 1** The 1st 0.05sec portion of the reconstructed sampled values of uniform interval \(T=1/2048\text{sec}\) from a uniformly sampled sequence of a sinusoidal signal of frequency 250Hz with sampling frequency \(f_s=512\text{Hz}\).

**Figure 2** The 1st 0.05sec portion of the reconstructed sampled values of uniform interval \(T=1/2048\text{sec}\) from a uniformly sampled sequence of an FM signal of instantaneous frequencies changing from 50Hz to 250Hz with sampling frequency \(f_s=512\text{Hz}\), by taking parameters as \(f_0=150\text{Hz}\) and \(\alpha=100\text{Hz}\) in Eq. (11).
pled values, for example, around the 1st 0.01sec portion of the signals, in contrast to the conventional interpolation schemes such as using spline functions, polynomial interpolation, Lagrange one, and so on, which smoothly connect the given sampled values. This is one of the special features of the proposed algorithm based on the Fourier theory.

4.2 Results of the Reconstruction from Non-uniformly Sampled Sequences

4.2.1 Fundamental Results of the Reconstruction

After the validation of the proposed reconstruction algorithm from uniformly sampled sequences being ascertained, examination for reconstruction of uniformly sampled sequences from a non-uniformly sampled version is carried out. Two examples of the reconstructed results are shown in Figs. 3 and 4, where the same representation of the values as in the previous figures are used. Fig. 3 shows the result for a sinusoidal signal of frequency 200Hz, while Fig. 4 is that for an FM signal of instantaneous frequencies 5Hz to 195Hz.

As the results of the non-uniform sampling in time domain and the uniform frequency sampling, relative RMS errors of the reconstructed values in Figs. 3 and 4 are about -29.5dB and -30.8dB, respectively.

Relative RMS errors of reconstructed values for all the experiments due to the above samplings are, however, at most -23.0dB. If we want to establish the more precise reconstruction, a usual technique for windowing the spectrum before the inverse discrete Fourier transformation may be expected to be effective for the purpose 3).

4.2.2 Some Characteristic Results of the Reconstruction

By introducing some standard windows before the inverse discrete Fourier transformation, effect of the windows is evaluated from the viewpoint of relative RMS errors of the reconstructed values from a non-uniformly sampled sequence of a sinusoidal signal. At that time, frequency of the sinusoidal signal is parametrically changed from 50Hz to 200Hz, while reconstructed uniform time interval T is kept as 1/2048sec. The experimentally evaluated results of relative reconstructed RMS errors against the sinusoidal frequency are shown in Fig. 5, where 4 standard windows, i.e., rectangular, tapering, Hanning, and Blackman ones are used for comparison.

It is seen from the results that relative RMS errors of the reconstructed sequences increase approximately from -57dB to -29dB, with signal frequencies, i.e., relative interpolation rates, since the non-uniform sampling of sinusoidal signals and reconstruction interval are fixed. Moreover, the RMS errors are improved by about 2.0dB to 2.5dB with introduction of the windows. This fact is surmised as the result of reduced impulse response of the equivalent interpolation filter through windows of low-pass characteristics decaying smoothly to zero.

Fig. 6 shows similar results for relative RMS errors of the reconstructed sequences versus reconstructed data points, when a non-uniformly sampled sequence of a sinusoidal signal of frequency 200Hz is used with standard windows, and reconstruction uniform interval T is changed from 1/1024sec to 1/8196sec.

In these cases, relative RMS errors of the reconstructed sequences are also improved by about 1.5dB to 2.5dB with introduction of the windows, and further the more reconstructed data points become, the more the RMS errors are rather to be improved. The latter means that the more interpolation rate is increased, the more
Fig. 5 Relative RMS errors of reconstructed sequences versus signal frequency, when sinusoidal signals are sampled non-uniformly with sampling frequency $f_s$ being changed linearly with time from 414Hz to 612Hz, and reconstructed uniform interval $T$ is fixed as $1/2048$ sec.

The precise reconstruction of uniformly sampled sequences can be expected, which is the special feature of the proposed algorithm and cannot be established by any conventional interpolation using spline functions and polynomials.

Fig. 6 Relative RMS errors of reconstructed sequences versus reconstructed data points, when a non-uniformly sampled sequence of a sinusoidal signal of frequency 200Hz is used with standard windows, and reconstructed uniform interval $T$ is changed from $1/1024$ sec to $1/8196$ sec.

5. Conclusions

In this paper, after a new non-uniform sampling theorem for band-limited signals is given, together with an associated interpolation formula of non-uniformly sampled sequences under the aliasing error free condition, a practical algorithm for reconstructing sampled sequences of any uniform interval from a non-uniformly one is proposed by using the frequency sampling technique and FFT algorithm, the special features as well as effectiveness of the proposed algorithm are ascertained through numerical experiments with sinusoidal or FM signals of finite bandwidths, and the following main results are obtained:

1. The proposed non-uniform sampling theorem for band-limited signals and algorithm for reconstructing uniformly sampled sequences completely include the ordinary uniform theorem and the conventional rate transformation as a special case of uniform sampling, respectively.

2. The proposed algorithm works well for all the examined sinusoidal or FM signals under the relative RMS errors of reconstructed values less than -23dB, and the errors can be reduced to less than about -30dB by simply introducing standard windows before the inverse discrete Fourier transformation.

3. The proposed algorithm has the following special features, compared with the conventional interpolation using spline functions and polynomials:

   (i) Correct amplitudes of sinusoidal or pseudo-sinusoidal components can be reconstructed even from consecutive small sampled sequences in a manner which may be optimum from the viewpoint of the Fourier theory.

   (ii) The precision in the relative RMS error of reconstructed values is rather improved with interpolation rate.

   As mentioned previously, the proposed sampling theorem itself is classified into the oversampling, so that it has no value from viewpoint of data quantity required for sampling band-limited signals, compared with the ordinary uniform or Paley-Wiener non-uniform one, and moreover the independence among sampled values is lost in the interpolation formula.

   The above special features, however, are the results of the redundant sampling, and this is the very nature for emphasizing the non-uniform sampling theorem in this paper, since many applications such as a new flexible 2D-image reconstruction from degraded projections without any interpolation proposed in reference 12), possibility of precise information extraction in wavelet transform suggested in reference 8), and so on, may be reduced to the redundancy. Some results will be reported in near future, but most of them are left for the future works.

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References

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Appendix A. Proof of the Theorem 2

In order to preserve a norm of integral, by using an associated time interval $\Delta_n \equiv (\Delta t_n + \Delta t_{n-1})/2$ to a sampled value $x(t_n)$, and Dirac's delta function, express a sampled sequence $\{x(t_n)\}$ as a continuous function $x_s(t)$ formally as follows:

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(t_n) \Delta_n \delta(t - t_n). \tag{A.1}$$

Then, its Fourier spectrum $X_s(f)$ is given by

$$X_s(f) = \sum_{n=-\infty}^{\infty} x(t_n) \Delta_n \exp(-j2\pi fn). \tag{A.2}$$

Consider the adjacent 2 terms in the sum in the right side of Eq.(A.2), denote it by $\Delta X_s(f; t_n, t_{n+1})$, and modify it as

$$\Delta X_s(f; t_n, t_{n+1}) = x(t_n) \exp(-j2\pi f t_n) + x(t_{n+1}) \exp(-j2\pi f t_{n+1})$$

$$\Delta X_s(f; t_n, t_{n+1}) = x(t_n) \exp(-j2\pi f t_n) - x(t_{n+1}) \exp(-j2\pi f t_{n+1})$$

$$\Delta X_s(f; t_n, t_{n+1}) = \exp(-j2\pi f t_n) \Delta X_s(f; t_n, \Delta t_n). \tag{A.3}$$

The 2nd factor in the right side of Eq.(A.3), is denoted as $\Delta X_s(f; t_n, \Delta t_n)$ in Eq.(A.4). It reflects the essential Fourier spectral component associated with the 2 sampled values $x(t_n)$ and $x(t_{n+1})$, while the 1st linear phase factor represents information on temporal location of the spectral component.

By the way, $\Delta X_s(f; t_n, \Delta t_n)$ is periodic with a period $1/\Delta t_n$ with respect to $f$. Thus, an aliasing may be aroused with this period. Since $X_s(f)$ can be expressed by a half of the infinite sum of these $\{\Delta X_s(f; t_n, t_{n+1})\}$, the aliasing free condition can be reduced to

$$\Delta t_n \leq \frac{1}{2f_{\text{max}}}, \quad (\text{for all } n). \tag{A.5}$$

That is, the sampling condition given by the inequality (5) should be satisfied. As far as the condition is satisfied, no information on the original signal $x(t)$ is lost by the non-uniform sampling.

In actual, by defining $F_0 \equiv 1/(2f_{\text{max}})$, extracting information on $X_s(f)$ only over frequency interval $[-F_0, F_0]$ with truncation of $X_s(f)$, and inverse Fourier transforming, we obtain the final interpolation formula given by Eq.(6). This completes the proof of the theorem 2.

(Q.E.D.)