On the Closed-Loop Structure of $H^\infty$ Control Systems

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In this paper, we show that the order of the closed-loop $H^\infty$ control systems using central controllers is determined by the sum of the numbers of stable invariant zeros of $P_{12}(s)$ and $P_{21}(s)$. This fact gives a sharp contrast with the LQG case where the order of the closed-loop system is always identical to that of the plant. Furthermore, using this result, we derive a new explicit form of the closed-loop transfer function of $H^\infty$ control systems based on the chain-scattering approach, which clarifies the fundamental structure of $H^\infty$ control systems.

Key Words: $H^\infty$ control, chain-scattering approach, closed-loop system, McMillan degree, invariant zero

1. Introduction

In the 1980's, the standard $H^\infty$ control problem has been studied by many researchers\(^1\)~\(^4\),\(^16\),\(^17\),\(^19\). As a result, various methods and algorithms for constructing $H^\infty$ controller have been established and $H^\infty$ control has become a powerful tool for robust control. Actually, inexpensive software packages for computing $H^\infty$ controllers are now commercially available for most users and they enable us to easily apply $H^\infty$ control to real control systems using them. Furthermore, in order to deal with more complicated systems, a variety of $H^\infty$ control techniques has been developed in the last decade\(^5\)~\(^10\).

It is questionable, however, that nothing serious remains to be solved. In particular, the closed-loop structure of $H^\infty$ control systems are still to be exploited, according to our opinion. Our investigation in this paper will allow us to enhance the understanding of $H^\infty$ control theory, which is quite rich in a logical structure itself. Such a theoretical background of $H^\infty$ control will give us a deep insight and be helpful to deal with the application for more complicated systems.

In LQG control\(^12\)~\(^15\), or in any other control scheme that uses quasi state feedback, i.e., state feedback with an observer, the order of a closed-loop system is identical to that of a plant. The insertion of a controller does not increase the intrinsic complexity of a control system. This is a remarkable property of modern control methods. In $H^\infty$ control, however, the situation is different. The order of a closed-loop system is no longer identical to that of a plant. Instead, the McMillan degree of a closed system is determined by a different factor. In the so-called one-block case, i.e., if both $P_{12}$ and $P_{21}$ are square but not necessarily of the same size, this problem has been fully analyzed based on the classical interpolation theory\(^16\),\(^17\). Furthermore, it has been pointed out that all stable invariant zeros of $P_{12}$ and $P_{21}$ are hidden modes of the generalized $H^\infty$ control systems\(^6\).

In this paper, we derive an explicit form of the closed-loop transfer function of an $H^\infty$ control system, which has not been derived so far within the knowledge of the authors\(^1\). This derivation is carried out in the chain-scattering framework\(^11\). There, the solvability of $H^\infty$ control problem is reduced to the existence of $J$-lossless factorization for the chain-scattering form of a plant. Our result is derived by calculating the $J$-lossless factor directly, which is discussed in Section 3 in detail. Based on our representation of a closed-loop system, we prove the fact pointed out by Liu and Mita\(^6\) in a more explicit way. Our result, which holds for a four-block case, too, is a generalization of the results by Limebeer et al.\(^16\),\(^17\).

As a consequence of this result, we shall also provide a new interpretation of maximum augmentation, which is an important tool in the chain-scattering approach\(^11\).

Notation: We write the transfer function derived from a state-space representation as
\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
:= D + C(sI - A)^{-1}B.
\]

We write $H^\infty$ norm, a maximum singular value and a maximum eigenvalue as $\|\cdot\|_\infty$, $\sigma(\cdot)$ and $\lambda_{\text{max}}(\cdot)$, respectively. We denote a set of all rational stable proper matrix whose norm is less than 1 as $BH^\infty$. The homographic transformation is given by
\[
HM\left(\begin{bmatrix}
\Theta_{11} & \Theta_{12} \\
\Theta_{21} & \Theta_{22}
\end{bmatrix}; S\right):= (\Theta_{11}S + \Theta_{12})(\Theta_{21}S + \Theta_{22})^{-1}.
\]

This transformation has the following cascade property, which is one of advantages to use the chain-scattering approach rather than the LFT approach to

\(^1\) Our previous paper\(^18\) contains incorrect result. This paper is the modified version and also provides some new results.

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$H^\infty$ control. For two systems $\Theta_1$ and $\Theta_2$, there holds

$$HM(\Theta_1; HM(\Theta_2; S)) = HM(\Theta_1\Theta_2; S).$$

2. Preliminaries

Consider the plant

$$\begin{bmatrix} z \\ y \\ w \\ u \end{bmatrix} = P(s) \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{bmatrix} \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} \begin{bmatrix} z \\ w \\ u \end{bmatrix},$$

(1)

where $z$ is a control output ($\dim(z) = m$), $y$ is an observation output ($\dim(y) = q$), $w$ is an exogenous input ($\dim(w) = r$) and $u$ is a control input ($\dim(u) = p$), respectively. The state-space form of the plant is represented as

$$P(s) = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{bmatrix} = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix}.$$  

(2)

Consider the controller $u = K(s)y$. Let $\Phi(s)$ be the closed-loop transfer function from $w$ to $z$ (Fig. 1), which is given by

$$\Phi(s) = P_{11}(s) + P_{12}(s)K(s)(I - P_{22}(s)K(s))^{-1}P_{21}(s).$$  

(3)

Fig. 1 A closed-loop system

The $H^\infty$ control problem is to find all controllers $K(s)$ that satisfy $\|\Phi(s)\|_\infty < \gamma$ and internally stabilize the closed-loop system. A solvability condition has been obtained by various methods$^{11),19),20)$, In this paper, we take the chain-scattering approach$^{11) to investigate the closed-loop structure of $H^\infty$ control systems.

We make the following assumptions.

(A1) $(A, B_2)$ is stabilizable and $(C_2, A)$ is detectable.

(A2) $P_{12}(s)$ and $P_{21}(s)$ have no invariant zeros on the $j\omega$-axis.

Before stating the solvability condition of the $H^\infty$ control problem, we introduce some notation which is used throughout this paper:

$$B := [B_1, B_2], \quad C_2 := \begin{bmatrix} C_1 \\ 0 \end{bmatrix}, \quad D_2 := \begin{bmatrix} D_{11} & D_{12} \\ I_r & 0 \end{bmatrix},$$

$$C := \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, \quad B_w := [B_1, 0], \quad D_w := \begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix},$$

$$J_r := \begin{bmatrix} I_m \\ 0 \end{bmatrix}, \quad J'_r := \begin{bmatrix} I_r \\ 0 \end{bmatrix}, \quad -\gamma I_m.$$

The following theorem gives the solvability condition for the standard $H^\infty$ control problem.

**Theorem 1.** The standard $H^\infty$ control problem is solvable if and only if the following five conditions are satisfied.

(i) $\gamma^2 I_r - D_{11}^T(I_m - D_{12}(D_{12}^T D_{12})^{-1} D_{12}^T)D_{11} > 0$.  

(ii) $\gamma^2 I_m - D_{11}(I_r - D_{21}^T(D_{21} D_{21}^T)^{-1} D_{21})^T I_m > 0$.  

(iii) There exists a solution $X \geq 0$ of the Riccati equation

$$XA + A^T X + C_2^T C_z - (C_2^T D_{12}^* + XB_1) (D_{12}^* D_{12})^{-1} (D_{12}^* C_z + B^T X) = 0$$

(6)

such that $A + BF$ is stable, where

$$F = \begin{bmatrix} F_o \\ F_w \end{bmatrix} := -(D_{12}^* J_r D_{12})^{-1} (D_{12}^* C_z + B^T X).$$

(iv) There exists a solution $Y \geq 0$ of the Riccati equation

$$YA^T + AY + B_w D_{12}^*$$

$$- (B_w D_{12}^* Y C_2^T) (D_{12} W D_{12}^T + CY) = 0$$

(7)

such that $A + LC$ is stable, where

$$L = \begin{bmatrix} L_z & L_y \end{bmatrix} := -(B_w D_{12}^* Y C_2^T) (D_{12} W D_{12}^T + CY)^{-1}.$$

(v) $\lambda_{\text{max}}(XY) < \gamma^2$.  

**Proof:** See THEOREM 8.12 (p. 210) in 11).

3. Chain-scattering approach to $H^\infty$ control

In this section, we briefly introduce the chain-scattering approach to $H^\infty$ control. More details of this approach are found in 11).

If $P_{21}(s)$ is invertible, a necessary and sufficient condition for the solvability is that the chain-scattering form of $P(s)$ has J-lossless factorization

$$\begin{bmatrix} z \\ w \end{bmatrix} = \text{CHAIN}(P) \begin{bmatrix} u \\ y \end{bmatrix} = \Theta \Pi \begin{bmatrix} u \\ y \end{bmatrix},$$

(9)

where $\Theta$ is J-lossless (see the section 4.4 in 11)) and $\Pi$ is unimodular. Then, the $H^\infty$ controller is constructed as

$$K(s) = HM(\Pi^{-1}; S), \quad S \in BH^\infty.$$  

(10)

Due to the cascade property of $HM(\cdot; \cdot)$ (see the notation in Section 1), we can obtain the representation of $\Phi(s)$ without using a representation of the closed-loop transfer function $\Pi^{-1}$ explicitly, i.e.,

$$\Phi(s) = HM(\Theta; S),$$  

(11)

which is described in Fig. 2.

**Fig. 2** Structure of $H^\infty$ control

If neither $P_{21}(s)$ nor $P_{12}(s)$ is invertible, i.e., in the so-called four-block case, we need to augment the plant to derive chain-scattering representations of $P(s)$. For the plant given by (2), we consider a fictitious output

$$y' = C_2 x + D_{21} w,$$

(12)

where $D_{12}^{-1}$ is any matrix that makes $[D_{21}]$ invertible.

The augmented plant $P_o$ is described by

$$\begin{bmatrix} z \\ y' \end{bmatrix} = P_o \begin{bmatrix} w \\ u \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & 0 \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}:$$

(13)

where $C_1 := C_2$, $D_{21} := [D_{21} D_{22}]$.

We obtain the chain-scattering representation of $P_o$ as
Hereafter, the matrices which include the augmentation $(C'_2, D'_21)$ are designated by a symbol "^*".

Now, we introduce partial plants which are given by

$$P_1 := \begin{bmatrix} P_{11} & P_{12} \\ \textrm{I}_r & 0 \end{bmatrix} = \begin{bmatrix} A & B \\ C_3 & D_3 \end{bmatrix}, \quad \tilde{P}_y := \begin{bmatrix} 0 & \textrm{I}_r \\ \tilde{P}_{21} & \tilde{P}_{22} \end{bmatrix} = \begin{bmatrix} A & B \\ C_y & D_y \end{bmatrix},$$

where $\tilde{C}_y := \begin{bmatrix} 0 \\ C_y \end{bmatrix}$, $\tilde{D}_y := \begin{bmatrix} 0 \\ D_y \end{bmatrix}$.

Using these partial plants, we can obtain an alternative representation of $\text{CHAIN}(P_0)$ in (14) as

$$\begin{bmatrix} A & B \\ C_2 & D_2 \end{bmatrix} = \begin{bmatrix} A - B \tilde{D}_2^{-1} C_2 & B \tilde{D}_2^{-1} \\ C_2 - D_2 \tilde{D}_2^{-1} C_2 & D_2 \end{bmatrix}.$$  \hspace{1cm} (15)

Next, we consider the $J$-lossless factorization (9) for the augmented plant (16). For this purpose the following theorem is now available\textsuperscript{11}).

**Theorem 2.** Let (16) be a minimal realization of $\text{CHAIN}(P_0)$ with $D_2 \in \mathbb{R}^{(m+r) \times (p+r)}$. It has a $(J_2, J_2)$-lossless factorization if and only if the following three conditions are satisfied.

(i) There exists a nonsingular matrix $E_2$ satisfying

$$D_2^T J_2 E_2 = E_2^T J_{2r} E_2.$$  \hspace{1cm} (17)

(ii) There exists a solution $X \geq 0$ of the Riccati equation

$$X A_2 + A_2^T X - (C^T_2 J_2 D_2 + X B_2)(D_2^T J_2 D_2)^{-1},$$  

$$= \left(\begin{array}{cc} D_2^T J_2 E_2 + C_2^T & B_2 \\ C_2 & 0 \end{array}\right)$$

such that $\tilde{A}_2 := A_2 + B_2 F_2$ is stable, where

$$F_2 = -(D_2 J_2 D_2)^{-1}(D_2 J_2 C_2 + B_2 X).$$  \hspace{1cm} (18)

(iii) There exists a solution $\tilde{X} \geq 0$ of the Riccati equation

$$\tilde{X} A_2^T + A_2^T \tilde{X} + \tilde{X} C_2^T J_2 C_2 $$

such that $\tilde{A}_2 := A_2 + \tilde{X} C_2^T J_2 C_2$ is stable, and

$$\lambda_{\text{max}}(\tilde{X} \tilde{X}) < 1.$$  \hspace{1cm} (19)

In this case, the $(J_{2r}, J_{2r})$-lossless factor is given by

$$\Theta := \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} = \begin{bmatrix} A_2 & 0 \\ A_2 + B_2 F_2 & \tilde{X} - I & 0 \\ \tilde{X} - I & C_2 \tilde{X} & D_2 \end{bmatrix},$$  \hspace{1cm} (20)

where $\Theta_{12}$ and $\Theta_{22}$ are given by (21).

**Proof:** See THEOREM 6.6 (p. 139) in Kimura\textsuperscript{11}).

In this paper, we deal with the closed-loop transfer function resulting from the central solution. This is equivalent to putting $S = 0$ in (11), which implies

$$\Phi(s) = HM \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} = \Theta_{12} \Theta_{22}^{-1}.$$  \hspace{1cm} (22)

$$\text{where } \Theta_{12} \text{ and } \Theta_{22} \text{ are given by (21).}$$

Now, we introduce the notion of maximum augmentation\textsuperscript{11}). Though Theorem 2 is not a sufficient condition for the solvability of $H^\infty$ control problem for the original plant, it is shown that a sufficient condition is derived for a special augmentation which we call maximum augmentation.

**Definition 1.** The output augmentation $(C'_2, D'_21)$ in (12) is said to be maximum augmentation, if $(C'_2, D'_21)$ satisfies

$$D'_21 \begin{bmatrix} B_1 & L \end{bmatrix}^{T} D'_21 + C'_2 Y = 0,$$  \hspace{1cm} (23)

where $Y$ is the solution of the Riccati equation (7).

Similarly, if $P_{12}(s)$ is not invertible, we can consider a dual notion of an output augmentation in (12) as $(B'_2, D'_12)$. We define an input augmented plant as

$$P_1(s) := \begin{bmatrix} A & B_1 \\ C_1 & D_1 \end{bmatrix} = \begin{bmatrix} A & B_1 \\ D_1 \end{bmatrix},$$  \hspace{1cm} (24)

The maximum input augmentation which is a complete dualization of (23) is also defined as follows.

**Definition 2.** The input augmentation $(B'_2, D'_12)$ in (24) is said to be maximum augmentation, if $(B'_2, D'_12)$ satisfies

$$C_1 \left[ D_1 \right] \left[ D_1 \right] \left( F \right)^T D_1 + X B_2 = 0,$$  \hspace{1cm} (25)

where $X$ is the solution of the Riccati equation (6).

We shall also give a new interpretation of the maximum augmentation in the next section.

4. **Main results**

In this section we show that the McMillan degree of the closed-loop transfer function is determined by the sum of the number of stable invariant zeros of $P_{12}(s)$ and $P_{21}(s)$, which are parts of the original plant.

4.1 **Closed-loop structure of the central $H^\infty$ controller**

In order to prove the main theorem, we show a few lemmas which give several properties of maximum augmentation of the plant. Note that, though we only deal with the output augmentation $(C'_2, D'_21)$ in this section, the complete dualization (input augmentation) of our results also holds.

We consider a plant $P(s)$ in (2) for which $H^\infty$ control problem is solvable. Assume that both $P_{12}(s)$ and $P_{21}(s)$ are not invertible, i.e., consider the four-block case. Then, provided the maximum augmentation in (23) is chosen for $(C'_2, D'_21)$, the $H^\infty$ control problem on the augmented plant $P(s)$ in (13) is also solvable. Hence, the solvability conditions (i)-(v) in Theorem 2 are satisfied. Substituting (16) into (20) and $\tilde{A}_2$ of the condition (iii), we have

$$Y C_2 \left( D_2 J_2 D_2 \right) C_2^T \left( D_2 J_2 C_2 + B_2 X \right) = 0,$$  \hspace{1cm} (26)

$$\tilde{A}_2 = A - B_2 C_2 \left( D_2 J_2 D_2 \right) C_2^T \left( D_2 J_2 C_2 + B_2 X \right).$$  \hspace{1cm} (27)

Here, in order to derive (26) and (27), we used the identities.
As regards (26) and (27), we can prove the following lemmas.

**Lemma 1.** If \((C'_2, D'_2)\) in \(P_0(s)\) is the maximum augmentation given by (23), the eigenvalues of \(A - B_1D_1^{-1}C_2\) are independent of the augmentation \((C'_2, D'_2)\).

**Proof:** The Hamiltonian matrix corresponding to the Riccati equation (26) is given by

\[
H := \begin{bmatrix} (A - B_1D_1^{-1}C_2)^T \quad -C^T(D_1J_1^1D_1^{-1}C_1)^{-1}C_2 \\ 0 \end{bmatrix}.
\]

(29)

Obviously, we have \(\lambda(H) = \lambda(A - B_1D_1^{-1}C_2) \cup \lambda(-(A - B_1D_1^{-1}C_2))\).

From (30) and (35), we have

\[
\lambda(H) = \lambda(A + LD_1^{-1}C) \cup \lambda(-(A + LD_1^{-1}C)).
\]

(36)

which is independent of \((C'_2, D'_2)\).

**Lemma 2.** The Riccati equation (26) has a stabilizing solution \(Y \geq 0\) only if \(A - B_1D_1^{-1}C_2\) has no eigenvalue on the imaginary axis. The rank of \(Y\) is equal to the number of unstable eigenvalues of \(A - B_1D_1^{-1}C_2\) and the eigenspace of \(A - B_1D_1^{-1}C_2\) corresponding to the stable eigenvalues is equal to \(\text{Ker} Y\).

**Proof:** Apply LEMMA 3.6 (p. 49)\(^{11}\) to (26). \(\Box\)

**Remark:** In the \(H^\infty\) control problem, the partial plant \(P_{21}\) (\(P_{12}\)) is generally fat (tall) and is not invertible. Hence, \(P_{21}\) (\(P_{12}\)) may not have \(n\) invariant zeros, whereas the augmented plant \(\hat{P}_{21}\) (\(\hat{P}_{12}\)) has \(n\) invariant zeros. The above lemma implies that the maximum augmentation does not add new stable zeros to the zeros of \(\hat{P}_{21}\) and \(\hat{P}_{12}\). Hence, due to Lemma 2, the rank of the stabilizing solution \(Y(X)\) is equal to \(n - \#\{\text{the stable zeros of } P_{21}(P_{12})\}\). \(\Box\)

Now, we are ready to show one of our main results, which is on the degree of the closed-loop system.

**Theorem 3.** In addition to the assumptions (A1) and (A2), assume that an \(H^\infty\) control problem for a plant \(P(s)\) is solvable. Let the number of the stable invariant zeros of \(P_{12}(s)\) and \(P_{21}(s)\) be \(\alpha_1\) and \(\alpha_2\), respectively. Then, the McMillan degree of the closed-loop transfer function is at most \(2n - (\alpha_1 + \alpha_2)\).

**Proof:** Due to the assumption (A2), we can always find nonsingular matrices \(T_Y\) and \(T_X\) such that

\[
T_Y(A - B_1D_1^{-1}C_2)T_Y^T = \begin{bmatrix} \lambda Y_+ & 0 \\ 0 & \lambda Y_- \end{bmatrix},
\]

(42)

\[
T_X(A - B_2D_2^{-1}C_2)T_X = \begin{bmatrix} \lambda X_+ & 0 \\ 0 & \lambda X_- \end{bmatrix},
\]

(43)

where both \(-\lambda_+\) and \(\lambda_-\) are stable. From Lemma 1, \(-\lambda_+\) and \(\lambda_-\) are independent of the augmentations \((C'_2, D'_2)\) and \((B'_2, D'_2)\), respectively. From Lemma 2,
the size of $\Lambda_X$ is $n - \rho_{12}$ and the size of $\Lambda_Y$ is $n - \rho_{21}$. Divide $TY$ as

$$T_Y = \begin{bmatrix} T_{Y1} & T_{Y2} \end{bmatrix},$$

(44)

where the number of columns of $T_{Y1}$ is $n - \rho_{21}$. From (42), it follows that

$$(A - B_1 \tilde{D}_{12}^{-1} C_2)^2 T_{Y2} = T_{Y2} \Lambda_T^{-}.$$  

(45)

Due to Lemma 2, we obtain $YT_{Y2} = 0$. Therefore, we have

$$T_{Y1}^T Y_{Y1} = \begin{bmatrix} \Lambda_T^+ & 0 \\ 0 & \Lambda_T^- \end{bmatrix} T_{Y1}^T Y_{Y1} = \begin{bmatrix} \Lambda_T^+ & 0 \\ 0 & \Lambda_T^- \end{bmatrix}.$$  

(46)

Furthermore, we can show that $Y_0 := T_{Y1} Y_{Y1}$ is positive definite. Dualization of (46) yields

$$T_{Y1}^T X_{T1} = \begin{bmatrix} X_0 & 0 \\ 0 & 0 \end{bmatrix},$$

(47)

where the size of $X_0$ is $n - \rho_{12}$.

For the $J$-lossless system $\Theta(s)$ given by (21), we will consider its similarity transformation using $T := \begin{bmatrix} T_Y & 0 \\ 0 & X_0 \end{bmatrix}$ in the sequel. Using (27), (42) and (46), we have

$$T_Y A_T T_Y^T = T_{Y1}^T (A - B_1 \tilde{D}_{12}^{-1} C_2) Y_{Y1} = -T_{Y1} Y_{Y1}^{-1} \tilde{C}^T (\tilde{D}_{12}^{-1} \tilde{D}_{12}^{-1})^{-1} \tilde{C} T_{Y1}^T$$

$$= \begin{bmatrix} \Lambda_T^+ & 0 \\ 0 & \Lambda_T^- \end{bmatrix} T_{Y1}^T Y_{Y1} = \begin{bmatrix} \Lambda_T^+ & 0 \\ 0 & \Lambda_T^- \end{bmatrix} Y_{Y1}$$

(49)

where $T_{Y1} := \begin{bmatrix} I_{n - \rho_{12}} & 0 \end{bmatrix} T_{Y1}^{-1}$. Dualization of (49) yields

$$T_{X1} (A_{X,-} + B_2 F_2) T_X$$

$$= \begin{bmatrix} \Lambda_{X,+} & 0 \\ 0 & \Lambda_{X,-} \end{bmatrix} - T_{Y1}^{-1} \tilde{B}^T (\tilde{D}_{12}^{-1} \tilde{D}_{12}^{-1})^{-1} \tilde{B}^T T_{X1} X_0$$

$$= \begin{bmatrix} \Lambda_{X,+} & 0 \\ 0 & \Lambda_{X,-} \end{bmatrix} - T_{X1} (A_{X,-} + B_2 F_2) T_X$$

(50)

where $T_{X1} := \begin{bmatrix} I_{n - \rho_{12}} & 0 \end{bmatrix} T_{X1}^{-1}$. Therefore, the A-matrix of $\Theta$ in (21) is transformed as

$$T^{-1} \left[ \begin{array}{ccc} \Lambda_T^+ & 0 & 0 \\ 0 & \Lambda_T^- \\ 0 & 0 & \Lambda_T^- \end{array} \right] T$$

$$= \begin{bmatrix} \Lambda_T^+ & 0 & 0 \\ 0 & \Lambda_T^- & 0 \\ 0 & 0 & \Lambda_T^- \end{bmatrix}$$

(51)

From dualization of (40), we obtain the identity

$$\{C_2 - D_2 (D_2^T J_2 D_2) - D_2^T C_2\} T_{X2} = 0,$$

(52)

where $T_{X2} := T_X \begin{bmatrix} 0 \ I_{\rho_{12}} \end{bmatrix}^T$ consists of the eigenvectors of $A - \tilde{B}_2 \tilde{D}_{12}^{-1} C_1$ corresponding to the stable part $\Lambda_X$ in (43).

On the other hand, routine calculations using (16) and (19) yield

$$F_2 = \tilde{C}_y + \tilde{D}_y F.$$  

(53)

Using this relation and (52), we have

$$(C_2 + D_2 F_2) T_X = \{C_2 - D_2 \tilde{D}_{12}^{-1} \tilde{C}_y + D_2 \tilde{D}_{12}^{-1} (\tilde{C}_y + \tilde{D}_y F)\} T_X$$

$$= (C_2 + D_2 F) T_X$$

$$= \{C_2 - D_2 (D_2^T J_2 D_2) - D_2^T C_2\} T_X X_1 \begin{bmatrix} X_0 & 0 \\ 0 & 0 \end{bmatrix},$$

(54)

where $T_{X1} := T_X \begin{bmatrix} I_{n - \rho_{12}} & 0 \end{bmatrix}^T$. Therefore, from (46) and (54), the C-matrix of $\Theta$ is transformed as

$$\left[ -\gamma^{-2} C_2 Y T_Y (C_2 + D_2 F_2) T_X \right]$$

$$= \left[ -\gamma^{-2} C_2 Y T_Y \right] \begin{bmatrix} X_0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}.$$  

(55)

Hence, all modes corresponding to $-\Lambda_T^+$ and $\Lambda_X$ in (51) are unobservable. As a result, we obtain the $J$-lossless system $\Theta(s)$ whose McMillan degree is at most $2n - (\rho_{12} + \rho_{21})$. It can be shown that the closed-loop system with the central $H^\infty$ controller does not have a greater McMillan degree than the $J$-lossless system $\Theta(s)$. The proof is completed.

For the one-block case, we can obtain the following result.

**Corollary 1.** Under the assumptions (A1) and (A2), assume that an $H^\infty$ control problem in the one-block case is solvable. Let the number of unstable invariant zeros of $P_{12}(s)$ and $P_{21}(s)$ be $\rho_{12}$ and $\rho_{21}$, respectively. Then, the McMillan degree of the closed-loop transfer function given by (61) is at most $\rho_{12} + \rho_{21}$.

As a direct consequence of the Theorem 4, the case where $\rho_{12} = \rho_{21} = n$ is of particular interest.

**Corollary 2.** Under the assumptions (A1) and (A2), assume that the $H^\infty$ control problem is solvable. If both $P_{12}(s)$ and $P_{21}(s)$ have $n$ stable invariant zeros, i.e., $\rho_{12} = \rho_{21} = n$, then the closed-loop transfer function is given by

$$\Phi(s) = \begin{bmatrix} I_{m - D_{12}} - (D_{12}^T D_{12})^{-1} D_{12}^T \end{bmatrix} D_{11},$$

(56)

which has no dynamics. In particular, the closed-loop transfer function satisfies $\Phi(s) = 0$ in the one-block case, which shows that the $H^\infty$ control problem is solvable for any $\gamma > 0$.

**Proof:** Since both $\tilde{P}_{12}(s)$ and $\tilde{P}_{21}(s)$ have no unstable zeros, we have $X = Y = 0$ and

$$T_{X1}^T (A - B_2 \tilde{D}_{12}^{-1} C_1) T_X = \Lambda_X^-,$$

(57)

$$T_{X1}^T (A - B_1 \tilde{D}_{12}^{-1} C_1) T_X = \Lambda_X^-.$$  

(58)

Using the similarity transformation $T$ in (48), the $J$-lossless system $\Theta(s)$ in (81) is given by

$$\Theta = \begin{bmatrix} -\Lambda_T^+ & 0 & 0 \\ 0 & \Lambda_T^- & 0 \\ 0 & 0 & \Lambda_T^- \end{bmatrix} T^{-1} \begin{bmatrix} * & B_{d1} \\ * & B_{d2} \\ * & D_{d1} \end{bmatrix} = \begin{bmatrix} * & D_{d1} \\ * & D_{d2} \\ * & D_{d2} \end{bmatrix}.$$  

(59)
Hence, we obtain the closed-loop transfer function as
\[ \Phi(s) = \Theta_{12} \Theta_{22}^{-1} = D_{c2} \left( D_{12}^T D_{12} \right)^{-1} D_{12}^T D_{11}. \]

**Remark:** Theorem 3 demonstrates an interesting link between $H^\infty$ control and $J$-lossless conjugation\(11)\). Theorem 3 shows that, in $H^\infty$ control, the order of closed-loop transfer function depends on the number of the stable invariant zeros of $P_{21}(s)$ and $P_{12}(s)$. It may not be equal to the order of the plant. In fact, $2n - (\rho_{12} + \rho_{21})$ equals to the sum of the numbers of pole extractions and zero extractions for $CHAIN(P(s))$, which are key procedures of $J$-lossless factorization based on $J$-lossless conjugation. This is a significant characteristic feature of $H^\infty$ control.

**Remark:** The result of Theorem 3 holds in the generic case (A property on the original plant, $(A, B_1, B_2, C_1, C_2, D_{11}, D_{12}, D_{21})$, is said to be generic if the property holds for almost all plants except special ones.). In fact, we can find an example whose degree of the closed-loop system is exactly $2n - (\rho_{12} + \rho_{21})$, which implies that $2n - (\rho_{12} + \rho_{21})$ is lower bound in the generic sense.

### 4.2 Representation of the closed-loop transfer function with the central $H^\infty$ controller

Now, we are ready to give an explicit state-space representation for the closed-loop transfer function $\Phi(s)$ in (22). Note that this representation does not include the augmentation whereas the $J$-lossless system given by (21) depends on the augmentation.

**Theorem 4.** Under the assumptions (A1) and (A2), if $H^\infty$ control problem is solvable, the closed-loop transfer function for the central controller in (22) is described by
\[
\Phi(s) = \Theta_{12} \Theta_{22}^{-1} = \begin{bmatrix} A_{c2} & B_{c2} \\ C_{c2} & D_{c2} \end{bmatrix},
\]

where
\[
A_{c2} = \begin{bmatrix} A_{c2} & B_{c2} \\ C_{c2} & D_{c2} \end{bmatrix},
B_{c2} = \begin{bmatrix} B_{c2} \\ C_{c2} \end{bmatrix},
D_{c2} = \begin{bmatrix} D_{c2} \end{bmatrix},
\]

where $T_Y := T_Y \begin{bmatrix} I_{n-\rho_{21}} \\ 0 \end{bmatrix}$, $T_X := T_X \begin{bmatrix} I_{n-\rho_{21}} \\ 0 \end{bmatrix}$, $T_{Y1} := [I_{n-\rho_{12}}]$, $T_{X1} := [I_{n-\rho_{12}}]$, $T_{Y1}^T := [I_{n-\rho_{12}}]$, $T_{X1}^T := [I_{n-\rho_{12}}]$, $D_{c2} := \left( D_{12}^T D_{12} \right)^{-1} D_{12}^T$, $W := (I - \gamma^{-2}YX)^{-1}$,

\[
B_{c2} = -T_{Y1} (I - \gamma^{-2}XY)^{-1} \left\{ X(B_1 - B_2 D_{12}^T D_{11}) \right\},
\]

\[
B_{c2} = -T_{X1} \left\{ (B_2 + L_2 D_{12}) D_{12}^T D_{11} \right\},
\]

\[
C_{c1} = \left\{ -L_x^2 + \gamma^{-2} \left( I_{n-\rho_{12}} - D_{12} D_{12}^T \right) \right\}.
\]

**Proof:** The proof consists of the following three steps. First, we transform the $J$-lossless system $\Theta$ in (21) by using the similarity transformation $T$ given by (48). We show that the $J$-lossless system $\Theta$ whose order is $2n - (\rho_{12} + \rho_{21})$ does not include any augmentations. Finally, using this representation of $\Theta$, we derive the closed-loop system $\Phi$ from (22).

**Step 1** $A$-matrix of $\Theta$: From (16) and (53), it follows that
\[
A_{c2} + B_{c2} F_{x2} = A - B D_{12}^{-1} \hat{C}_y + B \hat{D}_{y}^{-1} (\hat{C}_y + \hat{D}_y F) = A + B F.
\]
\[
T^{-1} \begin{bmatrix} -A_{c2} & 0 \\ 0 & A_{c2} + B_{c2} F_{x2} \end{bmatrix} T
\]
\[
= \left\{ -T_Y^T (A + LC) T_Y^{-1} \right\} T_X^{-1} (A + BF) T_X,
\]

where the similarity transformation $T$ is given by (48). Based on Theorem 3, we eliminate the unobservable part of (63) to have
\[
-\gamma T_Y (A + LC) T_Y^{-1} \begin{bmatrix} 0 & 0 \\ 0 & T_X^{-1} (A + BF) T_X \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & A_{c2} \end{bmatrix},
\]

where the definition of $T_1$ and $T_2$ is given in Theorem 4.

**B-matrix of $\Theta$:** From (16) and (17), we have
\[
D_{12}^{-1} D_{12}^T J_{12} D_{y1}^{-1} = E_{c2}^T J_{pr} E_{c2}.
\]

Let $E_{c2}$ be a solution of $D_{12}^T J_{12} D_{y1}^{-1} = E_{c2}^T J_{pr} E_{c2}$. We have
\[
D_{12}^T J_{12} D_{y1}^{-1} \text{ and } D_{12}^T J_{12} D_{y1}^{-1}
\]

where $R := \gamma^2 I_r - D_{12}^T (I_{m-1} - D_{12}^T D_{12}^{-1}) D_{12}$. Since $D_{12}^T J_{12} D_{y1}$ in (66) has $r$ negative eigenvalues, due to Sylvester's inertia law, $R$ must be positive definite. Let $U \in \mathbb{R}^{(r \times p)}$ and $V \in \mathbb{R}^{(r \times r)}$ be nonsingular matrices such that

\[
U^T U = D_{12}^T D_{12}, \quad V^T V = R,\text{ respectively.}
\]

Equation (66) implies
\[
E_{c2} = \begin{bmatrix} U D_{12}^T D_{12} U \\ V \end{bmatrix} \begin{bmatrix} I_r \\ 0 \end{bmatrix},
\]

which is independent of augmentations. Hence, $E_{c2}$ in (65) is given by $E_{c2} = E_{c2} D_{y1}$. Now, let us write the transform of the $B$-matrix of $\Theta$ in (21) as
\[
B_{c2}^{-1} = \left\{ \gamma Y^{-1} X^{-1} \right\} T_Y^{-1} \left\{ (I - \gamma^{-2}XY)^{-1} \right\} \left\{ X(B_1 - B_2 D_{12}^T D_{11}) \right\},
\]

\[
B_{c2}^{-1} = -T_{Y1} (I - \gamma^{-2}XY)^{-1} \left\{ X(B_1 - B_2 D_{12}^T D_{11}) \right\},
\]

\[
B_{c2}^{-1} = -T_{X1} \left\{ (B_2 + L_2 D_{12}) D_{12}^T D_{11} + L_2 D_{11} \right\},
\]

\[
C_{c1} = \left\{ -L_x^2 + \gamma^{-2} \left( I_{n-\rho_{12}} - D_{12} D_{12}^T \right) \right\} D_{11}.
\]
which implies that $T_Y^{-1}(I - \frac{1}{\gamma}XY)^{-1}T_Y$ has a lower triangular form.

Since the identity (33) holds in the case of the maximum augmentation (23), we have

$$T_Y^{-1}YT_Y - T_Y^{-1}CT = -T_Y^{-1}(B_w + LD_w J'_y)D_y.$$  

(70)

Using (44) and (46), we can rewrite the relation (70) as

$$T_Y^{-1}CT = Y_0^{-1}T_Y^{-1}(B_w + LD_w J'_y)D_y.$$  

(71)

From (28) and (71), it follows that

$$T_Y^{-1}C_T = \begin{bmatrix} Y_0^{-1}T_Y^{-1} & 0 \\ I\gamma & -I \gamma \end{bmatrix} \begin{bmatrix} B_w + LD_w J'_y & 0 \\ 0 & 1 \end{bmatrix}.$$  

(72)

Therefore, (67), (69) and (72) enable us to rewrite the second term of $B^{(u)}$ in (68) as

$$T_Y^{-1}(I - \frac{1}{\gamma}XY)^{-1}T_Y^{-1}C_T J_- J_+ D_1 E_2^{-1}.$$

$$= -T_Y^{-1}(I - \frac{1}{\gamma}XY)^{-1}T_Y^{-1}Y_0^{-1}T_Y^{-1} \begin{bmatrix} 0 & I \gamma \\ -I \gamma & 0 \end{bmatrix} J_- J_+ D_1 E_2^{-1}.$$  

From (28) and (71), it follows that

$$T_Y^{-1}Y_0^{-1}T_Y^{-1} = \begin{bmatrix} 0 & I \gamma \\ -I \gamma & 0 \end{bmatrix}.$$  

(73)

Based on Theorem 3, we eliminate the unobservable part of $B^{(u)}$ to have

$$[I_{n - p_21} \ 0] B^{(u)} = T_Y^{-1}(I - \frac{1}{\gamma}XY)^{-1}.$$  

$$\cdot \begin{bmatrix} 0 & I \gamma \\ -I \gamma & 0 \end{bmatrix} J_- J_+ D_1 E_2^{-1}.$$  

(74)

Next, $B^{(l)}$ in (68) is simplified as follows. Equations (28) and (33) give

$$B_S = \frac{1}{\gamma} Y C_T J_- D_S = \begin{bmatrix} B - \frac{1}{\gamma} (B_w + LD_w J'_y) \\ 0 \end{bmatrix} J_- J_+ D_1 E_2^{-1}.$$  

(75)

Here, we used the identities

$$B_w = \begin{bmatrix} 0 & -I \gamma \\ I_m & 0 \end{bmatrix} J_- J_+ \begin{bmatrix} 0 & -I \gamma \\ I_m & 0 \end{bmatrix} J_+ J_- \begin{bmatrix} 0 & -I \gamma \\ I_m & 0 \end{bmatrix}.$$  

Equations (67) and (75) imply that $B^{(l)}$ in (68) is given by

$$B_S^{(l)} = T_X W \begin{bmatrix} 0 & -I \gamma \\ I_m & 0 \end{bmatrix} D_1 E_2^{-1}.$$  

(76)

$C$-matrix of $\Theta$: From (72), it follows that

$$-\frac{1}{\gamma} C_T Y T_Y = \begin{bmatrix} 0 & 0 \\ \frac{1}{\gamma} & -I \gamma \end{bmatrix} (B_w + LD_w J'_y)^T T_Y Y_0.$$  

(77)

Routine calculations using the definition of $B_w, D_w$ and $L$ yield

$$\begin{bmatrix} C_{q11} & 0 \\ C_{q21} & 0 \end{bmatrix} = \begin{bmatrix} C_{q11} & 0 \\ C_{q21} & 0 \end{bmatrix} C_T Y T_Y^T = \begin{bmatrix} 0 \\ \frac{1}{\gamma} \end{bmatrix} (B_1 + L_z D_{11} + L_y D_{21})^T T_Y Y_0.$$  

(78)

Furthermore, from (54), we have

$$\begin{bmatrix} C_{q12} & 0 \\ C_{q22} & 0 \end{bmatrix} = (C_1 + D_z F_2) T X = \begin{bmatrix} C_{q12} & 0 \\ C_{q22} & 0 \end{bmatrix} C_T Y T_Y^T = \begin{bmatrix} 0 \\ \frac{1}{\gamma} \end{bmatrix} (B_1 + L_z D_{11} + L_y D_{21})^T T_Y Y_0.$$  

(79)

$D$-matrix of $\Theta$: From (16) and (67), it follows that

$$D_S E_2^{-1} = \begin{bmatrix} D_1 D_2 & 0 \\ I & 0 \end{bmatrix} U^{-1} D_1 D_2 U^{-1}.$$  

(80)

[Step 2] Now, to derive the closed-loop transfer function $\Phi$, we describe the $(2n - (p_2 + p_1))$-th order $J$-lossless system. Using (64), (74), (76), (78), (79) and (80), we can rewrite (21) as

$$\Theta = \begin{bmatrix} C_{q11} & C_{q12} \\ C_{q21} & C_{q22} \end{bmatrix} = \begin{bmatrix} A_{q11} & 0 \\ B_{q1} & 0 \end{bmatrix}.$$  

(81)

[Step 3] Therefore, the closed-loop transfer function $\Phi(s)$ for the central $H^\infty$ controller is given by

$$\Phi(s) = \Theta_{12} \Theta_{12}^{-1}$$

$$= \begin{bmatrix} A_{q11} & 0 \\ B_{q1} & 0 \end{bmatrix} C_T D_S^{-1} D_T^{-1} C_T D_S^{-1} D_T^{-1} C_T D_S^{-1} D_T^{-1} C_T D_S^{-1} D_T^{-1} C_T D_S^{-1} D_T.$$  

(82)

(83)

where the similarity transformation

$$T_S = \begin{bmatrix} I_{2n - (p_2 + p_1)} \\ I_{2n - (p_2 + p_1)} \\ I_{2n - (p_2 + p_1)} \end{bmatrix} = \begin{bmatrix} I_{2n - (p_2 + p_1)} \\ I_{2n - (p_2 + p_1)} \end{bmatrix} D_T^{-1} C_T D_S^{-1} D_T^{-1} C_T D_S^{-1} D_T^{-1} C_T D_S^{-1} D_T.$$  

(84)

is used. Lengthy but straightforward calculations for (82) and (83) yield the representations of $A_{cl}, B_{cl}, C_{cl}$ and $D_{cl}$ in (61).

5. Conclusion

We have investigated the McMillan degree of closed-loop transfer function of $H^\infty$ control systems. Our result
implies that the intrinsic complexity of closed-loop transfer function is determined by the number of the stable invariant zeros of $P_{12}(s)$ and $P_{21}(s)$. Based on this result, we have obtained an explicit form of the $(2n-(\rho_{12}+\rho_{21}))$-th order closed-loop transfer function of $H^\infty$ control systems. We have also provided a new interpretation of the notion of the maximum augmentation. Our results based on the chain-scattering approach provide clear insights into the closed-loop structure of $H^\infty$ control systems.

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