Stability Analysis for a Class of Switched Systems[†]

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In this paper, we study stability property for a class of switched systems composed of several subsystems, where each subsystem's vector field is composed of a linear time-invariant part and a nonlinear norm-bounded perturbation part. It is assumed that the linear subsystem matrices are commutative pairwise, and there exists a linear convex stable combination of unstable linear subsystem matrices. First, in the case of no perturbations, we propose a switching law under which the entire switched system is globally exponentially stable. In the switching law, Hurwitz stable subsystems (if exist) are activated arbitrarily while unstable ones are activated in sequence with their duration time periods satisfying a specified ratio. Secondly, under the same switching law, we analyze qualitative property of the switched system in the case where nonlinear norm-bounded perturbations exist. Some numerical examples are given in the paper to demonstrate the results.

Key Words: switched system, exponential stability, commutative pairwise, switching law, nonlinear perturbations

1. Introduction

By a switched system, we mean a hybrid dynamical system that is composed of a family of continuous-time subsystems and a rule orchestrating the switching between the subsystems. Recently, there has been increasing interest in stability analysis and switching control design of such systems; see 1)~13) and the references cited therein. The motivations for studying such switched systems come from the fact that many practical systems are inherently multimodal in the sense that several dynamical systems are required to describe their behavior dependent on various environmental factors^{1), 2)}, and the methods of intelligent control design are based on the idea of switching between different controllers^{1), 3), 4)}.

In this paper, we study stability property for a class of switched systems, where each subsystem's vector field consists of a linear time-invariant part $A_i x$ and a nonlinear norm-bounded perturbation part, and the matrix A_i 's are commutative pairwise. For this kind of switched systems with no perturbations existing, it has been shown that when all subsystem matrices are Hurwitz stable (i.e., all eigenvalues lie in the left-half complex plane), there exists a common quadratic Lyapunov function for all subsystems⁵⁾ and the entire switched system is exponentially

stable for any switching law^{5), 6)}. Then, a natural question arises: how should we do when not all subsystems are Hurwitz stable? For this case, a switching law was given in 6) which guaranteed exponential stability of the entire switched system by specifying the duration time ratio between Hurwitz stable subsystems and unstable ones. However, the switching law there does not work when all subsystems are Hurwitz unstable, and restricted the total duration time of Hurwitz unstable subsystems compared with stable ones.

Here, we first consider the case of no perturbations in the switched system. Based on the assumption that there exists a stable linear convex combination of unstable A_i 's, we propose a switching law under which the entire switched system is globally exponentially stable. In the proposed switching law, Hurwitz stable subsystems (if exist) are activated arbitrarily while unstable ones are activated in sequence with their duration time periods satisfying a specified ratio. Then, for the case where nonlinear vanishing or nonvanishing perturbations exist, we study qualitative property of the switched system under the same switching law. It is shown that the exponential convergence property of the systems's trajectories is preserved if the norm-bounded perturbations are relatively small. We note here that the idea of a stable convex combination of unstable subsystem matrices has been proposed in 7) for more general switched systems where commutative property of subsystem matrices was not assumed. In that context, a time average control strategy and a finite time switched control strategy were proposed. However, the time average control strategy pro-

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posed there required rapid switchings, which potentially excite high frequency unmodelled dynamics and other unsavory phenomena associated with chattering. The finite time switched control strategy in 7) endeavored to avoid possible chattering by forcing the state to enter the interior of a conic region before subsequent switching, but the state-dependent switching law there is involved, restrictive and not easy to be extended to the case of more than two subsystems. Compared with the switching methods proposed in 7), our switching law does not result in chattering phenomena, does not restrict the number of subsystems, and the time interval between consecutive switchings can be adjusted whenever it is necessary. As can be seen later, our switching law includes a wide variety of switching methods in practice.

This paper is organized as follows. In Section 2, we state our control problem and give a motivation example. Section 3 proposes a switching law for the case of no perturbations, and shows that the linear switched system is globally exponentially stable under the proposed switching law. An example is given to demonstrate the switching law. In Section 4, we study the above switched system with norm-bounded nonlinear perturbations. A sufficient condition is derived on strength of the perturbations so that the solutions of the nonlinear switched system converge to the origin exponentially. An example shows that those linear switched systems that do not satisfy the commutative property can also be dealt with by introducing appropriate perturbations in our framework. Finally Section 5 concludes the paper.

2. Problem Formulation

Consider the switched system composed of N nonlinear subsystems described by

$$\dot{x}(t) = A_i x(t) + f_i(t, x(t)), \quad x(t_0) = x_0, \quad (1)$$

where $x(t) \in R^n$ is the state, $t_0 \geq 0$ is the initial time, x_0 is the initial state and $N > 1$ is the number of subsystems. $A_i \in R^{n \times n}$ is a constant matrix describing the linear part of the i -th subsystem, and $f_i(t, x(t))$ is the nonlinear perturbation term. Throughout this paper, we assume

- (1) not all A_i 's are Hurwitz stable;
- (2) A_i 's are commutative pairwise, i.e.,

$$A_i A_j = A_j A_i, \quad \forall i \neq j. \quad (2)$$

The first assumption is made since the case where all A_i 's are Hurwitz stable and no perturbations exist has been studied precisely in 5), 6). The second assumption is due

to technical reason, and an example in Section 4 will show how to deal with the case where it is not satisfied.

Without loss of generality, we denote the set of Hurwitz unstable subsystem matrices by $\mathcal{A}^+ = \{A_1, \dots, A_r\}$. This is reasonable because we can rearrange the order of the subsystems. Here, r is an integer satisfying $1 < r \leq N$, which means that there may not exist Hurwitz stable subsystems (when $r = N$), and there should be more than two unstable subsystems ($r > 1$).

For the set $\mathcal{A}^+$, we define

$$\Lambda \triangleq \left\{ (\lambda_1, \dots, \lambda_r) \mid A_\lambda = \sum_{j=1}^r \lambda_j A_j \text{ is Hurwitz stable}, \right. \\ \left. 0 < \lambda_j < 1, \quad \sum_{j=1}^r \lambda_j = 1 \right\}, \quad (3)$$

which is the set of coefficients that make a linear convex combination of unstable subsystem matrices Hurwitz stable. We assume in this paper that Λ is not empty, i.e., there is at least one $(\lambda_1, \dots, \lambda_r) \in \Lambda$ making A_λ Hurwitz stable.

Then, the control problem in this paper is described as: *determine a switching law for the switched system (1) so that the entire system is exponentially stable when the subsystems are activated according to the switching law.* Though switching laws can generally be dependent on the state ^{1), 8)~10)} or the output ^{1), 11)} of subsystems, or be designed using supervisors ¹²⁾, we consider the switching law depending on time directly.

The following example motivated by 6) shows that when not all subsystems are Hurwitz stable, the behavior of the switched system depends greatly on the time switching laws even when no perturbations exist. Therefore, time switching control problems are far from trivial.

Fig. 1 Trajectories of the switched system in example 1

Example 1. Consider the switched system (1) consisting of the following two subsystems

$$A_1 = \begin{bmatrix} 2 & 4 \\ 0 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -4 & -5 \\ 0 & 1 \end{bmatrix} \quad (4)$$

with no perturbations. A_1 and A_2 are unstable and commutative pairwise. Then, with any time unit $T_0 > 0$, the switched system with the initial condition $x_0 = [1 \ 1]^T$ is (see Fig.1)

(a) exponentially stable if subsystems A_1 and A_2 are activated alternatively with the same duration T_0 ;

(b) uniformly stable but not exponentially stable if subsystems A_1 and A_2 are activated alternatively with durations T_0 and $2T_0$, respectively;

(c) unstable if subsystems A_1 and A_2 are activated alternatively with durations T_0 and $3T_0$, respectively.

3. Linear Switched Systems

In this section, we focus our attention on the case where no perturbations exist, i.e., $f_i(t, x(t)) = 0$, $i = 1, \dots, N$.

Suppose that we have found a vector $(\lambda_1, \dots, \lambda_r) \in \Lambda$. Then, we propose the following switching law.

Switching Law. Activate the Hurwitz stable subsystems (if exist) arbitrarily, and activate the Hurwitz unstable subsystems in sequence with the duration time periods $\lambda_i T_0$, $i = 1, 2, \dots, r$, respectively, where T_0 is an arbitrarily specified nonnegative time unit.

If all subsystems are Hurwitz stable, then the switched system (1) without perturbations is globally exponentially stable for arbitrary switching^{5), 6)}. However, this case will not appear here according to the assumptions made in Section 2. If some subsystems are Hurwitz stable while other subsystems are not, a possible switching method is to activate only the Hurwitz stable subsystems, which corresponds to $T_0 = 0$ in the above switching law. Together with the fact that the time unit T_0 can be chosen arbitrarily and the fact that the activation order of all subsystems can be arbitrary, which will be shown later, we see that the above switching law includes a wide variety of switching methods in practice.

Theorem 1. The switched system (1) without perturbations is globally exponentially stable under the above switching law.

Proof. For the trivial case of activating only the Hurwitz stable subsystems while ignoring the unstable ones, it has been shown in 5), 6) that the switched system (1) is globally exponentially stable for arbitrary switching. Here, we consider the case of activating both the Hurwitz stable subsystems (if exist) and the unstable ones in $\mathcal{A}^+$.

For any $t > t_0$, we divide the interval $[t_0, t)$ according to the switching law as $t - t_0 = nT_0 + \hat{t} + \sum_{i>r} T_i(t)$, where n is a nonnegative integer, $0 \leq \hat{t} < T_0$. $T_i(t)$ denotes the total duration time of the i -th Hurwitz stable subsystem (if exists), and $nT_0 + \hat{t}$ is the total duration time of the

Hurwitz unstable subsystems. Since A_i 's are commutative pairwise, for any initial condition $x(t_0) = x_0$,

$$x(t) = e^{\lambda_1 A_1 T_0 + \dots + \lambda_{q-1} A_{q-1} T_0 + \lambda_q A_q \hat{t}} \times e^{n A_\lambda T_0} e^{\sum_{i>r} (A_i T_i(t))} x_0 \quad (5)$$

holds for some integer q satisfying $1 \leq q \leq r$ and $0 \leq \hat{t} < T_0$.

Since A_λ and A_i 's ($i > r$) are Hurwitz stable, there always exist positive scalars α_1 , α_2 and k_i 's, $i = 0, 1, \dots, N$ such that

$$\|e^{A_\lambda t}\| \leq k_0 e^{-\alpha_1 t}$$

$$\|e^{A_i t}\| \leq \begin{cases} k_i e^{\alpha_2 t}, & i \leq r \\ k_i e^{-\alpha_1 t}, & i > r \end{cases} \quad (6)$$

holds for $\forall t \geq 0$. Then, we obtain from (5) and (6) that

$$\|x(t)\| \leq \left(\prod_{i=0}^N k_i \right) e^{\alpha_2 \hat{t}} e^{-\alpha_1 (nT_0 + \sum_{i>r} T_i(t))} \|x_0\|$$

$$= \left(\prod_{i=0}^N k_i \right) e^{(\alpha_1 + \alpha_2) \hat{t}} e^{-\alpha_1 t} \|x_0\|$$

$$\leq c e^{-\alpha_1 t} \|x_0\|, \quad (7)$$

where $c = (\prod_{i=0}^N k_i) e^{(\alpha_1 + \alpha_2) T_0}$. This implies that the switched system (1) is globally exponentially stable. $\square$

The following numerical example shows how to do the switching when both Hurwitz stable subsystems and unstable ones exist.

Fig. 2 Trajectory of the switched system in example 2

Example 2. Consider the switched system (1) with no perturbations and

$$A_1 = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -4 & -3 \\ 0 & 2 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (8)$$

We see that $A_i A_j = A_j A_i$, $i, j = 1, 2, 3$, and A_1 and A_2 are unstable while A_3 and $\frac{5}{7} A_1 + \frac{2}{7} A_2$ are Hurwitz stable. Then, according to the proposed switching law, we activate A_1 and A_2 with the duration time ratio being 5 : 2,

and activate A_3 arbitrarily. The solution trajectory of the switched system with the initial condition $x_0 = [1 \ 1]^T$ is shown in Fig. 2, where the three subsystems are activated alternatively with the durations 0.25, 0.1 and 0.05, respectively.

Remark 1. Since A_i 's are commutative pairwise, the activation order of the subsystems can be arbitrary in the proposed switching law. Also, there is no restriction on the time unit T_0 , thus fast switching is not necessary and chattering phenomena will not appear here. $\square$

Remark 2. How to find the combination coefficient λ_i 's $\in \Lambda$ was discussed for the case of $r = 2$ by using a brute-force interval search⁷⁾. Unfortunately, efficient solutions for the case of $r > 2$ do not exist in general. $\square$

Remark 3. Though the set Λ in (3) was defined only using Hurwitz unstable subsystems, Hurwitz stable ones can also be taken into account. In fact, when there is only one unstable subsystem or it is difficult to find a $\lambda = \{\lambda_1, \dots, \lambda_r\} \in \Lambda$, we should consider Hurwitz stable subsystems in (3). More precisely, we try to find scalars λ_j , $j = 1, 2, \dots, N$ satisfying $\sum_{j=1}^N \lambda_j = 1$ such that the convex combination $\sum_{j=1}^N \lambda_j A_j$ is Hurwitz stable. In that case, the switching law should be modified as: "Activate all subsystems in sequence with the duration time periods $\lambda_i T_0$, $i = 1, 2, \dots, N$, respectively, where T_0 is an arbitrarily specified positive time unit." $\square$

Remark 4. From the proof of Theorem 1, it is easy to understand that the proposed switching law can also be modified to the following: "Arbitrarily choose a time sequence $T^0 < T^1 < \dots < T^k < T^{k+1} < \dots$. Then, activate the Hurwitz stable subsystems (if exist) arbitrarily, and activate the Hurwitz unstable subsystems with the duration time periods $\lambda_i T_0^k$ during each time interval $[T^k, T^{k+1})$, where $i = 1, 2, \dots, r$ and T_0^k is an arbitrarily chosen positive number satisfying $T_0^k \leq T^{k+1} - T^k$." $\square$

Remark 5. It is quite easy to confirm that if a set of $(\lambda_1, \dots, \lambda_r)$ makes A_λ unstable, then the proposed switching law results in instability. Similarly, if a set of $(\lambda_1, \dots, \lambda_r)$ makes A_λ stable but not exponentially stable, then uniformly stable but not exponentially stable behavior occurs provided not activating Hurwitz stable subsystems. $\square$

Using Theorem 1 and Remark 5, Example 1 in Section 2 can be explained quite well. In Example 1, both A_1 and A_2 are unstable while $\frac{1}{2}A_1 + \frac{1}{2}A_2$ is Hurwitz stable. Therefore, according to Theorem 1, the switched system is exponentially stable if the two subsystems are activated alternatively with the same duration. On the

contrary, if the subsystems A_1 and A_2 are activated alternatively with durations T_0 and $3T_0$, then the stability property of the switched system is determined by the matrix $\frac{1}{4}A_1 + \frac{3}{4}A_2 = \begin{bmatrix} -\frac{5}{2} & -\frac{11}{4} \\ 0 & \frac{1}{4} \end{bmatrix}$, thus instability occurs according to Remark 5. Also, if the subsystems A_1 and A_2 are activated alternatively with durations T_0 and $2T_0$, then the stability property of the switched system is determined by the matrix $\frac{1}{3}A_1 + \frac{2}{3}A_2 = \begin{bmatrix} -2 & -2 \\ 0 & 0 \end{bmatrix}$, thus the entire system is uniformly stable but not exponentially stable.

4. Nonlinear Switched Systems

In this section, we consider the system (1) with the nonlinear (certain or uncertain) perturbations $f_i(t, x(t))$ satisfying the following norm bounded conditions

$$\|f_i(t, x(t))\| \leq \gamma \|x(t)\| + \beta(t), \quad (9)$$

where γ is a known nonnegative scalar, $\beta(t)$ is a nonnegative Lebesgue integrable function such that $\int_{t_0}^{\infty} e^{\alpha_1 \tau} \beta(\tau) d\tau < \infty$, and α_1 is the same as in (6). The kind of perturbation in (9) is called in (14) vanishing when $\beta(t) \equiv 0$, and nonvanishing when $\beta(t) \not\equiv 0$.

First, we state a well known result which will be used later in Theorem 2.

Lemma. (Gronwall-Bellman Inequality)¹⁵⁾ Suppose that $u \geq 0, v \geq 0$ are two continuous functions, and $m \geq 0$ is a given scalar. Under these conditions, if

$$u(t) \leq m + \int_{t_0}^t u(\tau) v(\tau) d\tau, \quad \forall t \geq t_0, \quad (10)$$

then

$$u(t) \leq m \exp \left[\int_{t_0}^t v(\tau) d\tau \right]. \quad (11)$$

We are now in the position to state the main result in this section.

Theorem 2. Suppose that the switched system (1) is controlled by the switching law proposed in Section 3. Then, the inequality

$$\|x(t)\| \leq c \left[\|x_0\| + \int_{t_0}^{\infty} e^{\alpha_1 \tau} \beta(\tau) d\tau \right] e^{-(\alpha_1 - \gamma c)(t - t_0)} \quad (12)$$

holds for any initial condition $x(t_0) = x_0$, where c is the same as appeared in the proof of Theorem 1. Therefore, under the condition $\gamma < \frac{\alpha_1}{c}$,

- (1) if $\beta(t) \equiv 0$, then $x_e = 0$ is an equilibrium of the nonlinear switched system (1) which is globally exponentially stable;
- (2) if $\beta(t) \not\equiv 0$, then the solutions of the nonlinear

switched system (1) are uniformly bounded and converge to the origin exponentially.

Proof. For any $t > t_0$, according to the proposed switching law, we divide the time interval $[t_0, t)$ as $[t_0, t_1) \cup [t_1, t_2) \cup \dots \cup [t_{k-1}, t_k) \cup [t_k, t)$, where the i_j -th subsystem is supposed to be activated during $[t_j, t_{j+1})$, $j = 0, 1, \dots, k$. Then, using the ordinary differential equation theory, we have

$$\begin{aligned} x(t_1) &= e^{A_{i_0}(t_1-t_0)}x_0 + \int_{t_0}^{t_1} e^{A_{i_0}(t_1-\tau)}f_{i_0}(\tau, x(\tau))d\tau, \\ x(t_2) &= e^{A_{i_1}(t_2-t_1)}x(t_1) + \int_{t_1}^{t_2} e^{A_{i_1}(t_2-\tau)}f_{i_1}(\tau, x(\tau))d\tau, \\ &= e^{A_{i_1}(t_2-t_1)+A_{i_0}(t_1-t_0)}x_0 \\ &\quad + \int_{t_0}^{t_1} e^{A_{i_1}(t_2-t_1)+A_{i_0}(t_1-\tau)}f_{i_0}(\tau, x(\tau))d\tau \\ &\quad + \int_{t_1}^{t_2} e^{A_{i_1}(t_2-\tau)}f_{i_1}(\tau, x(\tau))d\tau, \end{aligned} \quad (13)$$

and by induction,

$$\begin{aligned} x(t) &= e^{A_{i_k}(t-t_k)+A_{i_{k-1}}(t_k-t_{k-1})+\dots+A_{i_0}(t_1-t_0)}x_0 \\ &+ \int_{t_0}^{t_1} e^{A_{i_k}(t-t_k)+A_{i_{k-1}}(t_k-t_{k-1})+\dots+A_{i_0}(t_1-\tau)}f_{i_0}(\tau, x(\tau))d\tau \\ &+ \int_{t_1}^{t_2} e^{A_{i_k}(t-t_k)+A_{i_{k-1}}(t_k-t_{k-1})+\dots+A_{i_1}(t_2-\tau)}f_{i_1}(\tau, x(\tau))d\tau \\ &+ \dots + \int_{t_{k-1}}^{t_k} e^{A_{i_k}(t-t_k)+A_{i_{k-1}}(t_k-\tau)}f_{i_{k-1}}(\tau, x(\tau))d\tau \\ &+ \int_{t_k}^t e^{A_{i_k}(t-\tau)}f_{i_k}(\tau, x(\tau))d\tau. \end{aligned} \quad (14)$$

Noting that in the righthand of the above equation, the first term corresponds to the case of switching linear subsystems during $[t_0, t)$ under the proposed switching law, and the integral terms $e^{A_{i_k}(t-t_k)+\dots+A_{i_j}(t_{j-1}-\tau)}$ correspond to the switching during $[\tau, t)$ under the same switching law. Then, using the estimation (7) in the proof of Theorem 1 and the norm condition (9), we obtain

$$\begin{aligned} \|x(t)\| &\leq ce^{-\alpha_1(t-t_0)}\|x_0\| \\ &+ \int_{t_0}^{t_1} ce^{-\alpha_1(t-\tau)}(\gamma\|x(\tau)\| + \beta(\tau))d\tau \\ &+ \dots + \int_{t_{k-1}}^{t_k} ce^{-\alpha_1(t-\tau)}(\gamma\|x(\tau)\| + \beta(\tau))d\tau \\ &+ \int_{t_k}^t ce^{-\alpha_1(t-\tau)}(\gamma\|x(\tau)\| + \beta(\tau))d\tau, \end{aligned} \quad (15)$$

and thus,

$$\begin{aligned} \|x(t)\|e^{\alpha_1 t} &\leq c \left[e^{\alpha_1 t_0}\|x_0\| + \int_{t_0}^{\infty} e^{\alpha_1 \tau}\beta(\tau)d\tau \right] \\ &+ \gamma c \int_{t_0}^t \|x(\tau)\|e^{\alpha_1 \tau}d\tau. \end{aligned} \quad (16)$$

Here, the integral interval of the positive term $e^{\alpha_1 \tau}\beta(\tau)$

has been enlarged from $[t_0, t)$ to $[t_0, \infty)$. By the Gronwall-Bellman Inequality, we obtain

$$\|x(t)\|e^{\alpha_1 t} \leq c \left[e^{\alpha_1 t_0}\|x_0\| + \int_{t_0}^{\infty} e^{\alpha_1 \tau}\beta(\tau)d\tau \right] e^{\gamma c(t-t_0)}. \quad (17)$$

Then the inequality (12) is obtained by multiplying both sides of the above inequality by $e^{-\alpha_1 t}$.

From (12), it is straightforward that the nonlinear switched system is globally exponentially stable when $\beta(t) \equiv 0$, and the solutions converge to the origin exponentially when $\beta(t) \not\equiv 0$ (noting that the origin is not an equilibrium of the system in this case). $\square$

Observe that when $\beta(t) \equiv 0$ and $\gamma = 0$, Theorem 2 shrinks to Theorem 1. Therefore, Theorem 2 is an extension to Theorem 1.

We note that another important motivation for studying the nonlinear switched systems (1) originates from the observation that linear switched systems can be considered in this framework when commutative property is not satisfied. In that case, what we need to do is to decompose the subsystem matrices into two parts so that one part corresponds to A_i and another corresponds to the perturbation term f_i in (1). Let us give an example to show how this method works, which also demonstrates the effectiveness of Theorem 2.

Fig. 3 Trajectory of the switched system in example 3

Example 3. Consider the switched system composed of the following two subsystems

$$\dot{x} = \begin{bmatrix} 4.5 & 6 \\ 0 & -8.5 \end{bmatrix} x, \quad \dot{x} = \begin{bmatrix} -6 & -3.5 \\ 0.5 & 2 \end{bmatrix} x, \quad (18)$$

both of which are unstable and do not satisfy the commutative property. The existing switching methods up to now can not tell us how to construct the switching law for this system.

Here, we first decompose the above two subsystems into

the form of (1) with

$$\begin{aligned} A_1 &= \begin{bmatrix} 4 & 6 \\ 0 & -8 \end{bmatrix}, \quad f_1 = \begin{bmatrix} 0.5 & 0 \\ 0 & -0.5 \end{bmatrix} x \\ A_2 &= \begin{bmatrix} -6 & -4 \\ 0 & 2 \end{bmatrix}, \quad f_2 = \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \end{bmatrix} x. \end{aligned} \quad (19)$$

It is easy to know that $A_1 A_2 = A_2 A_1$, and $\frac{1}{2}A_1 + \frac{1}{2}A_2$ is Hurwitz stable in (19). After some simple calculations, we obtain that $\alpha_1 = 1$, $\alpha_2 = 4$, $k_0 = k_1 = k_2 = 1$, and $\gamma = 0.5$ for the system (19). Then, we choose the time unit T_0 as 0.04, and activate the two subsystems alternatively with the same duration time periods 0.02. Since $\gamma = 0.5 < \frac{\alpha_1}{c} = \frac{1}{e^{0.2}} = 0.8187$, according to Theorem 2, the entire switched system (19) should be globally exponentially stable. The trajectory of the switched system is depicted in Fig.3, where the initial condition is $x_0 = [1 \ 1]^T$.

Remark 6. The inequality (12) is almost the same as the one given in 6). However, the switching law we proposed is quite different from the one in 6), where the switching law should guarantee that $\frac{T^-(t_0, t)}{T^+(t_0, t)} \geq q$, and $T^-(t_0, t)$ (resp., $T^+(t_0, t)$) denotes the total time period of Hurwitz stable subsystems (resp., unstable subsystems) during $[t_0, t]$, and q is a known positive scalar. Therefore, the switching law there restricts the activation time period of unstable subsystems compared with stable ones, and more importantly, it does not work when no Hurwitz stable subsystems exist. $\square$

Remark 7. If there exists some $\lambda = (\lambda_1, \dots, \lambda_r) \in \Lambda$ of (3), then there is a neighborhood $\mathcal{N}_\lambda$ around this λ such that $\mathcal{N}_\lambda \subset \Lambda$, and thus the proposed switching law, Theorem 1 and Theorem 2 remain valid for any $\lambda \in \mathcal{N}_\lambda$. This fact tells us that the set Λ in (3) is an open one, and that we do not have to use the proposed switching law very exactly in the sense that small shift is admissible in the choice of λ . $\square$

Remark 8. The results in this paper can be easily extended to the case of discrete-time switched system composed of N subsystems $z(k+1) = F_i z(k) + g_i(k, z(k))$, where the constant matrices F_i are commutative pairwise, and a combination $\Pi_{i=1}^N F_i^{\lambda_i}$ of F_i 's is schur stable. $\square$

5. Conclusion

In this paper, we have studied the stability property for a class of nonlinear switched systems, where commutative property holds for the subsystems' linear parts. When no perturbations exist and a linear convex combination of

unstable subsystem matrices is Hurwitz stable, we have proposed a switching law which is based on the convex combination coefficients so that the entire switched system is globally exponentially stable. The switching law activates the Hurwitz stable subsystems (if exist) arbitrarily, and activates the unstable subsystems with their activation time periods satisfying a specified ratio. When nonlinear perturbations exist in the switched systems, we have discussed quantitatively the strength of the perturbations such that the desirable stability property is preserved. Finally, we demonstrated the effectiveness of the switching law when perturbations exist by giving an example, which in the meantime showed how to deal with those linear switched systems that do not satisfy the commutative property in our framework.

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