Convergence Properties of the Membership Set in the Presence of Disturbance and Parameter Uncertainty[†]

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The size of the membership set is investigated in the presence of bounded disturbance and l_2 bounded parameter uncertainty. A tight upper bound of the diameter of the membership set is derived in a deterministic setting, which indicates that the diameter does not converge to zero in general. Then, a probabilistic upper bound of the diameter is derived in a stochastic setting, where the disturbance and the parameter uncertainty are assumed to be random variables and to take a value near the worst ones with a probability. This probabilistic bound leads to the fact that the diameter converges to zero with probability one as the number of samples tends to infinity.

Key Words: Set membership identification, uncertainty, diameter, convergence, stochastic setting.

1. Introduction

The membership set represents the entire set of system parameters that are consistent with measured data and a given bound of disturbance, and it is useful for robustness analysis of system identification. Nowadays, many algorithms of set membership identification are available^{1)~5)}, which produce the membership set itself or an optimal parameter in it. On the other hand, there are a few studies focused on fundamental properties of the membership set^{6)~9)}. In the studies, except for the authors' previous work⁹⁾, the properties have been explored for systems with additive bounded disturbance but no uncertainty in the system parameter.

In this paper, we concentrate our attention on a fundamental property, that is, convergence property of the membership set with respect to the number of samples. We investigate the membership set in the presence of not only disturbance but also parameter uncertainty as the authors' previous work⁹⁾. However, instead of an l_1 bounded uncertainty⁹⁾, we here consider an l_2 bounded uncertainty. Since the l_2 norm is more compatible with persistently exciting condition than the l_1 norm, we can expect to derive a simple formula that is easy to understand, which is demonstrated in this paper.

We first introduce a system with an additive bounded

disturbance and an l_2 bounded parameter uncertainty, and define the membership set. Then, we derive a characterization of the membership set, which has a convenient formula. Based on this characterization, a tight upper bound of the diameter of the membership set is directly obtained in a deterministic setting, which indicates that the diameter does not converge to zero in general. This motivates us to provide another upper bound of the diameter in a stochastic setting^{7)~9)}, where we assume that the disturbance and the parameter uncertainty are random variables and their bounds are *tight*, that is, they take a value near the worst ones with a probability. The obtained probabilistic upper bound of the diameter indicates that the diameter converges to zero with probability one if the bounds of the disturbance and the parameter uncertainty are tight and the regressor is deterministic and persistently exciting. The result also indicates the necessary number of samples which guarantees that the resultant diameter of the membership set is less than a given bound with a given probabilistic confidence. Notice that there are some researches on finite sample analysis^{10)~12)}, but they have not investigated convergence property of the membership set. A numerical example illustrates the obtained result.

2. The Membership Set

Let us describe a single-input single-output (SISO) discrete time system as

$$y_i = \phi_i^T(\theta + \eta_i) + v_i \quad (1)$$

where $y_i \in \mathbf{R}$ is the system output, $\phi_i \in \mathbf{R}^m$ is the measurable regressor vector, $\theta \in \mathbf{R}^m$ is the true but unknown parameter to be identified, $v_i \in \mathbf{R}$ is an unknown additive disturbance, and $\eta_i \in \mathbf{R}^m$ is an unknown parameter

[†] The preliminary version of this paper was presented in the 41st SICE Annual Conference (2002·8)

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(Received July 8, 2002)

(Revised December 9, 2002)

uncertainty at time $i = 1, 2, \dots, N$. The disturbance and parameter uncertainty are bounded, that is,

$$|v_i| \leq \epsilon, \quad \|\eta_i\|_2 \leq \delta \quad (2)$$

where the bounds ϵ and δ are given, and $\|\bullet\|_2$ is the l_2 norm

$$\|\eta_i\|_2 \doteq (\eta_i^T \eta_i)^{1/2}.$$

Then, the membership set is defined as all estimates $\hat{\theta}$ that are consistent with the given data and the given bounds of the disturbance and the parameter uncertainty, that is,

$$\Omega_0^N(\epsilon, \delta) \doteq \bigcap_{i=1}^N \{\hat{\theta} \in \mathbf{R}^m : \exists v_i, \exists \eta_i \text{ s.t. } y_i = \phi_i^T(\hat{\theta} + \eta_i) + v_i, |v_i| \leq \epsilon, \|\eta_i\|_2 \leq \delta\}. \quad (3)$$

To obtain a convenient representation of the membership set, we first rewrite the system (1) as

$$y_i - \phi_i^T \theta = v_i + \phi_i^T \eta_i$$

where the constraints (2) imply

$$|v_i + \phi_i^T \eta_i| \leq \epsilon + \delta \|\phi_i\|_2.$$

Notice that the equality in the above can be attained with

$$v_i = \pm \epsilon, \quad \eta_i = \pm \frac{\delta}{\|\phi_i\|_2} \phi_i$$

which meet (2). Thus, we obtain another characterization of the membership set:

$$\Omega_1^N(\epsilon, \delta) \doteq \bigcap_{i=1}^N \{\hat{\theta} \in \mathbf{R}^m : -(\epsilon + \delta \|\phi_i\|_2) \leq y_i - \phi_i^T \hat{\theta} \leq \epsilon + \delta \|\phi_i\|_2\}. \quad (4)$$

Note that $\Omega_0^N(\epsilon, \delta) = \Omega_1^N(\epsilon, \delta)$ and the quantifiers in the definition of $\Omega_0^N(\epsilon, \delta)$ are eliminated in the characterization $\Omega_1^N(\epsilon, \delta)$. In the rest of this paper, we use this convenient characterization $\Omega_1^N(\epsilon, \delta)$.

We remark that the parameter uncertainty behaves as an additional disturbance and the resultant bound $\epsilon + \delta \|\phi_i\|_2$ depends on the regressor and is time varying. Hence, to identify this membership set, we may use the existing algorithms with some modifications.

3. The Size of the Membership Set

Let us define the diameter of the membership set as

$$\text{dia} \Omega_1^N(\epsilon, \delta) \doteq \sup_{\hat{\theta}_1, \hat{\theta}_2 \in \Omega_1^N(\epsilon, \delta)} \|\hat{\theta}_1 - \hat{\theta}_2\|_2.$$

The diameter is an index of the size of the membership set. In this section, we take a deterministic setting, that is, we derive the diameter based only on the bounds ϵ and δ . On the other hand, we will take a stochastic setting in the following sections, where v_i and η_i are assumed to be random variables.

The following is a standard definition of persistently exciting^{7),8)} in set membership identification.

Definition 1. The regressor ϕ_i is said to be persistently exciting (PE) if there exist some α, β , and $n_0 \geq m$ such that

$$\alpha^2 I \leq \sum_{i=i_0}^{i_0+n_0-1} \phi_i \phi_i^T \leq \beta^2 I \quad (5)$$

for all $i_0 \geq 1$. Here, the matrix inequality $A \leq B$ means $x^T(B - A)x \geq 0$ for all x .

Note that both deterministic and stochastic regressors can be PE.

The PE condition ensures the boundedness of the membership set, and is assumed throughout this paper. For simplicity, let $N = \ell n_0$, where ℓ is an integer. Then, we obtain the result.

Theorem 1. Consider the system (1). Assume that the regressor ϕ_i is deterministic and PE. Then,

$$\text{dia} \Omega_1^N(\epsilon, \delta) \leq \frac{2\sqrt{n_0}}{\alpha}(\epsilon + \delta\beta). \quad (6)$$

Proof: Rewrite the given data as a compact form

$$y \doteq \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} \in \mathbf{R}^N, \quad \Phi^N \doteq \begin{bmatrix} \phi_1^T \\ \phi_2^T \\ \vdots \\ \phi_N^T \end{bmatrix} \in \mathbf{R}^{N \times m}.$$

The PE condition implies

$$(\Phi^N)^T \Phi^N = \sum_{i=1}^{\ell n_0} \phi_i \phi_i^T \geq \alpha^2 \ell I$$

thus we have

$$\|(\Phi^N)^T \Phi^N\|_2^{-1} \|\Phi^N\|_2 \leq \frac{1}{\alpha\sqrt{\ell}} \quad (7)$$

where $\|\bullet\|_2$ is the l_2 induced norm, that is,

$$\|A\|_2 \doteq \sup_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \sqrt{\bar{\lambda}(AA^T)}$$

where A is a real matrix and $\bar{\lambda}(AA^T)$ is the largest eigenvalue of AA^T . Furthermore, for any i , there exists i_0 such that $i_0 \leq i \leq i_0 + n_0 - 1$. Then, the PE condition implies

$$\phi_i \phi_i^T \leq \sum_{i=i_0}^{i_0+n_0-1} \phi_i \phi_i^T \leq \beta^2 I$$

thus

$$\|\phi_i\|_2 \leq \beta$$

for all i . Hence, if $\hat{\theta}_1$ and $\hat{\theta}_2$ are in $\Omega_1^N(\epsilon, \delta)$, then

$$-(\epsilon + \delta\beta) \leq y_i - \phi_i^T \hat{\theta}_1 \leq \epsilon + \delta\beta$$

$$-(\epsilon + \delta\beta) \leq y_i - \phi_i^T \hat{\theta}_2 \leq \epsilon + \delta\beta.$$

This implies

$$-2(\epsilon + \delta\beta) \leq \phi_i^T(\hat{\theta}_1 - \hat{\theta}_2) \leq 2(\epsilon + \delta\beta)$$

thus we have

$$\|\Phi^N(\hat{\theta}_1 - \hat{\theta}_2)\|_2 \leq 2\sqrt{N}(\epsilon + \delta\beta). \quad (8)$$

Using the inequalities (7) and (8), we obtain

$$\begin{aligned} \|\hat{\theta}_1 - \hat{\theta}_2\|_2 &\leq \|(\Phi^N)^T \Phi^N\|^{-1} (\Phi^N)^T \|\epsilon\|_2 \\ &\quad \cdot \|\Phi^N(\hat{\theta}_1 - \hat{\theta}_2)\|_2 \\ &\leq \frac{2\sqrt{N}}{\alpha\sqrt{\ell}} (\epsilon + \delta\beta) \\ &= \frac{2\sqrt{n_0}}{\alpha} (\epsilon + \delta\beta) \end{aligned} \quad (9)$$

for all $\hat{\theta}_1, \hat{\theta}_2 \in \Omega_1^N(\epsilon, \delta)$. We therefore conclude the proof. $\square$

We remark that this theorem includes the existing result⁷⁾ as a special case. In fact, if there is no parameter uncertainty ($\delta = 0$), then the theorem reduces to the previous result.

This theorem gives only an upper bound of the diameter of the membership set. Thus the bound may be conservative. However, it should be noted that the bound is actually the diameter if $\delta = 0$ ⁷⁾ or if $\alpha = \beta$ and $n_0 = m$ (in Appendix A). In this sense, the bound is not conservative. We therefore see that the size of the membership set does not necessarily decrease whenever the number of the samples increases.

4. Convergence Property of the Membership Set

Now, we investigate convergence property of the membership set in a stochastic setting⁹⁾, where the disturbance and the parameter uncertainty are regarded as random variables.

The following definition is introduced in the reference⁷⁾, which is a probabilistic assumption on the disturbance.

Definition 2. Suppose that the disturbance v_i is a random variable satisfying $|v_i| \leq \epsilon$. The bound ϵ is said to be tight if for any $\rho > 0$ and each i

$$\text{Prob}\{-\epsilon \leq v_i \leq -(\epsilon - \rho)\} \geq p_v(\rho)$$

$$\text{Prob}\{\epsilon - \rho \leq v_i \leq \epsilon\} \geq p_v(\rho)$$

for some $1 \geq p_v(\rho) > 0$.

In this paper, we also introduce a probabilistic assumption on the parameter uncertainty.

Definition 3. Suppose that the parameter uncertainty η_i is a random vector satisfying $\|\eta_i\|_2 \leq \delta$. The bound δ is said to be tight if for any $\mu > 0$, each i , and any ϕ_i

$$\text{Prob}\{-\delta\|\phi_i\|_2 \leq \phi_i^T \eta_i \leq -(\delta - \mu)\|\phi_i\|_2\} \geq p_\eta(\mu)$$

$$\text{Prob}\{(\delta - \mu)\|\phi_i\|_2 \leq \phi_i^T \eta_i \leq \delta\|\phi_i\|_2\} \geq p_\eta(\mu)$$

Fig. 1 Tightness in the case $m = 2$

for some $1 \geq p_\eta(\mu) > 0$.

The tightness of the bounds of v_i and η_i means that v_i and η_i take a value around the surface of the possible values with nonzero probability. This is obvious for v_i , and can be seen for η_i if we rewrite the formulae in Definition 3 as

$$\text{Prob}\{-\delta \leq \frac{\phi_i^T}{\|\phi_i\|_2} \eta_i \leq -(\delta - \mu)\} \geq p_\eta(\mu)$$

$$\text{Prob}\{\delta - \mu \leq \frac{\phi_i^T}{\|\phi_i\|_2} \eta_i \leq \delta\} \geq p_\eta(\mu).$$

Actually, when $m = 2$, we can easily illustrate this probabilistic statement with Fig. 1. Suppose that the bound δ is tight with a parameter μ . Then, for a fixed ϕ_i , η_i takes a value in each shaded portion with probability $p_\eta(\mu)$. Since this holds for any ϕ_i , we see that, with a probability, η_i takes a value in the point patterned portion, that is, around the surface of the possible values.

Notice that the definition is quite different from that in the previous paper⁹⁾, where the parameter uncertainty is assumed to be bounded with l_1 norm and their extreme points are especially used in the definition of the tightness. On the other hand, in Definition 3, the surface of the parameter uncertainty are picked up in the entire set of the uncertainty, that is, all directions of the uncertainty are uniformly important.

The next lemma can be obtained immediately.

Lemma 1. Assume that the disturbance v_i and the parameter uncertainty η_i are independent random variables, and their bounds ϵ and δ are tight. Then,

$$\begin{aligned} \text{Prob}\{-(\epsilon + \delta\|\phi_i\|_2) \leq v_i + \phi_i^T \eta_i \\ \leq -(\epsilon - \rho) - (\delta - \mu)\|\phi_i\|_2\} \geq p_v(\rho)p_\eta(\mu) > 0 \end{aligned}$$

$$\begin{aligned} \text{Prob}\{\epsilon - \rho + (\delta - \mu)\|\phi_i\|_2 \leq v_i + \phi_i^T \eta_i \\ \leq \epsilon + \delta\|\phi_i\|_2\} \geq p_v(\rho)p_\eta(\mu) > 0. \end{aligned}$$

That is, the bound $\epsilon + \delta\|\phi_i\|_2$ of $v_i + \phi_i^T \eta_i$ is also tight.

For the sake of simplicity, suppose that the regressor ϕ_i in (1) is periodic with period n ($n \geq m$) and $N = \ell n$. Then, we can write

$$\phi_{i_p+kn} = \phi_{i_p}$$

where $i_p = 1, 2, \dots, n$, $k = 0, 1, 2, \dots, \ell - 1$, and $\ell \geq 1$. If the regressor is PE, we can assume that $n_0 = n$ in (5) without loss of generality, that is,

$$\alpha^2 I \leq (\Phi^n)^T \Phi^n \leq \beta^2 I \quad (10)$$

holds for some α and β , where

$$\Phi^n = \begin{bmatrix} \phi_1^T \\ \phi_2^T \\ \vdots \\ \phi_n^T \end{bmatrix} \in \mathbf{R}^{n \times m}.$$

The following theorem is the main result of this paper, where the size of the membership set is evaluated with a probabilistic confidence.

Theorem 2. Consider the system (1). Assume that the regressor ϕ_i is deterministic, periodic, and PE. In addition, v_i and η_i are independent random variables, and their bounds are tight. Then, the size of the membership set can be evaluated with a probabilistic confidence as

$$\text{Prob} \left\{ \text{dia } \Omega_1^N(\epsilon, \delta) \leq \frac{2\sqrt{n}}{\alpha}(\rho + \mu\beta) \right\} \geq c_p > 0 \quad (11)$$

where c_p is a positive constant satisfying

$$c_p \leq \left\{ 1 - 2(1 - p_v(\rho)p_\eta(\mu))^\ell \right\}^n. \quad (12)$$

Proof: We first fix i_p and consider the corresponding subsequence. For any $\hat{\theta}$, we have the identity

$$\phi_{i_p}^T(\theta - \hat{\theta}) = y_{i_p+kn} - \phi_{i_p}^T \hat{\theta} - (v_{i_p+kn} + \phi_{i_p}^T \eta_{i_p+kn})$$

by rewriting (1). For $\hat{\theta}$ in the membership set, we see

$$-(\epsilon + \delta \|\phi_{i_p}\|_2) \leq y_{i_p+kn} - \phi_{i_p}^T \hat{\theta} \leq \epsilon + \delta \|\phi_{i_p}\|_2.$$

Thus, if there exists k_u such that

$$\begin{aligned} \epsilon - \rho + (\delta - \mu) \|\phi_{i_p}\|_2 &\leq v_{i_p+k_u n} + \phi_{i_p}^T \eta_{i_p+k_u n} \\ &\leq \epsilon + \delta \|\phi_{i_p}\|_2 \end{aligned} \quad (13)$$

then we have

$$\phi_{i_p}^T(\theta - \hat{\theta}) \leq \rho + \mu \|\phi_{i_p}\|_2.$$

In the same way, if there exists k_l such that

$$\begin{aligned} -\epsilon - \delta \|\phi_{i_p}\|_2 &\leq v_{i_p+k_l n} + \phi_{i_p}^T \eta_{i_p+k_l n} \\ &\leq -(\epsilon - \rho) - (\delta - \mu) \|\phi_{i_p}\|_2 \end{aligned} \quad (14)$$

then we also have the inequality

$$-(\rho + \mu \|\phi_{i_p}\|_2) \leq \phi_{i_p}^T(\theta - \hat{\theta}).$$

Hence, when such k_u and k_l both exist, it turns out that

$$|\phi_{i_p}^T(\theta - \hat{\theta})| \leq \rho + \mu \|\phi_{i_p}\|_2$$

for any $\hat{\theta} \in \Omega_1^N(\epsilon, \delta)$. This implies

$$|\phi_{i_p}^T(\hat{\theta}_1 - \hat{\theta}_2)| \leq 2(\rho + \mu\beta) \quad (15)$$

for any $\hat{\theta}_1, \hat{\theta}_2 \in \Omega_1^N(\epsilon, \delta)$, where $\|\phi_{i_p}\|_2 \leq \beta$ follows (10). Therefore, we can see that

$$\begin{aligned} \text{Prob} \left\{ |\phi_{i_p}^T(\hat{\theta}_1 - \hat{\theta}_2)| \leq 2(\rho + \mu\beta), \forall \hat{\theta}_1, \hat{\theta}_2 \in \Omega_1^N(\epsilon, \delta) \right\} \\ \geq 1 - 2(1 - p_v(\rho)p_\eta(\mu))^\ell \end{aligned} \quad (16)$$

holds for this fixed i_p . This is because the probability that $v_{i_p+kn} + \phi_{i_p}^T \eta_{i_p+kn}$ does not fall in (13) (or (14)) during $k = 0, 1, \dots, \ell - 1$ is $(1 - p_v(\rho)p_\eta(\mu))^\ell$ from Lemma 1. Now, if (15) holds for all $i_p = 1, 2, \dots, n$, we have

$$\|\Phi^n(\hat{\theta}_1 - \hat{\theta}_2)\|_2 \leq 2\sqrt{n}(\rho + \mu\beta)$$

and thus we obtain

$$\begin{aligned} \|\hat{\theta}_1 - \hat{\theta}_2\|_2 &\leq \|(\Phi^n)^T \Phi^n\|^{-1} (\Phi^n)^T \|\Phi^n(\hat{\theta}_1 - \hat{\theta}_2)\|_2 \\ &\leq \frac{2\sqrt{n}}{\alpha}(\rho + \mu\beta) \end{aligned}$$

where $\|(\Phi^n)^T \Phi^n\|^{-1} (\Phi^n)^T\|_2 \leq 1/\alpha$ follows (10). Here, to ensure (15) for all i_p , (16) should hold for all i_p , $i_p = 1, 2, \dots, n$. We therefore conclude the proof. $\square$

Note that Theorem 2 says that

$$\text{Prob} \left\{ \text{dia } \Omega_1^N(\epsilon, \delta) \leq \frac{2\sqrt{n}}{\alpha}(\rho + \mu\beta) \right\} \rightarrow 1 \quad \text{as } \ell \rightarrow \infty$$

for all ρ and μ when we select

$$c_p = \left\{ 1 - 2(1 - p_v(\rho)p_\eta(\mu))^\ell \right\}^n.$$

Since ρ and μ can be chosen arbitrarily, it turns out that, with probability one,

$$\text{dia } \Omega_1^N(\epsilon, \delta) \rightarrow 0 \quad \text{as } \ell \rightarrow \infty.$$

Thus we conclude that the tightness implies convergence property of the membership set.

Although Theorem 2 gives a probabilistic upper bound of the diameter of the membership set, it also gives a deterministic upper bound if $c_p = 1$, that is, $p_v(\rho) = p_\eta(\mu) = 1$. This selection is possible when we set $\rho = 2\epsilon$, $\mu = 2\delta$. Then, we obtain

$$\text{dia } \Omega_1^N(\epsilon, \delta) \leq \frac{4\sqrt{n}}{\alpha}(\epsilon + \delta\beta).$$

Comparing the above with (6) in Theorem 1, we see that the deterministic bound derived from Theorem 2 is twice as conservative as that of Theorem 1.

Now, let us consider the case that c_p is the probabilistic confidence that we want to ensure and $2\sqrt{n}(\rho + \mu\beta)/\alpha$ is the bound of the diameter of the membership set that we want to specify. Note that ρ and μ determine the bound. Thus, it would be preferable that we can calculate the necessary number of repetition ℓ which satisfies

Table 1 Relation between confidence and tightness

$\hat{\mu}$	0.05	0.01	0.005	0.001
ℓ	608	6772	19143	213917
c_p (%)	90.11	90.08	90.09	90.08
c_m (%)	100	100	100	100

(11) for given c_p , ρ , and μ . This can be done as follows. Suppose that the distributions of the disturbance and the parameter uncertainty are known. Then, the probability of $p_v(\rho)p_\eta(\mu)$ can be computed. Solving the inequality (12), we have

$$\ell \geq \frac{\ln\left(\frac{1-(c_p)^{1/n}}{2}\right)}{\ln(1-p_v(\rho)p_\eta(\mu))}. \quad (17)$$

This gives the necessary number of repetition ℓ explicitly.

5. Numerical Example

Let us consider the case that no additive disturbance exists for simplicity, that is,

$$y_i = \phi_i^T(\theta + \eta_i),$$

$$\theta = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \phi_i = \begin{bmatrix} u_{i-1} \\ u_i \end{bmatrix}, \quad \eta_i = \begin{bmatrix} \eta_{1i} \\ \eta_{2i} \end{bmatrix}.$$

The parameter uncertainty η_i satisfies $\|\eta_i\|_2 \leq 0.1$ and is assumed to be an independent and uniformly distributed random variable. In this case, the probability on the tightness is calculated as

$$p_\eta(\mu) = \frac{1}{\pi} \left(\cos^{-1}(1 - \hat{\mu}) - (1 - \hat{\mu})\sqrt{2\hat{\mu} - \hat{\mu}^2} \right)$$

where $\hat{\mu} \doteq \mu/\delta$ is the index of tightness we choose later.

We used the following sequence of the regressor

$$\phi_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \phi_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \phi_3 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

which is periodic with period 3 and PE ($\alpha = \sqrt{3}, \beta = 5$).

For several indices of tightness $\hat{\mu}$, we specified the confidence c_p around 90% and calculated the necessary number of repetition ℓ satisfying (17). The results are shown in Table 1. Here c_m is the measured confidence, each of which is the average of 10,000 trials. The significant figure is four. Subsequently, choosing 5%, 10%, and 15% of ℓ in Table 1, we also calculated the corresponding confidence c_m by simulations. The results are shown in Table 2.

Table 1 shows that $c_p \leq c_m$, thus we confirmed Theorem 2. However, there is a gap between c_p and c_m . Note that a comment related to conservatism in Theorem 2 is stated in the second paragraph below the proof of the theorem. Table 2 shows how much conservative c_p is from the view point of the number of repetition. For this example, we needed at least 10% and at most 15% of ℓ satisfying (17) to ensure that c_m is around 90%.

Table 2 Relation between confidence and the number of repetition

$\hat{\mu}$	0.05	0.01	0.005	0.001
ℓ (5%)	31	339	958	10696
c_m (%)	62.87	63.17	63.33	63.05
ℓ (10%)	61	678	1915	21392
c_m (%)	89.70	89.92	89.48	89.52
ℓ (15%)	91	1016	2872	32088
c_m (%)	98.28	97.98	98.00	98.11

6. Concluding Remarks

In this paper, we have investigated convergence properties of the membership set in a deterministic and a stochastic settings. In particular, we have shown a quantitative relationship between the diameter of the membership set and the number of samples. We have also derived the necessary number of samples which meets the specification for the diameter with a specified probabilistic confidence.

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Appendix A. The case $\alpha = \beta$ and $n_0 = m$

Let us define

$$\Phi^N = \begin{bmatrix} \alpha I & \alpha I & \cdots & \alpha I \end{bmatrix}^T.$$

The PE condition with $\alpha = \beta$ and $n_0 = m$ obviously holds for this Φ^N . Furthermore, we see

$$(\Phi^N)^T \Phi^N = \beta^2 \ell I$$

and

$$\|\phi_i\|_2 = \beta$$

for all i . Thus, from the characterization (4) of the membership set, there exist $\theta_1, \theta_2 \in \Omega_1^N(\epsilon, \delta)$ such that

$$y - \Phi^N \theta_1 = -(\epsilon + \delta\beta)e, \quad y - \Phi^N \theta_2 = (\epsilon + \delta\beta)e$$

where

$$e = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T.$$

Then we have

$$\Phi^N(\theta_1 - \theta_2) = 2(\epsilon + \delta\beta)e.$$

We therefore see that

$$\begin{aligned} \|\theta_1 - \theta_2\|_2^2 &= (\theta_1 - \theta_2)^T \{(\Phi^N)^T \Phi^N\}^{-1} (\Phi^N)^T \Phi^N (\theta_1 - \theta_2) \\ &= \frac{1}{\beta^2 \ell} \|\Phi^N(\theta_1 - \theta_2)\|_2^2 \\ &= \frac{4N}{\beta^2 \ell} (\epsilon + \delta\beta)^2 \end{aligned}$$

that is,

$$\|\theta_1 - \theta_2\|_2 = \frac{2\sqrt{m}}{\beta} (\epsilon + \delta\beta).$$

This means

$$\text{dia } \Omega_1^N(\epsilon, \delta) \geq \|\theta_1 - \theta_2\|_2 = \frac{2\sqrt{m}}{\beta} (\epsilon + \delta\beta)$$

and with Theorem 1, we conclude

$$\text{dia } \Omega_1^N(\epsilon, \delta) = \frac{2\sqrt{m}}{\beta} (\epsilon + \delta\beta).$$

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