Unfalsified Control Approach to Parameter Space Design of PID Controllers[†]

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In this paper, unfalsified control concept is applied to the parameter space design problem of PID gains for mixed sensitivity control problem, and a new method of drawing unfalsified region on the parameter plane is proposed. This method does not require constructing a mathematical model for design, but draws the unfalsified region directly from the input-output responses of the plant. Therefore, it is expected that this method can be applied to complex nonlinear plants. By numerical experiments, it is found that sufficiently large region cannot be falsified by using a step response data, but can be falsified by using the responses of the plant excited by sinusoidal inputs for many frequencies. If the unfalsified region is empty, it can be concluded that no solution exist. Usefulness is demonstrated by a numerical example for a nonlinear plant with backlash.

Key Words: unfalsified control, PID control, parameter space design, mixed sensitivity problem, nonlinear plant

1. Introduction

When a plant contains nonlinear elements such as backlash, friction, and saturation, there is no design method that can be directly applied to such a complex plant¹²⁾. Therefore, unwillingly we approximate them by local linearization or describing function, or treat them as sector nonlinearities. Since this difficulty is fundamentally caused by our taking model-based design approach in control engineering, we can avoid it by developing a data-driven design method that does not require plant model. Unfalsified control concept proposed by Safonov¹⁾ is very interesting because it gives us such data-driven design methods. It has been applied to both adaptive control^{2),3)} and off-line controller design^{4),5)}.

On the other hand, the author has already proposed a parameter space design method of PID controllers for H_∞ control problem and robust sensitivity control problem^{6)~10)}. These are convenient methods of drawing a necessary and sufficient solution set on the parameter plane. However, these methods use the frequency response of the plant, and therefore cannot be applied to nonlinear plants.

In the reference²⁾, automatic tuning of PID controller is proposed for a mixed sensitivity performance index based on unfalsified concept, and it seems to be a promising

approach for parameter space design if the following difficulties are got rid of.

- a) In the unfalsified approach, a finite number of candidate controllers are prepared and some of them will be falsified adaptively by the input-output data. However, we want to treat infinite number of PID gains on the parameter plane.
- b) Unfalsified adaptive controller makes use of the inverse of the controller. But we cannot obtain a proper algorithm for drawing the solution set because of the inversion, and it is necessary to avoid it.
- c) Relation between the largeness of the falsified region and the class of input-output data has not been examined in the previous studies, and it is important to find a class of input-output data that make the falsified region as large as possible.

In the reference 5), the first two problems are cleared for an l_∞ norm performance index and a class of linear time-invariant controllers, and the design problem is reduced to a linear programming problem.

In this paper, we will propose a parameter space design method of PID controllers for a mixed sensitivity performance index via unfalsified control approach.

2. Problem formulation

Let us consider the feedback system described by

$$y = Pe \quad (1)$$

$$u = Ky \quad (2)$$

$$e = w - u \quad (3)$$

where P is a single-input single-output plant whose property is defined by a set of input output data, and K is

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Fig. 1 Mixed sensitivity problem

a controller. y, u, e, w are discrete time signals with sampling time ΔT where $u = \{u(1), u(2), \dots\}$ and other signals are defined in the same way. We assume that the input output data $e(i), y(i)$ of the plant are given where the initial condition of the plant is zero. The controller K is a discrete time PID controller of the form

$$K(z^{-1}) = p_1 + p_2 \frac{z^{-1}}{1 - z^{-1}} + p_3(1 - z^{-1}) \quad (4)$$

Introduce weighting functions $V(z^{-1}), W(z^{-1})$ and new variables

$$z_1 = Ve, z_2 = Wu \quad (5)$$

as shown in Fig. 1, and our design specification is given by the l_2 -gain condition: for all $w \in l_2$

$$\|z\|_2 < \gamma \|w\|_2 \quad (6)$$

where $z = (z_1, z_2)^T$ and

$$\|z\|_2 = \sqrt{\sum_{i=1}^{\infty} \{z_1^2(i) + z_2^2(i)\}} \quad (7)$$

$$\|w\|_2 = \sqrt{\sum_{i=1}^{\infty} w^2(i)} \quad (8)$$

Our problem is to obtain such PID gains p_1, p_2, p_3 that do not satisfy (6).

Remark 1 If the plant were linear time-invariant with a transfer function $P(z^{-1})$, the l_2 -gain condition (6) is equivalent to a mixed sensitivity control condition described by the H_∞ norm condition.

$$\left\| \begin{pmatrix} VS \\ WT \end{pmatrix} \right\|_\infty < \gamma \quad (9)$$

where the sensitivity function $S(z^{-1})$ and the complementary sensitivity function $T(z^{-1})$ are given by

$$S = \frac{1}{1 + PK}, \quad T = \frac{PK}{1 + PK} \quad (10)$$

In this case, namely when the plant is linear time-invariant and the frequency response is available, a necessary and sufficient region of PID gains that satisfy (9)

can be drawn on the parameter plane by the frequency domain method of the references 8) and 9).

3. Falsification condition

Lemma 1 (Theorem 2.1 of the reference 11)) Let F be a causal operator from the extended space W_e into itself. If F is bounded on W_e , then F maps W into itself and the bounds of F on W_e and W are equal. Conversely, if F maps W into itself and is bounded on W , then F is bounded on W_e and the bounds of F on W and W_e are equal.

This property has already been used to derive a falsification condition in the reference 1). By using this property, we will give a new unfalsification condition suitable for parameter space design of PID gain.

Represent the integral and derivative values of y as

$$y_I = \frac{z^{-1}}{1 - z^{-1}} y, \quad y_D = (1 - z^{-1})y \quad (11)$$

and the responses of W to y, y_I, y_D as

$$y_W = Wy, \quad y_{WI} = Wy_I, \quad y_{WD} = Wy_D \quad (12)$$

Then, put

$$v_1(i) = [y_W(i), y_{WI}(i), y_{WD}(i)]^T$$

$$v_2(i) = [y(i), y_I(i), y_D(i), e(i)]^T$$

and define the following matrices where $c_1 \in R$, $R \in R^{3 \times 3}$, $Q \in R^{3 \times 3}$, $q \in R^{3 \times 1}$, $c_2 \in R$.

$$c_1(m) := \sum_{i=1}^m z_1^2(i) \quad (13)$$

$$R(m) := \sum_{i=1}^m v_1(i)v_1(i)^T \quad (14)$$

$$\begin{pmatrix} Q(m) & q(m) \\ q^T(m) & c_2(m) \end{pmatrix} := \sum_{i=1}^m v_2(i)v_2(i)^T \quad (15)$$

Theorem 1 The l_2 gain condition (6) is satisfied for all $w \in l_2$, if and only if

$$p^T [Q(m) - \frac{1}{\gamma^2} R(m)] p + 2p^T q(m) + c_2(m) - \frac{1}{\gamma^2} c_1(m) > 0, m = 1, 2, \dots \quad (16)$$

is satisfied for all $w \in l_{2e}$ where p is a parameter vector given by

$$p = (p_1, p_2, p_3)^T \quad (17)$$

proof From Lemma 1, the l_2 gain condition (6) is satisfied, if and only if

$$\|z\|_{2[1,m]} < \gamma \|w\|_{2[1,m]}, m = 1, 2, \dots \quad (18)$$

is satisfied for all $w \in l_{2e}$ where $\|w\|_{2[1,m]}$ denotes the l_2 norm of the truncated signal;

$$\|w\|_{2[1,m]} = \sqrt{\sum_{i=1}^m w^2(i)} \quad (19)$$

Next, let us represent (18) by using the parameter vector p . Substitution of $z_2 = WKy$ and $w = Ky + e$ into (18) gives

$$\begin{aligned} \|z_1\|_{2[1,m]}^2 + \|WKy\|_{2[1,m]}^2 \\ < \gamma^2 \|Ky + e\|_{2[1,m]}^2 \end{aligned} \quad (20)$$

Since

$$Ky = p_1 y + p_2 y_I + p_3 y_D \quad (21)$$

$$WKy = p_1 y_W + p_2 y_{WI} + p_3 y_{WD} \quad (22)$$

, each term is represented as

$$\|z_1\|_{2[1,m]}^2 = c_1(m) \quad (23)$$

$$\|WKy\|_{2[1,m]}^2 = p^T R(m) p \quad (24)$$

$$\begin{aligned} \|Ky + e\|_{2[1,m]}^2 \\ = \begin{pmatrix} p^T & 1 \end{pmatrix} \begin{pmatrix} Q(m) & q(m) \\ q^T(m) & c_2(m) \end{pmatrix} \begin{pmatrix} p \\ 1 \end{pmatrix} \end{aligned} \quad (25)$$

Substituting these terms into (20) gives (16) immediately. This completes the proof.

Remark 2 Note that controller inversion does not appear in the above derivation. If we reverse the order of the plant and the controller of the feedback system in Fig. 1, which is used in the reference 1), a fictitious reference input w is given by $w = y + K^{-1}u$. In this case, K must be causally left invertible and a falsification condition suitable for drawing the solution set cannot be derived.

Remark 3 The region that satisfies (16) can be drawn on the parameter plane in the following way. Suppose that the plane is given by $p_3 = a$. First, substitute $p = (p_1, p_2, a)^T = (\tilde{p}^T, a)^T$ into (16) and represent it in the next form

$$\tilde{p}^T \tilde{Q} \tilde{p} + 2\tilde{p}^T \tilde{q}(m) + \tilde{c}(m) > 0 \quad (26)$$

where $\tilde{p} = (p_1, p_2)^T$. Then draw the falsified region by the algorithm given in Appendix. The shape of the region will be ellipse or hyperbola depending on the sign of the eigenvalues λ_1, λ_2 of the matrix Q , if λ_1, λ_2 are nonzero. If $\lambda_1 = 0$ and $\lambda_2 = 0$, the region is a half plane. If one of λ_i is zero, the boundary of the region is a parabola.

4. Input-output data selection

In this section, we will examine a proper class of input-output data that makes the falsified region larger by a numerical example. Let us consider the system shown in

Fig. 2 Step response $y(t)$ and $e(t)$

Fig. 3 Falsified region by data-driven method (a step response data)

Fig. 1 where

$$\begin{aligned} P(s) &= \frac{1}{s^2 + s + 1}, K(s) = 1 + \frac{0.01}{s} \\ V(s) &= \frac{100s + 1}{255s}, W(s) = \frac{0.5s + 1}{1.5} \end{aligned}$$

The unit step response $e(t), y(t)$ is obtained by Simulink in the interval $[0, 20]$ [sec] with the sampling time $\Delta T = 0.05$ [sec] as shown in Fig. 2. Then, by using the data, the falsified regions determined by (16) with $\gamma = 1$ are drawn on the parameter plane in Fig. 3 for $m = 1, 2, \dots, 400$.

On the other hand, since the plant is linear time-invariant, the exact region can be obtained by the frequency domain method⁸⁾ if the plant transfer function is supposed to be known. Discretization of $P(s), V(s), W(s)$ with the sampling time $\Delta T = 0.05$ gives

$$\begin{aligned} P_z(z^{-1}) &= \frac{0.001229z + 0.001209}{z^2 - 1.949z + 0.9512} \\ V_z(z^{-1}) &= \frac{0.3923z - 0.3921}{z - 1} \\ W_z(z^{-1}) &= \frac{14z - 12.67}{z + 1} \end{aligned}$$

and the exact region is shown in Fig. 4.

The unfalsified region in Fig. 3 is about ten times larger than the exact region in Fig. 4. We have also tested for sawtooth and random reference inputs, but the unfalsified

Fig. 4 Exact region by frequency domain method

Fig. 5 Response $y(t)$ and $e(t)$ for saw reference input Fig. 6 Falsified region by data-driven method
(a saw response data)

regions are much larger than the exact region similarly. The sawtooth case is shown in Fig. 5 and Fig. 6. This numerical experiment suggests that the region which can be falsified by truncating a single input-output data may be not large enough. Note that this does not deny the adaptive method where a single data is also used in real time, because the PID gain is fixed in our method but changed adaptively in the tuning method.

In order to overcome this difficulty, we will use sinusoidal inputs with many frequencies $w(t) = \sin \omega t$. We cannot prove this data selection works well for nonlinear systems generally, but at least we may justify the selection

for linear time-invariant stable systems as follows.

Let us consider a system $G(s) \in RH_\infty$. Then, the response $z(t) \in R^q$ of the system for the input $w(t) = \sin \omega t, t \geq 0$ satisfies

$$z(t) \rightarrow z_\infty(t) = A \sin(\omega t + \phi) \quad (t \rightarrow \infty) \quad (27)$$

where $A = |G(j\omega)|$ and $\phi = \angle G(j\omega)$, and the error $e(t) = z(t) - z_\infty(t)$ is bounded by

$$|e(t)| < M e^{-\alpha t} \quad (28)$$

for certain positive constants M and α . On this condition, the next theorem is satisfied.

Theorem 2 For a given $\delta > 0$,

$$\frac{\sqrt{\int_0^T z^2(t) dt}}{\sqrt{\int_0^T w^2(t) dt}} \leq |G(j\omega)| [1 + \delta] \quad (29)$$

is satisfied for a sufficiently large T that satisfies

$$T > \frac{1}{2\omega} + \frac{2}{2\delta + \delta^2} \left(\frac{1}{\omega} + \frac{1}{\alpha A^2} \left[2MA + \frac{M^2}{2} \right] \right) \quad (30)$$

Proof)

$$\begin{aligned} \int_0^T z^2(t) dt &= \int_0^T \{ z_\infty^2(t) + 2e(t)z_\infty(t) + e^2(t) \} dt \\ &\leq A^2 \int_{\phi/\omega}^{T+\phi/\omega} \sin^2 \omega \hat{t} d\hat{t} + 2MA \left(\frac{1}{\alpha} - \frac{1}{\alpha} e^{-\alpha T} \right) \\ &\quad + M^2 \left(\frac{1}{2\alpha} - \frac{1}{2\alpha} e^{-2\alpha T} \right) \\ &\leq A^2 \int_0^T w^2(t) dt + \frac{A^2}{\omega} + \frac{1}{\alpha} \left(2MA + \frac{M^2}{2} \right) \end{aligned}$$

Further,

$$\int_0^T w^2(t) dt = \frac{T}{2} - \frac{1}{4\omega} \sin 2\omega T \quad (31)$$

The left-hand side of (29) is bounded as

$$\frac{\sqrt{\int_0^T z^2(t) dt}}{\sqrt{\int_0^T w^2(t) dt}} \leq A \sqrt{1 + \frac{\frac{1}{\omega} + \frac{1}{\alpha A^2} (2MA + \frac{M^2}{2})}{\frac{T}{2} - \frac{1}{4\omega} \sin 2\omega T}} \quad (32)$$

The condition that the right-hand side of (32) is smaller than $A(1 + \delta)$ gives (30). This completes the proof.

This theorem implies that the time domain condition $\int_0^T z^2(t) dt < \gamma^2 \int_0^T w^2(t) dt$ comes closer to the frequency domain condition $|G(j\omega)| < \gamma, \omega \in R$ for a sufficiently large T . Therefore, the condition (16) is almost equal to the frequency domain condition for a sufficiently large m , namely $m = n$. Therefore we may only draw the region of (16) for $m = n$ instead of $m = 1, \dots, n$ for each frequency.

By applying this method to the above numerical example for 50 frequencies [rad/sec] logarithmically equally

Fig. 7 Falsified region by data-driven method (many sinusoidal reference inputs)

spaced between 0.0001 and 10, we obtain the falsified region as shown in Fig. 7. Each curve corresponds to each frequency and the falsified region is very close to the exact region.

Let us explain a procedure of finding a candidate PID gain. As an example, we try to make γ of (6) smaller.

Design procedure

(1) Obtain $e(k\Delta T)$ and $y(k\Delta T)$, $k = 1, \dots, n$ for the sinusoidal inputs $w(t) = A \sin(\omega_i t)$, $i = 1, \dots, N$ with the initial condition of the plant zero for a feedback system in Fig. 1. ω_i 's are N frequencies logarithmically equally spaced between 10^{d_1} and 10^{d_2} . The amplitude A may be determined by the range of the responses.

(2) Choose $p_3 = a$ (at first, usually $a = 0$) and draw the falsified region that does not satisfy (16) for $m = n$ on (p_1, p_2, a) plane for each frequency following Remark 3. The unfalsified region will shrink by setting γ smaller.

(3) Choose the value of $p_2 = b$ from the unfalsified region, and draw the falsified region on (p_1, b, p_3) plane for each frequency. The unfalsified region will shrink by setting γ smaller.

(4) After having iterated the above steps (2) and (3), we choose a candidate controller.

Remark 4 It is easier to get the data experimentally for mechanical systems than chemical processes. Even for mechanical systems, it is desirable to decrease the amount of data. Since the curves that form the boundary of unfalsified region is not so many as shown in Fig. 7, we can expect that it will be decreased. On the other hand, if we have a detailed model for simulation, our method can be applied to it without constructing a simple model for controller design.

Fig. 8 Mixed-sensitivity problem for a nonlinear plant

5. Numerical example : nonlinear plant case

Let us consider the nonlinear system with the performance weights as shown in Fig. 8 where a backlash nonlinearity with the width one is inserted at the plant input and $V(qs) = \frac{100qs+1}{255qs}$ is used instead of $V(s)$ for the tuning of the sensitivity weight. We examine the minimization of q with $\gamma = 1$.

Unfalsified region shrinks as q decreases from 1 as shown in Fig. 9, 10, 11, and finally it disappears at $q = 0.234$ for $p_1 = 0.38, p_2 = 0.028$. This means that no PI gain make q smaller than $q = 0.234$. If the backlash is removed, the unfalsified region disappears at $q = 0.0119$ for $p_1 = 0.55, p_2 = 0.0225$. Increase of q from 0.0119 to 0.234 suggests that the backlash deteriorates the performance. The step responses are shown in Fig. 12 where

a) Plant is linear and the controller is

$$K_1(z^{-1}) = 0.55 + \frac{0.0225}{1 - z^{-1}}$$

b) Plant contains backlash and the controller is $K_1(z^{-1})$,

c) Plant contains backlash and the controller is

$$K_2(z^{-1}) = 0.38 + \frac{0.028}{1 - z^{-1}}$$

6. Conclusion

In this paper, a design method of PID controllers on the parameter plane is proposed for a mixed sensitivity problem. Since the cost function can be expressed in terms of quadratic inequalities (ellipses and hyperbolas) in the PID parameter vector, the falsified region can be drawn easily on the parameter plane. An appropriate input-output data set that gives a larger estimate of the falsified region is examined by simulation, and it is found that a single step response data does not give a sufficiently large estimate, but a set of data given by sinusoidal reference inputs does. This method may be suitable for those mechanical systems to which a vibration test can be applied.

Fig. 9 Parameter plane for $q = 0.5$ Fig. 10 Parameter plane for $q = 0.25$ Fig. 11 Parameter plane for $q = 0.24$

Fig. 12 Step responses

Moreover, if we already have a detailed model for simulation, we can apply our method directly.

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Appendix

We will summarize how to draw the region Φ defined by

$$\Phi = \{p \in R^2 | p^T Q p + 2p^T q + r > 0\} \quad (33)$$

Decompose the symmetric matrix Q as

$$Q = U^T \Lambda U, \quad \Lambda = \text{diag}(\lambda_1, \lambda_2) \quad (34)$$

and represent (33) as

$$x^T \Lambda x + 2x^T \hat{q} + r > 0 \quad (35)$$

where $x = U^T p$ and $\hat{q} = U^T q$.

In the following, put $c_0 = \hat{q}^T \Lambda^{-1} \hat{q} - r$.

(1) $\lambda_1 > 0, \lambda_2 > 0$

(a) $c_0 < 0$, then all p

(b) $c_0 > 0$, then outside the ellipse given by

$$p = U \sqrt{c_0} \Lambda^{-\frac{1}{2}} y - U \Lambda^{-1} \hat{q}$$

where $y = (\sin \theta, \cos \theta)^T$ for $0 \leq \theta \leq 2\pi$

(2) $\lambda_1 < 0, \lambda_2 < 0$

(a) $c_0 > 0$, then empty set.

(b) $c_0 < 0$, then inside the ellipse given by

$$p = U \sqrt{-c_0} (-\Lambda)^{-\frac{1}{2}} y - U \Lambda^{-1} \hat{q}$$

where $y = (\sin \theta, \cos \theta)^T$ for $0 \leq \theta \leq 2\pi$

(3) $\lambda_1 > 0, \lambda_2 < 0$

(a) $c_0 > 0$, outer region of the hyperbolic defined by

$$p = U \sqrt{c_0} \text{diag}(\lambda_1^{-\frac{1}{2}}, (-\lambda_2)^{-\frac{1}{2}}) y - U \Lambda^{-1} \hat{q} \quad (36)$$

where $y = \frac{\pm 1}{\sqrt{\cos^2 \theta - \sin^2 \theta}} (\cos \theta, \sin \theta)^T$ for $-\pi/4 < \theta < \pi/4$.

(b) $c_0 < 0$, inner region of the hyperbolic defined by

$$p = U \sqrt{-c_0} \text{diag}(\lambda_1^{-\frac{1}{2}}, (-\lambda_2)^{-\frac{1}{2}}) y - U \Lambda^{-1} \hat{q} \quad (37)$$

where $y = \frac{\pm 1}{\sqrt{\sin^2 \theta - \cos^2 \theta}} (\cos \theta, \sin \theta)^T$ for $\frac{1}{4}\pi < \theta < \frac{3}{4}\pi$.

(4) $\lambda_1 < 0, \lambda_2 > 0$

(a) $c_0 > 0$, outer region of the hyperbolic defined by

$$p = U \sqrt{c_0} \text{diag}((-\lambda_1)^{-\frac{1}{2}}, (\lambda_2)^{-\frac{1}{2}}) y - U \Lambda^{-1} \hat{q} \quad (38)$$

where $y = \frac{\pm 1}{\sqrt{\cos^2 \theta - \sin^2 \theta}} (\cos \theta, \sin \theta)^T$ for $\frac{1}{4}\pi < \theta < \frac{3}{4}\pi$.

(b) $c_0 < 0$, inner region of the hyperbolic defined by

$$p = U \sqrt{-c_0} \text{diag}((-\lambda_1)^{-\frac{1}{2}}, (\lambda_2)^{-\frac{1}{2}}) y - U \Lambda^{-1} \hat{q} \quad (39)$$

where $y = \frac{\pm 1}{\sqrt{\sin^2 \theta - \cos^2 \theta}} (\cos \theta, \sin \theta)^T$ for $-\frac{1}{4}\pi < \theta < \frac{1}{4}\pi$.

The case of $\lambda_1 \lambda_2 = 0$ is given in the following, though it does not occur in generic.

(1) $\lambda_1 = 0, \lambda_2 = 0$

The unfalsified region is a half-plane given by

$$2p^T \hat{q} + r > 0$$

(2) $\lambda_2 = 0$

(a) $\hat{q}_2 \neq 0$

The boundary of the region, a parabola, is given by

$p = U(x_1, x_2)^T$ and

$$x_1 = X - \frac{\hat{q}_1}{\lambda_1}$$

$$x_2 = -\frac{\lambda_1}{2\hat{q}_2} Y - \left(r - \frac{\hat{q}_1^2}{\lambda_1} \right) \frac{1}{2\hat{q}_2}$$

where X and Y satisfy $Y = X^2$. If $\lambda_1 > 0$, the unfalsified region is outside the parabola. If $\lambda_1 < 0$, the unfalsified region is inside the parabola.

(b) $\hat{q}_2 = 0$

If $\lambda_1 > 0$ and $D_1 = \hat{q}_1^2 - \lambda_1 r < 0$, all points are unfalsified. If $\lambda_1 > 0$ and $D_1 \geq 0$, $x_1 \leq \underline{x}_1$ or $x_1 \geq \bar{x}_1$ where

$$\underline{x}_1 = \frac{-\hat{q}_1 - \sqrt{D_1}}{\lambda_1}, \quad \bar{x}_1 = \frac{-\hat{q}_1 + \sqrt{D_1}}{\lambda_1}$$

If $\lambda_1 < 0$ and $D_1 < 0$, all points are falsified. If $\lambda_1 < 0$ and $D_1 \geq 0$, $\underline{x}_1 \leq x_1 \leq \bar{x}_1$.

(3) $\lambda_1 = 0$

(a) $\hat{q}_1 \neq 0$

The boundary of the region, a parabola, is given by

$p = U(x_1, x_2)^T$ and

$$x_1 = -\frac{\lambda_2}{2\hat{q}_1} Y - \left(r - \frac{\hat{q}_2^2}{\lambda_2} \right) \frac{1}{2\hat{q}_1}$$

$$x_2 = X - \frac{\hat{q}_2}{\lambda_2}$$

where X and Y satisfy $Y = X^2$. If $\lambda_2 > 0$, the unfalsified region is outside the parabola. If $\lambda_2 < 0$, the unfalsified region is inside the parabola.

(b) $\hat{q}_1 = 0$

If $\lambda_2 > 0$ and $D_2 = \hat{q}_2^2 - \lambda_2 r < 0$, all points are unfalsified. If $\lambda_2 > 0$ and $D_2 \geq 0$, $x_2 \leq \underline{x}_2$ or $x_2 \geq \bar{x}_2$ where

$$\underline{x}_2 = \frac{-\hat{q}_2 - \sqrt{D_2}}{\lambda_2}, \quad \bar{x}_2 = \frac{-\hat{q}_2 + \sqrt{D_2}}{\lambda_2}$$

If $\lambda_2 < 0$ and $D_2 < 0$, all points are falsified. If $\lambda_2 < 0$ and $D_2 \geq 0$, $\underline{x}_2 \leq x_2 \leq \bar{x}_2$.