Simultaneous Estimation of States and Unknown Inputs for Multi-output Linear Systems

Takahiko ONO* and Tadashi ISHIHARA**

For a linear time-invariant discrete-time system such that the observability index is two, this paper proposes a new observer which estimates unmeasured states and unknown inputs simultaneously. Linear transformation plays an important role in design of the observer. It is performed to obtain an output error equation which is decoupled from the estimation errors of unmeasured state variables. Based on this error equation, the observer is designed by a state feedback control theory. Essentially, the unknown input is estimated without influence of the estimation error of the state. If the unknown input is constant, both the state and the input can be reconstructed exactly in finite time. Under a certain condition, the observer works as a minimal-time deadbeat state observer which is completely insensitive to any unknown input.

Key Words: observer, state estimation, disturbance estimation, deadbeat observer

1. Introduction

A problem of estimating unmeasured states and unknown inputs of a system has been studied by many researchers for the purpose of robust control1),2) or system diagnosis8),10). As a solution to this problem, the state and input observer (SIO), which estimates states and unknown inputs simultaneously from known signals, has been developed. Depending on the estimation method of unknown inputs, the SIO can be classified into two types. The first type of SIO needs some assumption on the unknown inputs3)~8). If the assumption is matched to a practical situation, the states and the inputs can be estimated accurately. The second type of SIO needs no assumption on the unknown inputs9)~13). This type of SIO has a potential to estimate a wide variety of unknown inputs even in the situation that no information about unknown inputs can be obtained. However, all the SIOs of this type have some limitations: the inverse-dynamics based estimation9),12) is limited to minimum phase systems; the SIOs proposed by Ha and Trinh13) and Park and Stein10) require that the effect of unknown inputs appears explicitly in a measured output and its first derivative, respectively; in the method of Stein and Park11), an estimation accuracy is completely dependent on a system, so it cannot be adjusted freely.

This paper proposes a new SIO for discrete-time linear time-invariant (LTI) systems. Basically, it is classified into the first type of SIO. Under a certain condition, however, it works as the second type of SIO, that is, it can estimate the states and the unknown inputs without any assumption on the inputs. In design of the observer, linear transformation plays an important role. It is conducted to obtain an output error equation which is decoupled from the estimation errors of unmeasured state variables. The observer is designed based on this equation. Essentially, it requires that the number of outputs is more than or equal to the number of half of state variables. This is a restriction in application, but the observer has some interesting features: (i) it can be designed based on state feedback control theories; (ii) it can estimate the states and the unknown inputs exactly in finite time if the unknown inputs are constant; (iii) under a certain condition, it works as a minimal-time deadbeat state observer which is completely insensitive to any unknown input.

This paper is organized as follows. In Section 2, a class of systems dealt with in this paper is given. In Section 3, as a preliminary, the state observer design based on linear transformation, which is a key technique in this paper, is explained. Section 4 extends this state observer design to design of the SIO. Section 5 presents a numerical example. Conclusions are summarized in Section 6.

2. Problem Formulation

Consider a discrete-time LTI system modeled by

\[ x_{k+1} = Ax_k + Bu_k + Dd_k, \quad (k = 0, 1, 2, \ldots) \]

\[ y_k = Cx_k \]

where \( x_k \in \mathbb{R}^n \), \( u_k \in \mathbb{R}^m \), \( y_k \in \mathbb{R}^n \) and \( d_k \in \mathbb{R}^n \) are a state, a control input, a measured output and an un-
known input at time $k$, respectively. $A$, $B$, $C$ and $D$ are known constant real matrices of appropriate dimensions. The signals $u_k$ and $y_k$ are known and available in estimation at time $k$. It is assumed that the system satisfies the following conditions.

(C1) $A$ is nonsingular
(C2) $C$ is of full row rank
(C3) $\text{rank } [C^t (CA)^t] = n_y$
(C4) $D$ is of full column rank
(C5) $n_d \leq n_y$

The condition (C3) implies that the observability index is two. To meet (C3), the system must be observable and satisfy $2n_y \geq n_x$. Under these conditions, this paper considers design of an observer which estimates $x$ and $d$ simultaneously from the known signals $u$ and $y$. Throughout the paper, the following notations are used. For a signal $u_k$, an estimate of $u_k$ is denoted by $\hat{u}_k$ and its estimation error is denoted by $\delta u_k := u_k - \hat{u}_k$. A minimal time to achieve exact reconstruction of $x_k$ is defined as $k_{\text{min}} := \min \{ k_0 : x_k = x_k \text{ for } k \geq k_0 \}$.

3. Preliminary

Before developing a new SIO, a state observer design based on linear transformation is presented for unperturbed systems with $D = 0$. The SIO for perturbed systems with $D \neq 0$ is designed by extending this method.

Suppose that, by a linear transformation

$$ x_k = T^{-1} \begin{bmatrix} z_k \\ y_k \end{bmatrix} $$

(2)

the model (1) is changed into a form of

$$ \begin{bmatrix} z_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \begin{bmatrix} z_k \\ y_k \end{bmatrix} + \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix} u_k $$

$$ y_k = \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} z_k \\ y_k \end{bmatrix} $$

(3)

where $z \in R^{n_x - n_d}$ is an unmeasurable variable and

$$ \tilde{A}_{11} = MA_{21} $$

(4)

holds for a matrix $M \in R^{(n_x - n_d) \times n_y}$. For the system (3), consider design of a state observer which estimates $x_k$ as

$$ \dot{x}_k = T^{-1} \begin{bmatrix} \tilde{z}_k \\ y_k \end{bmatrix} $$

(5)

under the estimation equations

$$ \dot{z}_{k+1} = \tilde{A}_{11} \tilde{z}_k + \tilde{A}_{12} y_k + \tilde{B}_1 u_k + \nu_k $$
$$ \dot{y}_{k+1} = \tilde{A}_{21} \tilde{z}_k + \tilde{A}_{22} y_k + \tilde{B}_2 u_k + \nu_k $$

(6)

where $\nu_k$ is a function to reduce an estimation error. The estimate $\hat{x}_k$ converges to $x_k$ as $k$ increases if $\nu_k$ is given so that $\dot{z}_k$ converges to zero for the error system

$$ \dot{z}_{k+1} = \tilde{A}_{11} \tilde{z}_k + M \nu_k $$
$$ \dot{y}_{k+1} = \tilde{A}_{21} \tilde{z}_k + \nu_k $$

(7)

Recalling (4), this error system can be written as

$$ \dot{\tilde{z}}_k = M \tilde{y}_k $$
$$ \dot{\tilde{y}}_{k+1} = \tilde{A}_{21} \tilde{y}_k + \nu_k $$

(8a)

(8b)

for $k \geq 1$. Clearly, if $\tilde{y}_k \rightarrow 0$ as $k \rightarrow \infty$, $\tilde{z}_k \rightarrow 0$. So, by giving $\nu_k$ so that $\tilde{y}_k$ converges to zero for (8b), $\dot{\tilde{z}}_k$ converges to $\tilde{z}_k$. Noting that $\tilde{y}_k$ is available at time $k$ and its dynamic equation (8b) is decoupled from $\tilde{z}$, the problem of determining $\nu_k$ can be handled as a state feedback regulation problem for LTI systems characterized by $(\tilde{A}_{21} M, I, M, 0)$. Particularly, since the pair $(\tilde{A}_{21} M, I)$ is controllable, any linear or nonlinear state feedback control theory such as LQ control, pole placement method or sliding mode control can be utilized.

The essence of the above-mentioned design is the linear transformation which changes the system into the form of (3) and (4). Such transformation is defined as follows.

**Proposition 1.** (I) For a matrix $M \in R^{(n_x - n_d) \times n_y}$ and a full row rank matrix $N \in R^{(n_x - n_d) \times n_y}$, define

$$ T := \begin{bmatrix} MC + NCA^{-1} \\ C \end{bmatrix} $$

(9)

Then, the rank of $T$ is independent of $M$. Also, there exists $N$ such that $T$ has full rank if and only if the conditions (C1) through (C3) are satisfied. Particularly, when $2n_y = n_x$, $T$ has full rank for any full row rank matrix $N$.

(II) For a full rank matrix $T$, perform the linear transformation (2) to the model (1). Then, the model is transformed into the form of (3) and (4).

**Proof:** See Appendix A.

The set of the conditions (C1) through (C3) is the sufficient condition for changing the system into the form of (3) and (4). This paper focuses on the observer design via the linear transformation defined by (2) and (9). In this design, the design variables are $M$, $N$ and $\nu_k$. Hereafter, let $[H \ J] := T^{-1}$.

**Remark.** Since the rank of $T$ is independent of $M$, it is acceptable to set $M = 0$. Then, $\tilde{z}_k = 0$ for $k \geq 1$ and so the observer, which is constructed as

$$ \dot{\tilde{z}}_{k+1} = \tilde{A}_{12} y_k + \tilde{B}_1 u_k $$
$$ \dot{\tilde{y}}_{k+1} = H \tilde{z}_k + J y_k $$

(10)

is a deadbeat linear observer which attains $k_{\text{min}} = 1$ for any $\tilde{z}_0$. Generally, for a linear observer, a minimal time to achieve exact reconstruction of a state is associated
with the observability index \( \mu: k_{\min} = \mu - 1 \) for estimation. Since the model (1) has \( \mu = 2 \), the observer (10) is a minimal-time observer. Also, a minimal-order observer has the form
\[
\dot{z}_{k+1} = \hat{F}z_k + \hat{G}y_k + \hat{T}u_k \\
\hat{x}_k = H\dot{z}_k + J\hat{y}_k,
\] (11)
where, for \( \hat{W} \in \mathbb{R}^{(n_x-n_y) \times n_x} \),
\[
\hat{W}A - \hat{W}\hat{T} = \hat{G}C, \quad \hat{T} = \hat{W}B, \quad \hat{W}H + J\hat{C} = I
\] (12)
From (A.8) in the appendix, \( \hat{A}_{12} = N \) if \( M = 0 \). Thus, the observer (10) has \( \hat{F} = 0, \hat{G} = N, \hat{T} = \hat{B}_1, \hat{H} = H, \hat{J} = J \) and \( \hat{W} = NCA^{-1} \), which meet (12). So, this is also a minimal-order observer. The linear transformation (2) and (9) provides a method to design the minimal-time minimal-order state observer directly without any additional design variable such as an observer gain.

4. SIO Design

The state observer design in the previous section is extended to design of an SIO for the perturbed system with \( D \neq 0 \). In this section, a further condition is added.

(C6) \( D := (\hat{A}_{21}NC^{-1} + C)D \) has full column rank.

Under this condition, the pseudo-inverse matrix of \( D_d \) can be defined: \( D_d := (D_dD_d)^{-1}D_d \). This matrix is necessary to estimate \( d_k \) uniquely.

4.1 General Case

Give \( M \) and \( N \) to perform the linear transformation to the model (1). Then, it is transformed to
\[
z_{k+1} = \hat{A}_{11}z_k + \hat{A}_{12}y_k + \hat{B}_1u_k + D_1d_k \\
y_{k+1} = \hat{A}_{21}z_k + \hat{A}_{22}y_k + \hat{B}_2u_k + D_2d_k
\] (13)
where \( D_1 := (MC + NCA^{-1})D \) and \( D_2 := CD \). Now, let us give the estimation equations for \( z_k \) and \( y_k \) as
\[
\dot{z}_{k+1} = \hat{A}_{11}\dot{z}_k + \hat{A}_{12}\hat{y}_k + \hat{B}_1\hat{u}_k + M\nu_k + D_1\hat{d}_k \\
\dot{y}_{k+1} = \hat{A}_{21}\dot{z}_k + \hat{A}_{22}\hat{y}_k + \hat{B}_2\hat{u}_k + \nu_k + D_2\hat{d}_k,
\] (14)
where \( \hat{d}_k \) is an estimate of \( d_k \) and \( \nu_k \) is a function to reduce an estimation error. Since the equation (4) holds, the error system can be written as
\[
\dot{\tilde{z}}_k = M\hat{y}_k + NCA^{-1}D\hat{d}_{k-1} \\
\dot{\tilde{y}}_{k+1} = \hat{A}_{21}M\hat{y}_k + \nu_k + D_3\hat{d}_k + D_3\hat{d}_{k-1}
\] (15a)
(15b)
for \( k \geq 1 \), where \( D_3 := \hat{A}_{21}NCA^{-1}D \). The equation (15b) is decoupled from the unknown variable \( \tilde{x} \). For this reason, this paper calls it an output error equation. As seen from (15), \( \hat{d}_k \) and \( \nu_k \) should be given so that \( \hat{y}_k \) and \( \hat{d}_k \) are kept small. Such \( \hat{d}_k \) and \( \nu_k \) are given below.

First, an estimation equation of \( \hat{d}_k \) is given. Under (C6), the output error equation can be changed into
\[
d_{k-1} = D^1(D_2d_{k-1} + D_3\hat{d}_{k-2} - \hat{y}_k + \hat{A}_{21}M\hat{y}_{k-1} + \nu_{k-1}) + D^1D_3\Delta d_{k-1}
\] (16)
for \( k \geq 2 \), where \( \Delta d_k := \Delta d_k - d_k \). The first term of the right hand side of (16) is available at time \( k \). If \( \Delta d_{k-1} \) is small, it gives a good approximation of \( d_{k-1} \). Essentially, for the strictly proper system (1), \( d_k \) cannot be estimated exactly at time \( k \) since its effect appears in the future output at time \( k + 1 \). For these reasons, this paper gives \( \hat{d}_k \) as an estimate of \( d_{k-1} \) and calculates it as follows:
\[
\hat{d}_k := D^1(D_2d_{k-1} + D_3\hat{d}_{k-2} - \hat{y}_k + \hat{A}_{21}M\hat{y}_{k-1} + \nu_{k-1})
\] (17)
In this case, for \( k \geq 2 \),
\[
\hat{d}_k = d_{k-1} - D^1D_3\Delta d_{k-1}
\] (18)
or equivalently,
\[
\hat{d}_k = -\Delta d_k - D^1D_3\Delta d_{k-1}
\] (19)
which indicates that \( d \) is dependent only on \( \Delta d \), not \( \hat{y} \) or \( \hat{x} \). So, if \( d \) is constant, \( \hat{d}_k = d_k \) for \( k \geq 2 \).

Next, a general guideline for determining \( \nu_k \) is given. From (5), (13), (14) and (17), the whole error system is
\[
\hat{y}_{k+1} = \hat{A}_{21}M\hat{y}_k + \nu_k + D_3\hat{d}_k
\] (20a)
\[
\begin{bmatrix}
\dot{\tilde{z}}_k \\
\dot{\tilde{y}}_{k+1}
\end{bmatrix} =
\begin{bmatrix}
HM \\
0
\end{bmatrix} \begin{bmatrix}
\hat{y}_k \\
D_3\hat{d}_k
\end{bmatrix} + \hat{A}_{21}M\hat{y}_k + \nu_k + D_3\hat{d}_k
\] (20b)
for \( k \geq 2 \), where \( \Delta d_k := [\Delta d_{k} \quad \Delta d_{k-1} \quad \Delta d_{k-2}]^T \),
\[
D_z := \begin{bmatrix} 0 & NCA^{-1}D & NCA^{-1}DD^1D_3 \end{bmatrix}
\]
\[
D_{\nu} := -\begin{bmatrix} D_2 & (D_2D^1 + I)D_3 & D_3D_3D_3 \end{bmatrix}
\]
(21)
\[
D_{\nu} := \begin{bmatrix} I & D_3D_3 & 0 \end{bmatrix}
\]
From (20), it turns out that \( \nu_k \) can improve only \( \tilde{x}_k \). This function \( \nu_k \) is given so that the error system (20) is stable. Noting that \( \hat{y}_k \) is available at time \( k \), the error system can be viewed as an LTI system with a measurable state (\( \hat{y}_k \)) and an uncertain input (\( \Delta d_k \)). Thus, the problem of designing \( \nu_k \) can be handled as a state feedback regulation problem for LTI systems with uncertain inputs. Particularly, since the pair \( (\hat{A}_{21}M, I) \) is controllable, regardless of linear and nonlinear, any state feedback control theory can be applied.

Consequently, the SIO is composed of (5), (14) and (17). Its design variables are \( M, N \) and \( \nu_k \), which are given so that the error system (20) is stable. Noting that (16) is identical to (17) when \( \Delta d_k = 0 \), it can be interpreted that \( d_k \) is estimated under the assumption of \( \Delta d_k = 0 \). In other words, the SIO gives \( \hat{d}_k \) by regarding \( \Delta d_k \) as an uncertainty and ignoring it. The error system (20) is a
convenient form for designing \( \nu_k \) to reduce the influence of \( \Delta d \). On the other hand, when \( D^1D_3 = 0 \), the assumption of \( \Delta d_k = 0 \) is unnecessary in essence since \( d_k = d_{k-1} \) for any \( d_k \). In these senses, the proposed SIO works as the first type of SIO if \( D^1D_3 \neq 0 \) and works as the second type of SIO if \( D^1D_3 = 0 \).

4.2 Linear Design

Depending on the function \( \nu_k \), the SIO can be either a linear or nonlinear observer. This subsection considers a design of \( \nu_k \) which results in a linear SIO.

One of the forms of \( \nu_k \) is a static linear function of \( y_k \):

\[
\nu_k = -K\hat{y}_k
\]

where \( K \) is a constant matrix to be designed. In this case, the SIO is constructed as a linear observer to \( y_k \) and \( u_k \). Basically, \( K \) is chosen so that the error system (20) is stable, that is, \( \bar{A}_2M - K \) is stable. This subsection further considers how \( K \) should be given for accurate estimation.

Defining \( E_K = [y_k^t, d_{k-1}^t, d_{k-2}^t, d_{k-3}^t]^t \), the error system (20) can be changed into a linear form to \( d_k \) for \( k \geq 2 \):

\[
\begin{bmatrix}
\bar{x}_k \\
\hat{d}_k
\end{bmatrix} =
\begin{bmatrix}
\bar{A}_2 & \bar{B}_2 \\
0 & I
\end{bmatrix}
\begin{bmatrix}
\xi_k \\
- K
\end{bmatrix} +
\begin{bmatrix}
0 \\
I
\end{bmatrix}
\begin{bmatrix}
\xi_k \\
- K
\end{bmatrix}
\]

in which \( \bar{A}_2, \bar{B}_2, C_{\xi 1} \) and \( C_{\xi 2} \) are defined by (22). Recalling that \( \nu_k \) improves only \( \hat{x}_k \), it is sufficient to choose \( K \) so that \( \hat{x}_k \) is kept small. Accordingly, for more accurate estimation, it is necessary to choose \( K \) so that the error system \( (\bar{A}_2, \bar{B}_2, C_{\xi 1}, 0) \) is stable and has a low sensitivity in a range of frequencies which covers a spectrum of \( d_k \) or where one desires to minimize its influence on \( \hat{x} \).

4.3 Deadbeat state estimation

A special SIO, which achieves deadbeat state estimation, is considered.

It follows from (20) and (21) that, when \( NCA^{-1}D = 0 \), the error caused by \( \Delta d \) is reduced. In this sense, \( N \) can be regarded as a design variable for reinforcing robustness to \( d \). On the other hand, it is found from (20b) that \( \hat{x} \) is independent of \( \hat{y} \) if \( M = 0 \). For the unperturbed system with \( D = 0 \), the setting of \( M = 0 \) results in the deadbeat state estimation. For the perturbed system with \( D \neq 0 \), the deadbeat state estimation can be achieved in a limited situation. It is summarized as follows.

**Proposition 2.** (I) Suppose that the SIO is constructed under \( M = 0 \) and \( NCA^{-1}D = 0 \) and \( \nu_k = 0 \). Then, for any \( d_k \), it works as a deadbeat state observer which attains

\[
k_{\min} = \begin{cases}
1 & (NCA^{-1}D = 0) \\
3 & (NCA^{-1}D \neq 0)
\end{cases}
\]

for any \( \nu_k \) and any initial values of \( \hat{x}_k, \hat{y}_k \) and \( \hat{d}_k \).

(ii) Suppose that the SIO is constructed under \( M = 0 \) and \( NCA^{-1}D = 0 \). Then, for any \( d_k \), it works as a minimal-time deadbeat state observer which attains \( k_{\min} = 1 \) for any \( \nu_k \) and any initial values of \( \hat{x}_k, \hat{y}_k \) and \( \hat{d}_k \).

Proof: See Appendix B.

The SIO, which is designed under \( M = 0, NCA^{-1}D = 0 \) and \( \nu_k = 0 \), is the simplest structured linear observer in the sense that \( \bar{A}_1 = 0, D_1 = 0, D_3 = 0 \) and \( D^1D_2 = I \) in (14) and (17), that is, only the coefficient matrices of the transformed model is required to construct the observer. Moreover, it is completely robust to \( d_k \) in the sense that it achieves not only \( \hat{x}_k = x_k \) but also \( \hat{d}_k = d_{k-1} \) for any \( d_k \) in finite time. To design such a simple structured robust SIO, it is necessary to find \( N \) that satisfies \( NCA^{-1}D = 0 \), rank \( T = n_x \) and (C6). However, such \( N \) cannot always be found. So, let us consider an essential condition for constructing the robust SIO.

Consider the sufficient condition for existence of \( N \) such that \( NCA^{-1}D = 0 \). It is given by (C7) or (C8):

\[
\begin{align}
(C7) \quad CA^{-1}D & = 0 \\
(C8) \quad \dim \text{Ker}(CA^{-1}D)^I & \geq n_x - n_y
\end{align}
\]

The condition (C7) is obvious. It is equivalent to that a system characterized by \((A, D, C, 0)\) has the blocking zeros at the origin. However, such a system is quite limited. On the other hand, when (C8) holds, by assigning the basis vectors of \( \text{Ker}(CA^{-1}D)^I \) to the columns of \( N^1 \), \( NCA^{-1}D = 0 \). Compared to (C7), (C8) holds for a wider class of systems. For this reason, let us consider (C8) in more detail. Suppose that \( CA^{-1}D \) has full column rank:
rank $CA^{-1}D = n_d$. Under (C1) through (C5), this rank condition could be satisfied for many systems. When this condition holds, $\text{dim Ker}(CA^{-1}D)^c = n_y - n_d$. Combining this equation with (C8), the necessary condition for (C8) is given by

$$2n_y - n_x - n_d \geq 0 \quad (26)$$

So, more outputs are required for robust estimation. Noting that (C7) is a rare case, the inequality (26) gives an essential necessary condition for designing the robust SIO.

4.4 Design Summary

The design procedure of the proposed SIO is summarized as follows:

Step 1: Confirm the conditions (C1) through (C5).

Step 2: Give $M$ and $N$ to transform the system into (3) and (4) via the linear transformation (2) and (9).

Step 3: Confirm the condition (C6). If it is not satisfied, return to Step 2.

Step 4: Determine $\nu_k$ so that the error system (20) is stable. If $\nu_k$ is given as a static linear function of $y_k$ in (23), determine $K$ so that (24) is stable.

Step 5: The SIO is composed of (5), (14) and (17). If $N$ can be chosen so that $NCA^{-1}D = 0$, rank $T = n_x$ and (C6), the robust SIO, which achieves $\dot{x}_k = x_k$ and $\dot{d}_k = d_k$ to any $d_k$, can be designed by taking the design variables as $M = 0, NCA^{-1}D = 0$ and $\nu_k = 0$.

5. Numerical Example

The estimation performance of the proposed SIO is examined through a numerical example. Consider the ship steering control system \(^{14}\), which is modeled by

$$\dot{x}(t) = \begin{bmatrix} \alpha_1 & \alpha_2 & 0 & 0 \\ \alpha_3 & \alpha_4 & 0 & 1 \\ 1 & 0 & 0 & \alpha_5 \\ 0 & 1 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} \beta_1 \\ \beta_2 \\ 0 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} \delta_1 \\ \delta_2 \\ 0 \\ 0 \end{bmatrix} d(t)$$

where $\alpha_1 = -0.05, \alpha_2 = -6, \alpha_3 = -0.001, \alpha_4 = -0.15, \alpha_5 = 13, \beta_1 = \delta_1 = -0.2, \beta_2 = \delta_2 = 0.03$ and $\beta_3 = 1$. The transfer function matrix from $d$ to $y$ has one unstable zero.

Using the proposed SIO, let us estimate $x$ and $d$ from $y$ and $u$ at a rate of 10Hz. First, the discrete-time model of the system is obtained by the zero-order hold approximation. Then, the conditions (C1) to (C5) are satisfied. But, since neither (C7) nor (C8) holds, the robust SIO cannot be constructed. Thus, $x$ and $d$ are estimated under the assumption of $\Delta d_k = 0$. To examine how the estimation performance is influenced by the design variables, the SIO is designed under the following two cases:

Case 1: $M = I, N = I$ and $\nu_k = -K y_k$,

Case 2: $M = 0, N = I$ and $\nu_k = 0$

In both the cases, the SIO is linear to $y_k$ and $u_k$. Its estimation performance can be evaluated based on (24). In Case 1, to keep $\dot{x}$ as small as possible while $d$ is changing slowly, $K$ was chosen so that the error system $(A_{C1}, B_{C1}, C_{C1}, 0)$ is stable, that is, $A_{C1} M - K$ is stable, and has a low sensitivity to $d_k$ in a low frequency range. As a result, the eigenvalues of $A_{C1} M - K$ were placed at $-0.6368 \pm 0.0357$. In Case 2, the SIO achieves the deadbeat state estimation if $d$ is constant. So, in essence, $\dot{x}$ can be kept small while $d$ is changing slowly. In the performance analysis below, these SIOs are referred to as SIO1 and SIO2, respectively.

The primary factor to affect the estimation performance of the SIOs is the increment of $d$ or, roughly speaking, $d$. So, let us first examine the effects of $d_k$ and $\nu_k$ on the estimation accuracy in terms of sensitivity. Since $x_1$ and $x_3$ are directly observed, $\dot{x}_1$ and $\dot{x}_3$ are not considered here. Now, represent the input-output relationship of the error system (24) as

$$\begin{bmatrix} \dot{x}_2 \\ \dot{x}_4 \\ \dot{d}_1 \\ \dot{d}_3 \end{bmatrix} = \begin{bmatrix} S_{21} & S_{22} \\ S_{41} & S_{42} \\ S_{61} & S_{62} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

where the subsystems from $d$ to $\dot{x}_1$ and $\dot{x}_3$ are omitted. The sensitivity of (27) to $\dot{x}_1$ and $\dot{x}_3$ is quite low in a wide range of frequencies. Thus, they could estimate $x$ accurately unless $d$ changes extremely rapidly. This low sensitivity characteristics is achieved by $\nu_k$. So, it plays an important role in improving the accuracy of $\dot{x}$. On the other hand, the sensitivity of $d$ is identical for both SIO1 and SIO2, which cannot be improved by $\nu_k$. Compared to $\dot{x}$, $\dot{d}$ is much more influenced by $d$. This is because, as seen from (24), $\dot{d}_k$ is affected directly by $d_k$.

Next, the estimation performance is evaluated through a numerical simulation. The condition is as follows: the initial conditions are $x(0) = [0 0.2 0 0.2]^T, z_0 = 0, y_0 = 0$ and $d_0 = 0; d(t)$ is composed of constant, sinusoidal and rectangular elements; the system is controlled by the same input $u$ for both the simulations. Fig. 2 shows the simulation results for SIO1. Only the results on the unmeasured variables are illustrated. From these figures, it turns out
that $x_2$ and $x_4$ can be estimated accurately after transition even when $d$ changes rapidly. Given the results of the sensitivity analysis above, this is as expected. The disturbance $d$ can also be estimated accurately as long as it changes slowly. But, when $d$ changes rapidly, $\hat{d}$ lacks accuracy, as marked by asterisks in Fig. 2. The result of SIO2 is omitted, for $\hat{x}$ and $\hat{d}$ of SIO2 are almost the same as those of SIO1. The only difference between SIO1 and SIO2 can be seen in the transient response of $\hat{x}$. Fig. 3 magnifies $\hat{x}_2$ and $\hat{x}_4$ in transition. The error $\hat{d}$ in SIO2 decreases immediately. In contrast, $\hat{x}$ in SIO1 converges to zero as oscillating. This is mainly because the SIO1 has negative real parts of the eigenvalues of $A_{21}M - K$. On the whole, however, it could be concluded that both the SIOs have good estimation performance.

6. Conclusions

For discrete-time LTI systems, a new SIO was presented. Its feature is summarized as follows. (i) The design algorithm is quite simple. In the simplest case, only the coefficient matrices of the linearly transformed state-space model is required to design the observer. In a general case, the design problem amounts to the regulation problem for LTI systems. Then, any state feedback control theory can be used in determining $v_k$. (ii) The estimation accuracy of $\hat{d}$ is dependent on only $\Delta d$. So, for constant unknown inputs, the state is reconstructed exactly in finite time. For varying unknown inputs, the exact reconstruction of the state is achieved only when $D^TD_3=0$. But, even when this condition is not satisfied, the state can be estimated accurately as long as $\Delta d$ is small. These features come from the linear transformation which yields the output error equation (15b).

References

2) T. Ishihara, H. Guob and H. Takeda: Integral controller design based on disturbance cancellation: Partial LFR approach for non-minimum phase plants, Automatica, 41-12, 2083/2089 (2005)
3) G.H. Hostetter and J.S. Meditch: On the generalization of observers to systems with unmeasurable, unknown inputs, Automatica, 9-6, 721/724 (1973)
7) M. Corless and J.L. Tu: State and input estimation for a class of uncertain systems, Automatica, 34-6, 757/764 (1998)
Appendix A. Proof of Proposition 1

Proof of (I). First, the sufficiency is shown. Since the rank of a matrix is invariant under the elementary operation, rank $T = \text{rank} [(NCA^{-1})^t C]^t$. So, the rank of $T$ is independent of $M$. On the other hand, defining

\[ P := \begin{bmatrix} M & N \\ I & 0 \end{bmatrix}, \quad Q := \begin{bmatrix} C \\ CA^{-1} \end{bmatrix}, \quad T = PQ. \tag{A.1} \]

Then, according to Sylvester's inequality,

\[ \text{rank } P + \text{rank } Q - 2n_y \leq \text{rank } T \leq \min \{ \text{rank } P, \text{rank } Q \} \tag{A.2} \]

Also, it follows from (C3) that rank $Q = n_x$ and $2n_y \geq n_x$. Combining this fact with (A.2), it turns out that, when $2n_y = n_x$, $T$ has full rank for any full row rank matrix $N$. When $2n_y > n_x$, it cannot immediately be concluded from (A.2) that $T$ has full rank since the left hand side of (A.2) is less than $n_x$. So, let us show that there exists at least a full row rank matrix $N$ which ensures rank $T = n_x$. Let $C_i$ denote the $i$th row vector of $C$. Since rank $Q = n_x$, there exists at least a set of $n_x$ row vectors of $Q$ such that

\[ A^t C_{p(1)}, \ldots, A^t C_{p(n_x-n_y)}, C_{n_x-n_y}^t, \ldots, C_{n_y}^t \tag{A.3} \]

are linearly independent, where $p(i)$ indicates a subscript such that $1 \leq p(1) < p(2) < \cdots < p(n_x-n_y) \leq n_y$. Now, give a full row rank matrix $N$ as

\[ N = \{ n_{ij} \}, \quad n_{ij} = \begin{cases} 1 & \text{for } i = \ell, j = p(\ell) \\ 0 & \text{otherwise} \end{cases} \tag{A.4} \]

where $\ell = 1, 2, \ldots, n_x - n_y$. Taking $M = 0$,

\[ T = [A^t C_{p(1)}, \ldots, A^t C_{p(n_x-n_y)}, C_{n_x-n_y}^t, \ldots, C_{n_y}^t] \tag{A.5} \]

The column vectors of the right hand matrix are linearly independent. So, $T$ has full rank for (A.4).

Next, the necessity is shown. Clearly, the conditions (C1) and (C2) hold when $T$ is of full rank. So, it is sufficient to show (C3). Defining $R := [(CA)^t C]^t$, $TA = PR$. Thus,

\[ \text{rank } TA = \text{rank } PR \leq \text{rank } R \leq n_x \tag{A.5} \]

Since rank $T = n_x$ and rank $A = n_x$, rank $TA = n_x$. Combining this equation with (A.5), rank $R = n_x$. Noting that rank $R = \text{rank} [(C^t (CA))^t]^t$, the condition (C3) holds.

Proof of (II). It is sufficient to prove only (4). Partition $T, A$ and $\tilde{A} := TA^{-1}$ as

\[ T = \begin{bmatrix} W \\ C \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{A.6} \]

Since $TA = AT$,

\[ C_1 A_{11} + C_2 A_{12} = \tilde{A}_{21} W_1 + \tilde{A}_{22} C_1, \quad C_1 A_{12} + C_2 A_{22} = \tilde{A}_{21} W_1 + \tilde{A}_{22} C_2. \tag{A.7} \]

Using (A.6),

\[ WA = MCA + NC \]

\[ = [M A_{21}, MA_{22} + N] T \tag{A.7} \]

From (A.7),

\[ TA = \begin{bmatrix} WA \\ CA \end{bmatrix} = \begin{bmatrix} M A_{21} & M A_{22} + N \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} T. \tag{A.8} \]

Thus, the equation (4) is satisfied.

Appendix B. Proof of Proposition 2

From (19), if $d$ is constant, $\bar{d}_k = 0$ for $k \geq 2$. Therefore, from (15a),

\[ \min \{ k_0 : \bar{x}_k = 0 \text{ for } k \geq k_0 \} = \begin{cases} 1 & (NCA^{-1} D = 0) \\ 3 & (NCA^{-1} D \neq 0) \end{cases} \]

Since $\bar{x}_k = H \tilde{x}_k$, the minimal time to achieve exact reconstruction of $x_k$ is given by (25). On the other hand, when $M = 0$ and $NCA^{-1} D = 0$, it is obvious from (15a) that $\bar{x}_k = 0$ for $k \geq 1$ for any $d_k$, that is, $k_{\min} = 1$. This is equal to the minimal time for the deadbeat state observer (10). Accordingly, the observer, which is designed for $M = 0$ and $NCA^{-1} D = 0$, works as a minimal-time deadbeat state observer for any $d_k$.

---

Takahiko Ono (Member)

He received a Ph.D. degree in information science from Tohoku University, Sendai, Japan in 1999. He was a research associate at Tohoku University from 1999 to 2004. Currently, he is a lecturer in Hiroshima City University, Hiroshima, Japan. His research interests are robust control and estimation theory.
Tadashi ISHIHARA (Member)

He received a Ph.D. degree in electrical engineering from Tohoku University, Sendai, Japan in 1979. From 1987 to 2003, he was an associate professor at Tohoku University. Currently, he is a professor in the Faculty of Science and Technology, Fukushima University, Japan. His current research interests are robust control, stochastic adaptive control and control system design based on the principle of matching.