Robust Stability/Performance Analysis for Linear Time-Invariant Polynomially Parameter-Dependent Systems Using Polynomially Parameter-Dependent Lyapunov Functions

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This paper addresses robust stability, H_2 performance, and H_∞ performance analysis for Linear Time-Invariant Parameter-Dependent (LTIPD) systems using Parameter-Dependent Lyapunov Functions (PDLFs). The state-space matrices of the LTIPD systems are set to be parameter-dependent in negative as well as positive power series with respect to the parameters, and PDLFs are also set to be parameter-dependent in negative as well as positive power series with respect to the parameters. Our results are derived using “slack variables” and restricting them to be parameter-independent, therefore, they are only sufficient conditions for the original analysis problems. However, we demonstrate the effectiveness of our results with randomly generated numerical examples and several numerical examples which are borrowed from existing papers.

Key Words: robust stability, robust performance, linear time-invariant parameter-dependent systems, polynomially parameter-dependent systems.

1. Introduction

Since de Oliveira *et al.* proposed a new robust stability analysis method for Linear Time-Invariant Parameter-Dependent (LTIPD) systems using slack variables¹⁾, much research using slack variables has been conducted for this decade; robust stability analysis^{2)~6)}, robust D -stability analysis⁷⁾, robust H_2 performance analysis^{5), 8), 9)}, and robust H_∞ performance analysis^{4)~6)}. All their derived conditions are sufficient conditions for the original analysis problems because slack variables need to be parameter-independent to derive parametrically affine Linear Matrix Inequalities (LMIs). To reduce the conservatism, new methods using parameter-dependent slack variables have been proposed^{10)~14)}, some of which formulate sufficient conditions which asymptotically tend to be the necessary conditions if the parameter-dependency of Lyapunov functions is given *a priori*. Several other approaches for robust stability analysis and robust performance analysis have been also proposed; quadratic separator approach^{15), 16)}, sums-of-squares approach^{17), 18)}, and KYP lemma approach¹⁹⁾, some of which formulate the necessary and sufficient conditions for the original analysis problems if the parameter-dependency of Lyapunov functions is given *a priori*. Compared to

these methods, simple slack variable approach (not using parameter-dependent slack variables) has the drawback that derived conditions are only sufficient conditions for the original problems even if the parameter-dependency of Lyapunov functions is increased.

Most of existing papers for robust stability/performance analysis (except 2), 5), and 20), the last considers more general nonlinear models) tackle the stability/performance analysis for polytopically parameter-dependent systems. However, the representation of polytopically parameter-dependent systems sometimes leads to increased conservatism. Let consider the system defined as $\dot{x} = \begin{bmatrix} -(\theta + 1/\theta) & 1 \\ 1 & -\theta \end{bmatrix} x$, where x is the state vector and θ ($0 < \delta_{\min} \leq \theta \leq \delta_{\max}$) is a parameter representing the uncertainty. To treat this system as a polytopic system, we first make a polytope which covers the variation of the system matrix; that is, make a polytope which covers the variation of $\theta + 1/\theta$, e.g. a polytope with three vertices A , B and C or a polytope with four vertices A , B , D and E in **Fig. 1**. Although we can obtain tighter polytopes than these, they always contain the region between $\theta + 1/\theta$ and line AB which are depicted as a gray region in Fig. 1, which leads to conservatism because we have no need to cover it (this will be shown in Section 5). As another method, we can apply biquadratic stability proposed in 3) in the following sequence; we first set the number of parameters in this system to be two; θ and $\alpha := 1/\theta$ because biquadratic stability is applicable only for parametrically affine systems, then we apply bi-

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Fig. 1 Polytopic covers for $\theta + 1/\theta$

quadratic stability to this modified system. However, generally speaking, such modification leads to conservatism, because such modification neglects the relationship between θ and $1/\theta$ (this will be also shown in Section 5). Similarly, in 2), the authors have proposed another analysis method for polynomially parameter-dependent systems. However, they also use quadratically parameter-dependent Lyapunov functions, which subsequently leads to conservative analysis. In contrast to these methods, in 5), the author has proposed an analysis method applicable for polynomially parameter-dependent systems, which theoretically encompasses biquadratic stability in 3). However, the formulation has the drawback that LTIPD systems cannot have terms with more than three parameters multiplied. This restriction is removed in the present manuscript.

In this paper, we address robust stability, robust H_2 performance, and H_∞ performance analysis for LTIPD systems, in which the state-space matrices are parameter-dependent in negative as well as positive power series of parameters and all their by-products. Results are derived using Lyapunov functions which are parameter-dependent in negative as well as positive power series with respect to parameters and all their by-products. We adopt slack variable approach, therefore, our results are only sufficient conditions for the original analysis problems. However, we demonstrate the effectiveness of our methods with randomly generated examples and several numerical examples borrowed from existing papers. In other words, we demonstrate that simple slack variable approach is not so conservative compared to existing methods. The main contribution of this paper is to propose the general formulations using polynomially Parameter-Dependent Lyapunov Functions (PDLFs) not only for robust stability analysis but also for performance analysis for LTIPD systems via simple slack variable approach, in which the increased-order PDLFs lead to decreased conservatism is

guaranteed.

Hereafter, $\text{He}(X)$ is the shorthand notation of $X + X^T$, 0_n , $0_{n,m}$, I_n and $\mathbf{0}$ respectively denote an n -dimensional square zero matrix, an $n \times m$ -dimensional zero matrix, an n -dimensional identity matrix and an appropriately dimensional zero matrix, $\mathcal{R}^{n \times m}$ and $\mathcal{S}^n$ respectively denote sets of $n \times m$ dimensional real matrices and $n \times n$ dimensional symmetric real matrices, $\otimes$ denotes Kronecker product, $X^\perp \in \mathcal{R}^{(n-r) \times n}$ denotes a matrix satisfying $X^\perp X = \mathbf{0}$ and $X^\perp X^{\perp T} > 0$, where $X \in \mathcal{R}^{n \times p}$ and $\text{rank}(X) = r$, and the notation $\text{Tr}_n(X)$ denotes $\begin{bmatrix} \text{Tr}(X_{11}) \cdots \text{Tr}(X_{1i}) \\ \vdots \quad \ddots \quad \vdots \\ \text{Tr}(X_{i1}) \cdots \text{Tr}(X_{ii}) \end{bmatrix}$ for $X = \begin{bmatrix} X_{11} \cdots X_{1i} \\ \vdots \quad \ddots \quad \vdots \\ X_{i1} \cdots X_{ii} \end{bmatrix}$, where $X_{kl} \in \mathcal{R}^{n \times n}$ ($k, l = 1, \dots, i$).

2. Preliminaries

In this section, we make some preliminaries for our proposed methods.

2.1 Definitions

In this paper, we consider the following LTIPD system.

$$G(\theta) : \begin{cases} \dot{x} = A(\theta)x + B(\theta)w \\ z = C(\theta)x + D(\theta)w \end{cases} \quad (1)$$

where $x \in \mathcal{R}^n$ is the state vector, $w \in \mathcal{R}^{n_w}$ is the disturbance input vector, $z \in \mathcal{R}^{n_z}$ is the performance output vector, and matrices $A(\theta)$, $B(\theta)$, etc. are parameter-dependent matrices of appropriate dimensions. We assume that the LTIPD system has k independent scalar parameters $\theta = [\theta_1 \cdots \theta_k]^T$. They represent time-invariant plant uncertainties and are supposed to lie in a convex hull with N vertices $\theta \in \text{co}\{\theta^{v_1} \cdots \theta^{v_N}\}$. For each parameter θ_i ($i = 1, \dots, k$), define the vector of its negative and positive power series ranging from $-l_i$ to m_i as

$$\theta_i^{[-l_i, m_i]} = [\theta_i^{-l_i} \cdots \theta_i^{-1} \ 1 \ \theta_i \cdots \theta_i^{m_i}]^T \in \mathcal{R}^{\sigma_i},$$

where $\sigma_i = l_i + m_i + 1$. In the case that $l_i \neq 0$, we assume that θ_i^{-1} is well-defined, e.g. in the case that $l_i \neq 0$ ($i = 1, \dots, k$), $\text{co}\{\theta^{v_1} \cdots \theta^{v_N}\}$ does not include the origin. Define the vector of all monomials obtained as products of all θ_i^j elements with $i = 1, \dots, k$, $j = -l_i, \dots, m_i$:

$$\bar{\theta} = \theta_1^{[-l_1, m_1]} \otimes \cdots \otimes \theta_k^{[-l_k, m_k]} \in \mathcal{R}^{\pi^{[1, k]}},$$

where $\pi^{[q, k]} = \prod_{i=q}^k \sigma_i$. By definition, let $\pi^{[k+1, k]} = 1$.

Based on these definitions, the parameter-dependent system is modeled as

$$\begin{aligned} A(\theta) &= \bar{A} \bar{\theta}_n, \quad \bar{A} \in \mathcal{R}^{n \times n \pi^{[1, k]}} \\ B(\theta) &= \bar{B} \bar{\theta}_{n_w}, \quad \bar{B} \in \mathcal{R}^{n \times n_w \pi^{[1, k]}} \\ C(\theta) &= \bar{C} \bar{\theta}_n, \quad \bar{C} \in \mathcal{R}^{n_z \times n \pi^{[1, k]}} \\ D(\theta) &= \bar{D} \bar{\theta}_{n_w}, \quad \bar{D} \in \mathcal{R}^{n_z \times n_w \pi^{[1, k]}} \end{aligned} \quad (2)$$

where $\bar{\theta}_n = \bar{\theta} \otimes I_n$ and $\bar{\theta}_{n_w} = \bar{\theta} \otimes I_{n_w}$. Such representation models any power series from $-l_i$ to m_i of the parameters θ_i ($i = 1, \dots, k$), and all their by-products. The matrices $\bar{A}$, $\bar{B}$, etc. are row-block matrices, each of which represent coefficients associated with one monomial of $\bar{\theta}$.

For this LTIPD system, we will look for a quadratic PDLF $x^T P(\theta)x$ with

$$P(\theta) = \bar{P} \bar{\theta}_n = \bar{\theta}_n^T \bar{P}^T, \quad \bar{P} \in \mathcal{R}^{n \times n \pi^{[1,k]}}, \quad (3)$$

where $\bar{P}$ is a row-block matrix of symmetric matrices.

Remark 1 Note that taking the l_i and m_i integers larger than needed for modeling the system (i.e. by adding zero matrices in the relevant positions of $\bar{A}$, $\bar{B}$, etc.), the PDLF can be of higher-order than the model. Doing so reduces the conservatism of the following exposed results. This will be shown with several numerical examples in section 5.

Two additional notations are as well needed. Let the vector

$$e = \left[\underbrace{0 \dots 0}_{\sum_{i=1}^k (l_i \times \pi^{[i+1,k]})} \quad 1 \quad \underbrace{0 \dots 0}_{\sum_{i=1}^k (m_i \times \pi^{[i+1,k]})} \right]^T \in \mathcal{R}^{\pi^{[1,k]}}$$

and let $e_n = e \otimes I_n$. This notation works as an index for getting parameter-independent elements and gives the symmetrized version of $P(\theta)$ in (3):

$$P(\theta) = \frac{1}{2} \bar{\theta}_n^T (\bar{P}^T e_n^T + e_n \bar{P}) \bar{\theta}_n.$$

Let as well the affine parameter-dependent matrices ($i = 1, \dots, k$)

$$\eta_i(\theta) = \begin{bmatrix} \theta_i I_{\sigma_i-1} \\ \mathbf{0} \end{bmatrix} - \begin{bmatrix} \mathbf{0} \\ I_{\sigma_i-1} \end{bmatrix} \in \mathcal{R}^{\sigma_i \times (\sigma_i-1)},$$

which satisfy $\eta_i(\theta)^\perp = \theta_i^{[-l_i, m_i]}^T$. At last, define

$$\eta(\theta) = \left[\begin{array}{c} \eta_k(\theta) \otimes I_{\pi^{[k+1,k]}} \\ 0_{\pi^{[1,k]} - \pi^{[k,k]}, (\sigma_k-1) \times \pi^{[k+1,k]}} \end{array} \right] \dots \left[\begin{array}{c} \eta_2(\theta) \otimes I_{\pi^{[3,k]}} \\ 0_{\pi^{[1,k]} - \pi^{[2,k]}, (\sigma_2-1) \times \pi^{[3,k]}} \end{array} \right] \eta_1(\theta) \otimes I_{\pi^{[2,k]}} \right] \in \mathcal{R}^{\pi^{[1,k]} \times (\pi^{[1,k]}-1)},$$

$\eta_n(\theta) = \eta(\theta) \otimes I_n$ and $\eta_{n_w}(\theta) = \eta(\theta) \otimes I_{n_w}$, which prove to satisfy $\eta(\theta)^\perp = \bar{\theta}^T$, $\eta_n(\theta)^\perp = \bar{\theta}_n^T$ and $\eta_{n_w}(\theta)^\perp = \bar{\theta}_{n_w}^T$.

2.2 Basic Lemmas

Well-known stability, H_2 performance, and H_∞ performance analysis conditions for the LTIPD system (1) are now recalled. In these lemmas, Lyapunov functions are set to be $x^T P(\theta)x$.

Lemma 1 (Stability) The system (1) is robustly stable

for all admissible θ if and only if there exists $P(\theta) > 0$ that satisfies (4) for all admissible θ .

$$\text{He} \{P(\theta)A(\theta)\} < 0 \quad (4)$$

Lemma 2 (H_2 performance) The system (1) is robustly stable and its H_2 performance is bounded by γ_2 for all admissible θ if and only if $D(\theta) = \mathbf{0}$ and there exist $P(\theta) > 0$ and $N(\theta) > 0$ that satisfy (5), (6), and (7) for all admissible θ .

$$\text{He} \{P(\theta)A(\theta)\} + C(\theta)^T C(\theta) < 0 \quad (5)$$

$$N(\theta) - B(\theta)^T P(\theta) B(\theta) > 0 \quad (6)$$

$$\gamma_2^2 > \text{Tr}\{N(\theta)\} \quad (7)$$

Lemma 3 (H_∞ performance) The system (1) is robustly stable and its H_∞ performance is bounded by γ_∞ for all admissible θ if and only if there exists $P(\theta) > 0$ that satisfies (8) for all admissible θ .

$$\begin{bmatrix} \text{He} \{P(\theta)A(\theta)\} P(\theta)B(\theta) \\ B(\theta)^T P(\theta)^T & -\gamma_\infty^2 I_{n_w} \end{bmatrix} + \begin{bmatrix} C(\theta)^T \\ D(\theta)^T \end{bmatrix} [C(\theta) D(\theta)] < 0 \quad (8)$$

In Lemmas 1, 2, and 3, the positivity of $P(\theta)$ is redundant if the nominal system of (1) is stable^{(9), (21)}, which can be assumed without loss of generality. Thus, we do not consider the positivity condition of $P(\theta)$ hereafter.

3. Main Results

In this section, we show our proposed methods for stability analysis using Lemma 1, H_2 performance analysis using Lemma 2, and H_∞ performance analysis using Lemma 3 for LTIPD system (1) using PDLF defined in (3).

3.1 Basic Results

We first show our basic results for stability and performance analysis.

Theorem 1 (Stability Analysis) If there exist matrices $\bar{P} \in \mathcal{R}^{n \times n \pi^{[1,k]}}$ and $M \in \mathcal{R}^{n(\pi^{[1,k]}-1) \times n \pi^{[1,k]}}$ such that

$$\text{He} \{ \bar{P}^T \bar{A} + \eta_n(\theta) M \} < 0 \quad (9)$$

holds for all vertices $\theta = \theta^{v_1}, \dots, \theta^{v_N}$, then the LTIPD system (1) is robustly stable.

Proof 1 First note that (9) is affine with respect to θ . Therefore if it holds for all vertices then it also holds for all the set $\theta \in \text{co} \{ \theta^{v_1} \dots \theta^{v_N} \}$. Multiplying (9) from the

left and the right by $\bar{\theta}_n^T$ and $\bar{\theta}_n$ respectively leads to (4) directly. This completes the proof.

Remark 2 Note that the last operation corresponds to the elimination lemma²²⁾, which is not lossless in this case since M is assumed to be independent of θ .

Theorem 2 (H_2 Performance Analysis) If there exist a symmetric matrix $\bar{N} \in \mathcal{S}^{n_w \pi^{[1,k]}}$, matrices $\bar{P} \in \mathcal{R}^{n \times n \pi^{[1,k]}}$, $F \in \mathcal{R}^{n(\pi^{[1,k]}-1) \times n \pi^{[1,k]}}$, $M \in \mathcal{R}^{n_w(\pi^{[1,k]}-1)+n(\pi^{[1,k]}-1) \times (n_w \pi^{[1,k]}+n \pi^{[1,k]})}$ and $H \in \mathcal{R}^{(\pi^{[1,k]}-1) \times \pi^{[1,k]}}$, and a positive number γ_2 such that

$$\text{He} \{ \bar{P}^T \bar{A} + \eta_n(\theta) F \} + \bar{C}^T \bar{C} < 0, \quad (10)$$

$$\begin{bmatrix} \bar{N} & \bar{B}^T \bar{P} \\ \bar{P}^T \bar{B} & \frac{\bar{P}^T e_n^T + e_n \bar{P}}{2} \end{bmatrix} + \text{He} \left(\begin{bmatrix} \eta_{n_w}(\theta) & \mathbf{0} \\ \mathbf{0} & \eta_n(\theta) \end{bmatrix} M \right) > 0, \quad (11)$$

$$\gamma_2^2 e e^T - \text{Tr}_{n_w}(\bar{N}) + \text{He} \{ \eta(\theta) H \} > 0 \quad (12)$$

hold for all vertices $\theta = \theta^{v_1}, \dots, \theta^{v_N}$, then the LTIPD system (1) is robustly stable and an upper bound of its H_2 performance is given by γ_2 .

Proof of Theorem 2 is much alike that of Theorem 1 when choosing $N(\theta) = \bar{\theta}_{n_w}^T \bar{N} \bar{\theta}_{n_w}$, thus we omit it.

Theorem 3 (H_∞ Performance Analysis) If there exist matrices $\bar{P} \in \mathcal{R}^{n \times n \pi^{[1,k]}}$ and $M \in \mathcal{R}^{n(\pi^{[1,k]}-1)+n_w(\pi^{[1,k]}-1) \times (n \pi^{[1,k]}+n_w \pi^{[1,k]})}$, and a positive number γ_∞ such that

$$\begin{bmatrix} \text{He}(\bar{P}^T \bar{A}) & \bar{P}^T \bar{B} \\ \bar{B}^T \bar{P} & -\gamma_\infty^2 (e e^T \otimes I_{n_w}) \end{bmatrix} + \begin{bmatrix} \bar{C}^T \\ \bar{D}^T \end{bmatrix} [\bar{C} \bar{D}] + \text{He} \left(\begin{bmatrix} \eta_n(\theta) & \mathbf{0} \\ \mathbf{0} & \eta_{n_w}(\theta) \end{bmatrix} M \right) < 0 \quad (13)$$

holds for all vertices $\theta = \theta^{v_1}, \dots, \theta^{v_N}$, then the LTIPD system (1) is robustly stable and an upper bound of its H_∞ performance is given by γ_∞ .

Proof of Theorem 3 is much alike that of Theorem 1, thus we omit it.

3.2 Less Conservative Conditions

In the previous subsection, Lyapunov functions $P(\theta)$ are set to be as in (3). If we reduce the conservatism due to setting $P(\theta)$ to be in (3), increasing the parameter-dependency of $P(\theta)$, which was indicated in Remark 1, is effective. However, it leads to increased numerical complexity because the sizes of additionally introduced matrices M , etc. are increased.

In this subsection, we show another methodology to tackle the conservatism due to setting $P(\theta)$ to be in (3). We now set $P(\theta)$ to be as follows.

$$P(\theta) = \bar{P}(\theta) \bar{\theta}_n, \quad (14)$$

where $P(\theta)$ has the same definition as in (3) but with $\bar{P}(\theta)$ being chosen as an affine function of θ . This choice increases by one the power of the PDLF.

Remark 3 If $\bar{P}(\theta)$ is set to be parameter-independent, then $P(\theta)$ in (14) is the same as $P(\theta)$ in (3). Therefore, using $P(\theta)$ in (14) is no more conservative than using $P(\theta)$ in (3).

Using $P(\theta)$ defined in (14), we obtain the following theorem.

Theorem 4 (Stability Analysis) If there exist a parametrically affine matrix $\bar{P}(\theta)$ and a matrix M such that

$$\text{He} \{ \bar{P}(\theta)^T \bar{A} + \eta_n(\theta) M \} < 0, \quad (15)$$

holds for all vertices $\theta = \theta^{v_1}, \dots, \theta^{v_N}$, then the LTIPD system (1) is robustly stable.

Proof 2 Inequality (15) is affine with respect to θ . Therefore if it holds for all vertices then it also holds for all the set $\theta \in \text{co} \{ \theta^{v_1} \dots \theta^{v_N} \}$. Multiplying (15) from the left and the right by $\bar{\theta}_n^T$ and $\bar{\theta}_n$ respectively leads to (4) directly. This completes the proof.

We obtain the counterparts of Theorems 2 and 3.

Theorem 5 (H_2 Performance Analysis) If there exist a symmetric parametrically affine matrix $\bar{N}(\theta)$, a parametrically affine matrix $\bar{P}(\theta)$, matrices F , M and H , and a positive number γ_2 such that (10), (11) and (12), in which $\bar{P}$ and $\bar{N}$ are replaced by $\bar{P}(\theta)$ and $\bar{N}(\theta)$ respectively, hold for all vertices $\theta = \theta^{v_1}, \dots, \theta^{v_N}$, then the LTIPD system (1) is robustly stable and an upper bound of its H_2 performance is given by γ_2 .

Proof of Theorem 5 is much alike that of Theorem 4 when choosing $N(\theta) = \bar{\theta}_{n_w}^T \bar{N}(\theta) \bar{\theta}_{n_w}$, where $\bar{N}(\theta)$ is an affine function of θ . Thus we omit it.

Remark 4 If $\bar{P}(\theta)$ and $\bar{N}(\theta)$ are set to be parameter-independent, then Theorem 5 becomes Theorem 2. Therefore, Theorem 5 is no more conservative than Theorem 2.

Theorem 6 (H_∞ Performance Analysis) If there exist a parametrically affine matrix $\bar{P}(\theta)$, a matrix M , and a positive number γ_∞ such that (13), in which $\bar{P}$ is replaced by $\bar{P}(\theta)$, holds for all vertices $\theta = \theta^{v_1}, \dots, \theta^{v_N}$, then the LTIPD system (1) is robustly stable and an upper bound of its H_∞ performance is given by γ_∞ .

Proof of Theorem 6 is much alike that of Theorem 4, thus we omit it.

4. Observations

In this section, we first show that our stability analysis Theorem 1 encompasses the stability analysis based on quadratic stability for parametrically affine LTIPD systems. We next show that increasing parameter-dependency of Lyapunov functions leads to less conservative analysis in exchange for increased numerical complexity. We have obtained similar assertions for H_2 performance and H_∞ performance analysis, however, we omit them due to lack of space.

4.1 Inclusion Relation

In this subsection, we show that Theorem 1 encompasses quadratic stability. For simplicity, we assume that the number of parameters k is three. However, we can easily extend the following result to the general case.

We now consider parametrically affine system for (1). Under this assumption, $A(\theta)$ in (2) is given as follows.

$$A(\theta) = A_0 + \sum_{i=1}^3 \theta_i A_i = \bar{A} \bar{\theta}_n,$$

where $\bar{A} = [A_0 \ A_3 \ A_2 \ 0_n \ A_1 \ 0_{n,3n}]$.

We give the following assertion.

Proposition 1 *If there exists a symmetric matrix $P_0 \in S^n$ such that $\text{He}\{P_0 A(\theta)\} < 0$ holds for all vertices $\theta = \theta^{v_1}, \dots, \theta^{v_N}$, then there always exist matrices $\bar{P}$ and M such that (9) holds for all vertices $\theta = \theta^{v_1}, \dots, \theta^{v_N}$.*

Proof 3 We set $\bar{P}$ in Theorem 1 to be $[P_0 \ 0_{n,7n}]$, and set M in Theorem 1 to be as follows with sufficiently small positive scalars $\varepsilon_1, \varepsilon_2$ and ε_3 .

$$M = \begin{bmatrix} (P_0 A_3)^T & \varepsilon_3 I_n & 0 & 0 \\ (P_0 A_2)^T & 0 & \varepsilon_2 I_{2n} & 0 \\ 0_n & & & \\ (P_0 A_1)^T & 0 & 0 & \varepsilon_1 I_{4n} \\ 0_{3n,n} & & & \end{bmatrix} \quad (16)$$

In the following, we obtain the equivalent condition to (9). We first obtain the following inequalities after applying Schur complement to (9).

$$\begin{bmatrix} \text{He}\{P_0 (A_0 + \sum_{i=1}^3 \theta_i A_i)\} & \varepsilon_3 \theta_3 I_n \\ \varepsilon_3 \theta_3 I_n & -2\varepsilon_3 I_n \\ \varepsilon_2 \theta_2 I_{2n} & -2\varepsilon_2 I_{2n} \end{bmatrix} + \frac{\varepsilon_1}{2} \theta_1^2 I_{4n} < 0, \quad \varepsilon_1 > 0 \quad (17)$$

As the variation of θ_1 is bounded, (17) always hold with sufficiently small positive scalar ε_1 if the first term of the first inequality is negative definite. We next consider the following inequality instead of the first one in (17).

$$\begin{bmatrix} \text{He}\{P_0 (A_0 + \sum_{i=1}^3 \theta_i A_i)\} & \varepsilon_3 \theta_3 I_n \\ \varepsilon_3 \theta_3 I_n & -2\varepsilon_3 I_n \\ \varepsilon_2 \theta_2 I_{2n} & -2\varepsilon_2 I_{2n} \end{bmatrix} < 0 \quad (18)$$

After applying Schur complement to (18), we then obtain the following inequalities.

$$\begin{bmatrix} \text{He}\{P_0 (A_0 + \sum_{i=1}^3 \theta_i A_i)\} & \varepsilon_3 \theta_3 I_n \\ \varepsilon_3 \theta_3 I_n & -2\varepsilon_3 I_n \end{bmatrix} + \frac{\varepsilon_2}{2} \theta_2^2 I_{2n} < 0, \quad \varepsilon_2 > 0 \quad (19)$$

As the variation of θ_2 is bounded, the above inequalities always hold with sufficiently small positive scalar ε_2 if the first term of the first inequality is negative definite. After applying the same procedure, which is used in the transformation from (17) to (18), to the first inequality in (19), we obtain the following inequalities.

$$\text{He}\{P_0 (A_0 + \sum_{i=1}^3 \theta_i A_i)\} < 0, \quad \varepsilon_3 > 0 \quad (20)$$

Considering (17), (19) and (20), if there exist P_0 such that $\text{He}\{P_0 (A_0 + \sum_{i=1}^3 \theta_i A_i)\} < 0$ holds then (9) holds with $\bar{P} = [P_0 \ 0_{n,7n}]$ and M defined in (16). The condition $\text{He}\{P_0 (A_0 + \sum_{i=1}^3 \theta_i A_i)\} < 0$ is the condition based on quadratic stability. Thus, if the condition based on quadratic stability holds, then the condition of Theorem 1 always holds.

Conservatism reduction from quadratic stability is proved here for systems affine in three parameters. The same proof can be generalized for all parametrically affine models at the expense of recursive Schur complements, one for each parameter.

4.2 Numerical Complexity vs. Conservatism

In this subsection, we show that increasing parameter-dependency of Lyapunov functions leads to less conservative analysis in exchange for increased numerical complexity.

For simplicity, we assume that the number of parameters is unity. However, we can easily extend the following result to the case that the number of parameters is $k(\geq 2)$.

Matrix $A(\theta)$ of (2) is given as follows.

$$\begin{aligned} A(\theta) &= \bar{A} \bar{\theta}_n = [A_{-l_1} \ \dots \ A_0 \ \dots \ A_{m_1}] \bar{\theta}_n \\ &= \hat{A} \hat{\theta}_n = [A_{-l_1} \ \dots \ A_0 \ \dots \ A_{m_1} \ 0_n] [\bar{\theta}_n^T \ \theta_1^{m+1} I_n]^T \end{aligned}$$

We give the following assertion.

Proposition 2 *If there exist matrices $\bar{P}$ and M such that (9) holds for all vertices $\theta = \theta^{v_1}, \dots, \theta^{v_N}$, then there always exist matrices $\hat{P} \in \mathcal{R}^{n \times n(l_1+m_1+1+1)}$ and $\hat{M} \in \mathcal{R}^{(l_1+m_1+1) \times n(l_1+m_1+1+1)}$ such that (9), in which $\eta(\theta)$ is appropriately extended, holds for all vertices $\theta = \theta^{v_1}, \dots, \theta^{v_N}$.*

Proof 4 We assume that there exist matrices $\bar{P}$ and M such that (9) holds for all vertices $\theta = \theta^{v_1}, \dots, \theta^{v_N}$. We set $\hat{P}$ to be $[\bar{P} \ 0_n]$, and set M in Theorem 1 to be $\begin{bmatrix} M & 0 \\ 0 & \varepsilon I_n \end{bmatrix}$ with a sufficiently small positive scalar ε . We then obtain the following inequality which is equivalent to (9) with using $\hat{P}$ and $\hat{M}$.

$$\begin{bmatrix} \text{He} \{ \bar{P}^T \bar{A} + \eta_n(\theta) M \} & \begin{bmatrix} 0 \\ \varepsilon \theta_1 I_n \end{bmatrix} \\ \begin{bmatrix} 0 & \varepsilon \theta_1 I_n \end{bmatrix} & -2\varepsilon I_n \end{bmatrix} < 0,$$

where $\eta_n(\theta)$ has a compatible size. After applying Schur complement to the above inequality, we obtain the following inequalities.

$$\text{He} \{ \bar{P}^T \bar{A} + \eta_n(\theta) M \} + \begin{bmatrix} 0 & 0 \\ 0 & \frac{\varepsilon \theta_1^2}{2} I_n \end{bmatrix} < 0, \varepsilon > 0$$

As the variation of θ_1 is bounded, the first inequality of the above condition always holds with sufficiently small positive scalar ε if the first term of the first inequality is negative definite, which is the assumption. Therefore there always exist $\hat{P}$ and $\hat{M}$ that satisfy (9). This completes the proof.

Conservatism reduction by increasing the power of parameters is proved here for systems with one parameter. The same proof can be generalized for all the considered models at the expense of recursive Schur complements, one for each power (negative and positive) of each parameter.

Proposition 2 shows that enlargement of the size of $\bar{P}$ leads to less conservative analysis, which was indicated in Remark 1. However, the enlargement of the size of $\bar{P}$ leads to the enlargement of the size of M , i.e. enlargement of the size of $\bar{P}$ leads to increased numerical complexity.

5. Numerical Examples

In this section, we show several numerical examples to demonstrate the effectiveness of our analysis methods.

5.1 Polynomial Representation vs. Polytopic Representation

We show an example that demonstrates the effectiveness of polynomial representation of LTIPD system in (2).

We consider the following system.

$$\dot{x} = \begin{bmatrix} -(\theta + 1/\theta) & 1 \\ 1 & -\theta \end{bmatrix} x, \quad 0 < \delta_{\min} \leq \theta \leq \delta_{\max} = 100$$

Note that this system is robustly stable for any $\delta_{\min}$. We minimize $\delta_{\min}$ under the condition that the system is assured to be robustly stable using our Theorem 1, Theorem 2 of 6), and Theorem 1 of 23) which is illustrated to be less conservative than Theorem 2 of 6) on numerical examples. When applying Theorem 2 of 6) and Theorem 1 of 23),

Table 1 Minimized $\delta_{\min}$

Theorem 1 ($l_1 = m_1 = 1$)			
9.1880×10^{-4}			
	Theorem 2 of 6)	Theorem 1 of 23)	Gridding method
polytope ABC	0.9950	0.9950	0.9950 (40000 points)
polytope ABDE	0.5000	0.5000	0.5000 (80000 points)

Table 2 Minimized $\delta_{\min}$

Theorem 1 ($l_1 = 1, m_1 = 1$)	Biquadratic stability in 3)
9.1880×10^{-4}	0.9950

two polytopes are used to cover the variation of $\theta + 1/\theta$; polytope ABC and polytope ABDE in Fig. 1. Lyapunov functions used in Theorem 2 of 6) correspond to be cubically PDLFs with respect to the vertices; vertices A, B and C or vertices A, B, D and E. Lyapunov functions used in Theorem 1 of 23) are set to be parameter-dependent up to second-order with respect to the vertices and their all by-products. The obtained $\delta_{\min}$ are given in **Table 1**. For comparison, we calculate $\delta_{\min}$ by fine gridding method, and the numbers of gridding points are denoted in parentheses. Considering that Theorem 2 of 6) and Theorem 1 of 23) produce the same stability margins as gridding method, these two methods are non-conservative for polytopically represented systems. However, Theorem 1 gives much tighter stability margin than Theorem 2 of 6) and Theorem 1 of 23). This result shows that stability analysis for polynomially parameter-dependent systems using polytopic approach is sometimes more conservative than using polynomial approach.

5.2 Original Representation vs. Modified Representation

In this subsection, we show that some modification of the original representation leads to conservatism.

We now consider the same example as in the previous subsection. Biquadratic stability proposed in 3) cannot be directly applied to this system, because this system is not affine with respect to θ . To apply biquadratic stability to this system, we assume that the system has two parameters; θ and $\alpha := 1/\theta$, whose deviations are respectively set as $\delta_{\min} \leq \theta \leq \delta_{\max} = 100$ and $1/\delta_{\max} = 1/100 \leq \alpha \leq 1/\delta_{\min}$. We minimize $\delta_{\min}$ under the condition that the system is assured to be robustly stable using biquadratic stability. The obtained $\delta_{\min}$ is given in **Table 2**. For comparison, we also show the obtained $\delta_{\min}$ using Theorem 1 with setting $l_1 = m_1 = 1$, which was obtained in the previous subsection. For this example, Theorem 1 gives much tighter stability margin than biquadratic stability with assuming that the number of parameters to be two. This result shows that modifi-

cation of system representation sometimes leads to conservatism.

5.3 Randomly Generated Examples

In this subsection, we demonstrate the effectiveness of our methods with randomly generated examples.

5.3.1 Stability Analysis

We consider the following four scenarios.

#1: $A(\theta) = \sum_{i=0}^2 \theta_1^i A_i$, $A_i \in \mathcal{R}^{2 \times 2}$ ($i = 0, 1, 2$), $|\theta_1| \leq \delta$

#2: $A(\theta) = \sum_{i=0}^2 \theta_1^i A_i$, $A_i \in \mathcal{R}^{3 \times 3}$ ($i = 0, 1, 2$), $|\theta_1| \leq \delta$

#3: $A(\theta) = A_0 + \theta_1 A_1$, $A_i \in \mathcal{R}^{4 \times 4}$ ($i = 0, 1$), $|\theta_1| \leq \delta$

#4: $A(\theta) = \sum_{i=0}^1 \sum_{j=0}^1 \theta_1^i \theta_2^j A_{ij}$, $A_{ij} \in \mathcal{R}^{2 \times 2}$ ($i, j = 0, 1$), $|\theta_1| \leq \delta$, $|\theta_2| \leq \delta$

We generate 1000 examples for #1 ~ #4. In all examples, A_0 or A_{00} are generated as stable matrices, and all the elements of A_i or A_{ij} lie in $[-0.5; 0.5]$.

For each generated example, we maximize δ under the condition that the system is assured to be robustly stable using Theorem 1, and set obtained δ as $\delta_{\max}$. For comparison, we calculate the maximum value of δ using fine gridding method (1000 points for the deviation of θ_1 in scenarios #1 ~ #3 and each 200 points for the deviations of θ_1 and θ_2 in scenario #4), and set obtained δ as δ_{grid} . **Table 3** shows the average values of $\delta_{\max}/\delta_{grid}$. We also give the number of decision variables as N_{dv} . Table 3 shows that Theorem 1 gives almost the exact stability margin, which demonstrates the effectiveness of Theorem 1.

5.3.2 H_2 Performance Analysis

We consider the following scenario:

#1: $A(\theta) = \sum_{i=0}^1 \sum_{j=0}^1 \theta_1^i \theta_2^j A_{ij}$, $A_{ij} \in \mathcal{R}^{2 \times 2}$ ($i, j = 0, 1$), $|\theta_1| \leq \delta$, $|\theta_2| \leq \delta$, and other matrices $B(\theta) \in \mathcal{R}^{2 \times 2}$ and $C(\theta) \in \mathcal{R}^{2 \times 2}$ are also similarly generated.

We generate 1000 examples. In all examples, matrices A_{00} are generated as stable matrices, and all the elements of A_{ij} , B_{ij} and C_{ij} lie in $[-0.5; 0.5]$.

For each generated example, we first calculate the stability margin δ by fine gridding method (200²) and set δ as δ_{grid} , then calculate the upper bound of H_2 performance γ_2 by applying Theorems 2 and 5 with setting δ to be $0.8 \times \delta_{grid}$ and set γ_2 as $\gamma_{2\min}$. For comparison, we calculate the maximum value of H_2 performance γ_2 with setting δ to be $0.8 \times \delta_{grid}$ using fine gridding method (200²), and set γ_2 as γ_{2grid} . **Table 4** shows the average values of $\gamma_{2\min}/\gamma_{2grid}$. We also give the number of decision variables as N_{dv} . Table 4 shows that Theorem 5 gives more exact performance than Theorem 2, which demonstrates the less conservatism of Theorem 5 compared to

Table 3 Average values of $\delta_{\max}/\delta_{grid}$

	#1	#2	#3
Theorem 1 ($l_1 = 0, m_1 = 1$)	—	—	0.9986 ($N_{dv} = 52$)
Theorem 1 ($l_1 = 0, m_1 = 2$)	1.0000 ($N_{dv} = 33$)	0.9999 ($N_{dv} = 72$)	1.0000 ($N_{dv} = 126$)
Theorem 1 ($l_1 = 0, m_1 = 3$)	1.0000 ($N_{dv} = 60$)	1.0000 ($N_{dv} = 132$)	—
	#4		
Theorem 1 ($l_i = 0, m_i = 1, i = 1, 2$)	0.9990 ($N_{dv} = 60$)		
Theorem 1 ($l_i = 0, m_i = 2, i = 1, 2$)	0.9991 ($N_{dv} = 315$)		

Table 4 Average values of $\gamma_{2\min}/\gamma_{2grid}$

	#1
Theorem 2 ($l_i = 0, m_i = 1, i = 1, 2$)	1.0283 ($N_{dv} = 301$)
Theorem 5 ($l_i = 0, m_i = 1, i = 1, 2$)	1.0032 ($N_{dv} = 397$)

Table 5 Average values of $\gamma_{\infty\min}/\gamma_{\infty grid}$

	#1
Theorem 3 ($l_i = 0, m_i = 1, i = 1, 2$)	1.0075 ($N_{dv} = 205$)
Theorem 6 ($l_i = 0, m_i = 1, i = 1, 2$)	1.0053 ($N_{dv} = 229$)

Theorem 2.

5.3.3 H_∞ Performance Analysis

We consider the following scenario:

#1: $A(\theta) = \sum_{i=0}^1 \sum_{j=0}^1 \theta_1^i \theta_2^j A_{ij}$, $A_{ij} \in \mathcal{R}^{2 \times 2}$ ($i, j = 0, 1$), $|\theta_1| \leq \delta$, $|\theta_2| \leq \delta$, and other matrices $B(\theta) \in \mathcal{R}^{2 \times 2}$, $C(\theta) \in \mathcal{R}^{2 \times 2}$ and $D(\theta) \in \mathcal{R}^{2 \times 2}$ are also similarly generated.

We generate 1000 examples. In all examples, matrices A_{00} are generated as stable matrices, and all the elements of A_{ij} , B_{ij} , C_{ij} and D_{ij} lie in $[-0.5; 0.5]$.

For each generated example, we first calculate the stability margin δ by fine gridding method (200²) and set δ as δ_{grid} , then calculate the upper bound of H_∞ performance γ_∞ by applying Theorems 3 and 6 with setting δ to be $0.8 \times \delta_{grid}$ and set γ_∞ as $\gamma_{\infty\min}$. For comparison, we calculate the maximum value of H_∞ performance γ_∞ with setting δ to be $0.8 \times \delta_{grid}$ using fine gridding method (200²), and set γ_∞ as $\gamma_{\infty grid}$. **Table 5** shows the average values of $\gamma_{\infty\min}/\gamma_{\infty grid}$. We also give the number of decision variables as N_{dv} . Table 5 shows that Theorem 6 gives more exact performance than Theorem 3, which demonstrates the less conservatism of Theorem 6 compared to Theorem 3.

These randomly generated examples demonstrate that our proposed methods produce almost the exact stability/performance analysis.

5.4 Comparison with Existing Methods

5.4.1 Stability Analysis

Consider now the following polytopic system borrowed from 6).

Table 6 Maximized η

Theorem 1 ($l_i = 0, m_i = 1, i = 1, 2$)	Theorem 4 ($l_i = 0, m_i = 1, i = 1, 2$)	Gridding method (1000 points)	Theorem 2 of 6)
3.5518 ($N_{dv} = 132$)	3.5518 ($N_{dv} = 156$)	3.5518	3.5518 ($N_{dv} = 96$)

Table 7 Maximized η

Theorem 1 ($l_i = 0, m_i = 1, i = 1, 2, 3$)	Theorem 4 ($l_i = 0, m_i = 1, i = 1, 2, 3$)	Gridding method (40000 points)	Theorem 2 of 6)
2.0605 ($N_{dv} = 976$)	2.2238 ($N_{dv} = 1136$)	2.2238	1.9306 ($N_{dv} = 204$)

$$A(\theta) = \sum_{i=1}^2 \theta_i (A_0 + \eta A_i),$$

where $\theta^{v1} = [1 \ 0]^T$, $\theta^{v2} = [0 \ 1]^T$, and A_i ($i = 0, 1, 2$) are defined as follows:

$$A_0 = \begin{bmatrix} -2.0 & 1.0 & -1.0 \\ 2.5 & -3.0 & 0.5 \\ -1.0 & 1.0 & -3.5 \end{bmatrix}, A_1 = \begin{bmatrix} -0.7 & -0.5 & -2.0 \\ -0.8 & 0.0 & 0.0 \\ 1.5 & 2.0 & 2.4 \end{bmatrix}, A_2 = -A_1.$$

Table 6 shows the maximized η under the condition that the system is assured to be robustly stable. For comparison, we show the number of decision variables as N_{dv} . Theorem 1 with setting $l_i = 0, m_i = 1$ ($i = 1, 2$), Theorem 4 with setting $l_i = 0, m_i = 1$ ($i = 1, 2$), and Theorem 2 of 6) all give the exact stability margin. However, the numbers of decision variables in our methods are larger than that in Theorem 2 of 6). Thus, Theorem 2 of 6) is effective compared to our methods in terms of numerical complexity.

We next consider the following polytopic system borrowed from 6), which was originally published in 17).

$$A(\theta) = \sum_{i=1}^3 \theta_i (A_0 + \eta A_i),$$

where $\theta^{v1} = [1 \ 0 \ 0]^T$, $\theta^{v2} = [0 \ 1 \ 0]^T$, $\theta^{v3} = [0 \ 0 \ 1]^T$, and A_i ($i = 0, \dots, 3$) are defined as follows:

$$A_0 = \begin{bmatrix} -2.4 & -0.6 & -1.7 & 3.1 \\ 0.7 & -2.1 & -2.6 & -3.6 \\ 0.5 & 2.4 & -5.0 & -1.6 \\ -0.6 & 2.9 & -2.0 & -0.6 \end{bmatrix}, A_1 = \begin{bmatrix} 1.1 & -0.6 & -0.3 & -0.1 \\ -0.8 & 0.2 & -1.1 & 2.8 \\ -1.9 & 0.8 & -1.1 & 2.0 \\ -2.4 & -3.1 & -3.7 & -0.1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.9 & 3.4 & 1.7 & 1.5 \\ -3.4 & -1.4 & 1.3 & 1.4 \\ 1.1 & 2.0 & -1.5 & -3.4 \\ -0.4 & 0.5 & 2.3 & 1.5 \end{bmatrix}, A_3 = \begin{bmatrix} -1.0 & -1.4 & -0.7 & -0.7 \\ 2.1 & 0.6 & -0.1 & -2.1 \\ 0.4 & -1.4 & 1.3 & 0.7 \\ 1.5 & 0.9 & 0.4 & -0.5 \end{bmatrix}.$$

Table 7 shows the maximized η under the condition that the system is assured to be robustly stable. For this example, the method of 17) gives the exact stability margin, and Theorem 2 of 6) does not give the exact stability margin. Theorem 1 with setting $l_i = 0, m_i = 1$ ($i = 1, 2, 3$) does not give the exact stability margin. However, Theorem 4 with setting $l_i = 0, m_i = 1$ ($i = 1, 2, 3$) gives the exact stability margin.

This example demonstrates the effectiveness of our methods compared to Theorem 2 of 6).

5.4.2 H_∞ Performance Analysis

Consider the following polytopic system borrowed from 6).

$$\begin{bmatrix} A(\theta) & B(\theta) \\ C(\theta) & D(\theta) \end{bmatrix} = \sum_{i=1}^3 \begin{bmatrix} \theta_i A_i & \theta_i B_i \\ \theta_i C_i & \theta_i D_i \end{bmatrix},$$

where A_i ($i = 1, 2, 3$), etc. are defined as follows.

Table 8 Upper bound of H_∞ performance

Theorem 3 ($l_i = 0, m_i = 1, i = 1, 2, 3$)	Gridding method (40000 points)	Theorem 5 of 6)
1.8716 ($N_{dv} = 945$)	1.8715	1.8716 ($N_{dv} = 106$)

Table 9 Upper bound of H_∞ performance

Theorem 3 ($l_i = 0, m_i = 1, i = 1, 2$)	Gridding method (1000 points)	Theorem 5 of 6)
1.0140 ($N_{dv} = 217$)	1.0140	1.0258 ($N_{dv} = 91$)

$$A_1 = \begin{bmatrix} -0.5008 & -0.4073 & 0.0336 \\ 0.2194 & 0.3801 & 0.5698 \\ -0.4972 & -0.6033 & -0.2826 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -0.1603 & -0.1433 & -0.3133 \\ -0.1536 & -1.254 & 0.3023 \\ 0.5079 & 0.2159 & -0.1128 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -0.2436 & 0.1515 & -0.1856 \\ -0.5742 & 0.0975 & 0.6841 \\ -0.3823 & -0.4388 & -0.9544 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0.2190 \\ -0.6587 \\ 0.7503 \end{bmatrix}, B_2 = B_1, B_3 = B_1,$$

$$C_1 = [-0.0890 \ 0.3838 \ -0.0374], C_2 = C_1, C_3 = C_1,$$

$$D_1 = -0.4786, D_2 = D_1, D_3 = D_1$$

Table 8 shows the minimized upper bound of H_∞ performance. For this example, Theorem 3 and Theorem 5 of 6) both give the exact H_∞ performance analysis. However, our method needs more numerical complexity than Theorem 5 of 6). Therefore, our method is not so effective as Theorem 5 of 6) in terms of numerical complexity.

We finally consider the following polytopic system borrowed from 6)

$$\begin{bmatrix} A(\theta) & B(\theta) \\ C(\theta) & D(\theta) \end{bmatrix} = \sum_{i=1}^2 \begin{bmatrix} \theta_i A_i & \theta_i B_i \\ \theta_i C_i & \theta_i D_i \end{bmatrix},$$

where A_i ($i = 1, 2$), etc. are defined as follows.

$$A_1 = \begin{bmatrix} -0.4820 & 0.2475 & 0.5523 \\ 0.6564 & -0.0541 & 0.6678 \\ 0.4631 & -0.8679 & -0.7465 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -0.3280 & -0.4585 & -0.0199 \\ 0.2639 & 0.1258 & -0.0179 \\ -0.0156 & 0.1079 & -0.5112 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} -0.7089 \\ 0.0738 \\ 0.5886 \end{bmatrix}, B_2 = \begin{bmatrix} 0.0118 \\ -0.5969 \\ 0.0295 \end{bmatrix},$$

$$C_1 = [0.3126 \ 0.3343 \ -0.3119], C_2 = C_1,$$

$$D_1 = 0.2713, D_2 = D_1$$

Table 9 shows the minimized upper bound of H_∞ performance. For this example, Theorem 5 of 6) does not give the exact performance, however, Theorem 3 gives almost the exact H_∞ performance analysis. Therefore, our

method is more effective than Theorem 5 of 6).

6. Conclusions

We propose general formulations for robust stability, H_2 performance, and H_∞ performance analysis for linear time-invariant uncertain systems, whose state-space matrices are parameter-dependent in negative as well as positive power series of parameters, using Lyapunov functions which are parameter-dependent in negative as well as positive power series of parameters. Although our formulations are only sufficient conditions for the original analysis problems because we apply slack variable approach. However, randomly generated examples and several examples borrowed from existing papers demonstrate the effectiveness of our methods compared to existing methods.

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Robust Stability/Performance A
Polytopically Parameter-Dependent

Parameter-Dependent

Classical, L₂, and L₁

and L₂ Norms

This paper addresses robust stability and performance

performance of a class of systems with respect to

performance with respect to parameter-dependent

only additive conditions for the original system

are obtained using Lyapunov methods and

Key Words: robust stability, robust performance,

robust stability, robust performance

1. Introduction

Robust stability and performance

has been a topic of research in control theory

and robustness analysis for many years

in the context of robust stability analysis

method for linear time-invariant (LTI) systems

(LTI) systems using slack variables. This approach

is very simple compared to existing methods, e.g.,

the use of LMI techniques to reduce the conservatism

induced by the LMI approach, which is the

the slack variables have been widely employed in the

decade of robust control analysis. In particular, and

the use of LMI techniques to reduce the conservatism

induced by the LMI approach, which is the

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