Mean Values, Measurement of Fuzziness and Variance of Fuzzy Random Variables for Fuzzy Optimization

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Abstract - This paper discusses mean values and variance defined by fuzzy measures as evaluation methods of fuzzy numbers/fuzzy random variables, and the methods are applicable to decision making with both randomness and fuzziness. We find the method with λ-mean functions has proper properties. The variance and the corresponding co-variance and correlation are introduced and their fundamental properties are discussed. The measurement of fuzziness regarding fuzzy numbers is also presented, where fuzziness is another uncertainty different from randomness and comes from the imprecise of data. An example is given as an application in financial engineering of portfolio.

I. Introduction

In dynamical systems such as option pricing in financial engineering (Yoshida [10, 11]), we need to pay attention to criteria of objective functions with uncertainty since the evaluation is related to the precision and reliability of decisions. Estimation of uncertain quantities is important in dynamical decision making. In this paper we focus on evaluation of fuzzy/random quantities as uncertainty. The criterion should not be ad hoc and it should be given from some reasonable and theoretical viewpoint. We give the valuation of fuzzy/random quantities by a kind of mean values, and further we discuss a criterion to measure the fuzziness for decision making. The most popular methods are the defuzzification and ordering of fuzzy quantities, and many authors have examined the defuzzification method for fuzzy numbers in various applications ([6,7,1],[1],[4]). From the viewpoint of measure theory, Campos and Munoz [1] gave the following type evaluation of fuzzy numbers:

\[
\int_0^1 h(\alpha) \, dm(\alpha),
\]

where the function \( h(\alpha) \) is an estimation of the \( \alpha \)-cut of the fuzzy numbers and \( m \) is a probability measure. López-Díaz and Gil [5] studied this type of evaluation in a general form with randomness. When we use the defuzzification methods like (1) in decision making modeling, it is needed to discuss the meaning of the measure \( m \) on \([0, 1]\) and to give its reasonable construction. In decision making with fuzzy numbers/fuzzy random variables, the meaning of criteria is important, and we discuss it from the viewpoint of measure theory. To introduce the mean values of fuzzy numbers, we need to demonstrate the following three items:

(Q.1) How do we represent the values which fuzzy numbers \( \tilde{a} \) take?

(Q.2) How do we define a measure induced from the fuzzy numbers \( \tilde{a} \)?

(Q.3) How should we give the mean value and the variance of the fuzzy random variables by the measure?

In this proposed method, we estimate fuzzy numbers/fuzzy random variables by probabilistic expectation and fuzzy measures, which are called evaluation measures, and the results are given by mean values and measurement of fuzziness with the decision maker’s two subjective parameters, which are called a possibility-necessity weight for subjective estimation and a pessimistic-optimistic index for subjective decision. Especially we focus on the estimation methods with the possibility measure and the necessity measure for its numerical computation in modeling. Next this method is applied to fuzzy random variables, and we discuss variances, co-variance and correlation of fuzzy random variables. As a result we find the method with λ-mean functions has proper properties, and we derive fundamental properties regarding the variance and the corresponding co-variance and correlation. In the last section, we give an application example in financial engineering of portfolio.

II. Mean and Measurement of Fuzziness by Evaluation Measures

A. Fuzzy numbers and fuzzy measures

First we introduce some notations of fuzzy numbers. Let \( \mathbb{R} \) denote the set of all real numbers, and let \( B \) and \( I \) be the Borel σ-field of \( \mathbb{R} \) and the set of all non-empty bounded closed intervals respectively. A fuzzy number is denoted by its membership function \( \tilde{a} : \mathbb{R} \to [0, 1] \) which is normal, upper-semicontinuous, fuzzy convex and has a compact support. \( \mathcal{R} \) denotes the set of all fuzzy numbers, and \( \mathcal{R}_c \) also denotes the set of fuzzy numbers with continuous membership functions. In this paper, we identify fuzzy numbers with their corresponding membership functions. The \( \alpha \)-cut of a fuzzy number \( \tilde{a}(\in \mathcal{R}) \) is given by \( \tilde{a}_\alpha := \{ x \in \mathbb{R} \mid \tilde{a}(x) \geq \alpha \} \) \( (\alpha \in (0, 1]) \) and \( \tilde{a}_0 := \overline{\{ x \in \mathbb{R} \mid \tilde{a}(x) > 0 \}} \), where \( \overline{c} \) denotes the closure of an interval. The \( \alpha \)-cut is also written by closed intervals \( \tilde{a}_\alpha = [\tilde{a}_\alpha^- , \tilde{a}_\alpha^+ ] \) \( (\alpha \in [0, 1]) \). Hence we introduce
a partial order $\succeq$, so called the fuzzy max order, on fuzzy numbers $\mathcal{R}$: Let $\tilde{a}, \tilde{b} \in \mathcal{R}$ be fuzzy numbers. $\tilde{a} \succeq \tilde{b}$ means that $\tilde{a}_a^+ \geq \tilde{b}_a^+$ and $\tilde{a}_a^- \geq \tilde{b}_a^-$ for all $\alpha \in [0, 1]$. Then $(\mathcal{R}, \succeq)$ becomes a lattice. An addition, a subtraction and a scalar multiplication for fuzzy numbers are defined as follows (Zadeh [14]): For $\tilde{a}, \tilde{b} \in \mathcal{R}$ and $\zeta \geq 0$, the addition and subtraction $\tilde{a} \pm \tilde{b}$ of $\tilde{a}$ and $\tilde{b}$ and the scalar multiplication $\zeta \tilde{a}$ of $\zeta$ and $\tilde{a}$ are fuzzy numbers given by their $\alpha$-cuts $(\tilde{a} \pm \tilde{b})_\alpha := [\tilde{a}_a^- + \tilde{b}_a^-, \tilde{a}_a^+ + \tilde{b}_a^+]$, $(\zeta \tilde{a})_\alpha := [\zeta \tilde{a}_a^-, \zeta \tilde{a}_a^+]$, where $\tilde{a}_a^- = \{\alpha \in [0, 1] : \tilde{a}_a^-\}$.\footnote{\textcopyright 2016 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (http://creativecommons.org/licenses/by/4.0/).}

Definition 1 ([8]). A map $M : \mathcal{B} \mapsto [0, 1]$ is called a fuzzy measure on $\mathcal{B}$ if $M$ satisfies the following (M.i), (M.ii) and (M.iii) or (M.i), (M.ii) and (M.iv):

(M.i) $M(\emptyset) = 0$ and $M(\mathbb{R}) = 1$;
(M.ii) $M(I_1) \leq M(I_2)$ for $I_1, I_2 \in \mathcal{B}$ satisfying $I_1 \subseteq I_2$;
(M.iii) $M(\bigcup_{n=0}^{\infty} I_n) = \lim_{n \to \infty} M(I_n)$ for $\{I_n\}_{n=0}^{\infty} \subseteq \mathcal{B}$ satisfying $I_n \subseteq I_{n+1}$ for $n = 0, 1, 2, \cdots$;
(M.iv) $M(\bigcap_{n=0}^{\infty} I_n) = \lim_{n \to \infty} M(I_n)$ for $\{I_n\}_{n=0}^{\infty} \subseteq \mathcal{B}$ satisfying $I_n \supseteq I_{n+1}$ for $n = 0, 1, 2, \cdots$.

In this paper, we use fuzzy measures $M$ to evaluate a confidence degree that a fuzzy number takes values in an interval and we call them evaluation measures. First we deal with fuzzy numbers $\tilde{a}$ whose membership functions are continuous, i.e., $\tilde{a} \in \mathcal{R}_c$, and next we discuss about general fuzzy numbers $\tilde{a} \in \mathcal{R}$ whose membership functions are upper-semicontinuous but are not necessarily continuous.

B. Mean values of fuzzy numbers by evaluation measures

Using fuzzy measures, we present a method to estimate fuzzy numbers. Campos and Munoz [1] studied an evaluation of fuzzy numbers in the form (1). In decision making with fuzzy numbers, we discuss the meaning of the estimation from the viewpoint of measure theory, and then fuzzy measures are used to evaluate a confidence degree that a fuzzy number takes values in an interval.

Definition 2 ([8]). A map $g : \mathcal{I} \mapsto \mathbb{R}$ is called a mean function on $\mathcal{I}$ if $g$ satisfies the following (g.i), (g.ii) and (g.iii):

(g.i) $g(I) \in I$ for $I \in \mathcal{I}$;
(g.ii) $g(I_1) \leq g(I_2)$ for $I_1, I_2 \in \mathcal{I}$ satisfying $I_1 \subseteq I_2$;
(g.iii) $\lim_{n \to \infty} g(I_n) = g(I)$ for $\{I_n\}_{n=0}^{\infty} \subseteq \mathcal{I}$, $I \in \mathcal{I}$ satisfying $\lim_{n \to \infty} I_n = I$ with Hausdorff metric.

In this paper, the randomness is evaluated by the probabilistic expectation, and the fuzziness is evaluated by the mean values and evaluation measures. Let $g : \mathcal{I} \mapsto \mathbb{R}$ be a map such that

$$g([x, y]) := \lambda x + (1 - \lambda)y, \quad [x, y] \in \mathcal{I}, \quad \text{for some } \lambda \in [0, 1].$$

We note that $\lambda$ is a constant satisfying $0 \leq \lambda \leq 1$. This scalarization is used for the estimation of fuzzy numbers to give a mean value of the interval $[x, y]$ with a weight $\lambda$, and $\lambda$ is called a pessimistic-optimistic index and means the pessimistic degree in decision making ([3]). Then, $g$ is called a $\lambda$-mean function and $g([x, y])$ is called a $\lambda$-mean of the interval $[x, y]$. Let a fuzzy number $\tilde{a} \in \mathcal{R}_c$. We introduce mean values of the fuzzy number $\tilde{a}$ with respect to $\lambda$-mean functions $g$ and an evaluation measure $M_\lambda$, which depends on $\tilde{a}$, as follows

$$E^g(\tilde{a}) := \int_0^1 g(\tilde{a}_\alpha) M_\lambda(\tilde{a}_\alpha) d\alpha,$$

where $\tilde{a}_a = [\tilde{a}_a^-, \tilde{a}_a^+]$ is the $\alpha$-cut of the fuzzy number $\tilde{a}$. We note that (3) is normalized by $M(\tilde{a}_\alpha)(\alpha \in [0, 1])$. In a comparison with (1), $h(\alpha)$ is replaced with $g(\tilde{a}_\alpha)$ and the measure $\mu(\alpha)$ is taken as $M_\lambda(\tilde{a}_\alpha)$. In (3), $M(\tilde{a}_\alpha)$ means a confidence degree that the fuzzy number $\tilde{a}$ takes values in the interval $\tilde{a}_\alpha$ at each grade $\alpha$ (see Example 1).

Example 1. Let a fuzzy number $\tilde{a} \in \mathcal{R}_c$. An evaluation measure $M_\lambda$ is called the possibility evaluation measure and the necessity evaluation measure induced from the fuzzy number $\tilde{a}$ if it is given by the following (4) and (5) respectively:

$$M_p^\lambda(\tilde{a}) := \sup_{x \in I} \tilde{a}(x),$$

$$M_n^\lambda(\tilde{a}) := 1 - \sup_{x \notin I} \tilde{a}(x)$$

for $I \in \mathcal{B}$. We note that $M_p^\lambda$ and $M_n^\lambda$ satisfy Definition 1(M.i)(M.ii) (or (M.i)(M.ii)(M.iv)) since $\tilde{a}$ is continuous and has a compact support. Since $M_p^\lambda(\tilde{a}_\alpha) = 1$ and $M_n^\lambda(\tilde{a}_\alpha) = 1 - \alpha$ from (4) and (5), the corresponding mean values $E^g(\tilde{a})$ are reduced to

$$E^g_p(\tilde{a}) := \int_0^1 g(\tilde{a}_\alpha) d\alpha,$$

$$E^g_n(\tilde{a}) := \int_0^1 g(\tilde{a}_\alpha)(2 - 2\alpha) d\alpha.$$
For general fuzzy numbers \( \tilde{a} \in \mathbb{R} \), we define the \emph{mean values} for the general fuzzy number \( \tilde{a} \in \mathbb{R} \) by
\[
\bar{E}(\tilde{a}) := \lim_{n \to \infty} E(\tilde{a}^n),
\]
where \( E(\tilde{a}^n) \) are defined by (3) and \( \{\tilde{a}^n\}_{n=1}^{\infty} \subset \mathbb{R} \) is a sequence of fuzzy numbers whose membership functions are continuous and satisfy that \( \tilde{a}^n \uparrow \tilde{a} \) pointwise as \( n \to \infty \). The limiting value (9) is called well-defined if it is independent of the selection of the sequences \( \{\tilde{a}^n\}_{n=1}^{\infty} \subset \mathbb{R} \) (Yoshida [12]). From (6) and (7), by the bounded convergence theorem we obtain the mean values defined by the possibility evaluation measure and the necessity evaluation measure as follows.

\textbf{Lemma 1.} For general fuzzy numbers \( \tilde{a} \in \mathbb{R} \), it holds that
\[
\bar{E}^{P}(\tilde{a}) = \int_0^1 g(\tilde{a}_\alpha) \, d\alpha,
\]
\[
\bar{E}^{N}(\tilde{a}) = \int_0^1 g(\tilde{a}_\alpha) (2 - 2\alpha) \, d\alpha.
\]

Similarly to (10) and (11), under Assumption M we obtain the following representation (12) regarding general fuzzy numbers through the dominated convergence theorem.

\textbf{Lemma 2.} For general fuzzy numbers \( \tilde{a} \in \mathbb{R} \), it holds that
\[
\bar{E}(\tilde{a}) = \int_0^1 g(\tilde{a}_\alpha) \, w(\alpha) \, d\alpha / \int_0^1 w(\alpha) \, d\alpha.
\]

The mean value \( \bar{E}(\cdot) \) has the following natural properties regarding the linearity and the monotonicity for the fuzzy max order.

\textbf{Theorem 1.} Suppose Assumption M holds. For fuzzy numbers \( \tilde{a}, \tilde{b} \in \mathbb{R} \) and real numbers \( \theta, \zeta \in \mathbb{R} \) such that \( \zeta \geq 0 \), the following (i) - (iv) hold.
(i) \( \bar{E}(\tilde{a} \uparrow_1 \theta) = \bar{E}(\tilde{a}) + \theta \), where \( 1_B \) means the characteristic function of a set \( B \).
(ii) \( \bar{E}(\zeta \tilde{a}) = \zeta \bar{E}(\tilde{a}) \).
(iii) \( \bar{E}(\tilde{a} \uplus \tilde{b}) = \bar{E}(\tilde{a}) \uplus \bar{E}(\tilde{b}) \).
(iv) If \( \tilde{a} \geq_1 \tilde{b} \), then \( \bar{E}(\tilde{a}) \geq_1 \bar{E}(\tilde{b}) \) holds, where \( \geq_1 \) is the fuzzy max order.

\textbf{C. Measurement of fuzziness of fuzzy numbers}

The concept of the degree of fuzziness is given by the distance between fuzzy data and their nearest crisp data (Wang and Klir [8]). By using fuzzy measures, we present a method to measure the size of fuzziness regarding fuzzy numbers. Let \( \tilde{a} \in \mathbb{R} \) be a fuzzy number. A \emph{measurement of fuzziness} \( \bar{F}(\tilde{a}) \) of the fuzzy number \( \tilde{a} \) is given as follows: Let \( \alpha \in [0, 1] \). For an interval \( \tilde{a}_\alpha = [\tilde{a}_{\alpha}^-, \tilde{a}_{\alpha}^+] \) as a number with fuzziness, let \( y \in \tilde{a}_\alpha \) be a real number without fuzziness, which is taken temporarily as a true value estimated for \( \tilde{a}_\alpha \). Then, a size of fuzziness should be given by the distance between \( y \) and \( \tilde{a}_\alpha \):
\[
\max \{ \tilde{a}_\alpha^+ - y, y - \tilde{a}_\alpha^- \}.
\]

Therefore, the upper/lower measurements of fuzziness should be given by
\[
m^U(\tilde{a}_\alpha) := \max_{y \in \tilde{a}_\alpha} \{ \max \{ \tilde{a}_\alpha^+ - y, y - \tilde{a}_\alpha^- \} \} = \tilde{a}_\alpha^+ - \tilde{a}_\alpha^- \quad \text{and} \quad
\]
\[
m^L(\tilde{a}_\alpha) := \min_{y \in \tilde{a}_\alpha} \{ \max \{ \tilde{a}_\alpha^+ - y, y - \tilde{a}_\alpha^- \} \} = \frac{\tilde{a}_\alpha^+ - \tilde{a}_\alpha^-}{2}.
\]
Now we obtain the following natural results about the measurement of fuzziness $\tilde{F}(\cdot) = \tilde{F}^P(\cdot)$ or $\tilde{F}(\cdot) = \tilde{F}^N(\cdot)$ regarding the linearity and the inclusion.

**Theorem 2.** Suppose Assumption M holds. Let $\tilde{F} = \tilde{F}^P$ or $\tilde{F} = \tilde{F}^N$. For fuzzy numbers $\tilde{a}, \tilde{b} \in \mathbb{R}$ and real numbers $\theta, \zeta \in \mathbb{R}$, the following (i) – (iv) hold.

(i) $\tilde{F}(\tilde{a} + 1(\theta)) = \tilde{F}(\tilde{a})$.

(ii) $\tilde{F}(\zeta \tilde{a}) = |\zeta| \tilde{F}(\tilde{a})$.

(iii) $\tilde{F}(\tilde{a} \pm \tilde{b}) = \tilde{F}(\tilde{a}) + \tilde{F}(\tilde{b})$.

(iv) If $\tilde{a} \triangleright \tilde{b}$, then $\tilde{F}(\tilde{a}) \geq \tilde{F}(\tilde{b})$ holds, where $\triangleright$ means the inclusion in the sense of fuzzy sets.

**D. Possibility-necessity weights**

Let $\tilde{a} \in \mathbb{R}$ be a fuzzy number and let $\nu \in [0, 1]$ be a parameter. For applications of the mean values and measurement of fuzziness in actual problems, we introduce a mean value and a measurement of fuzziness with a parameter $\nu$:

$$
\bar{E}^\nu(\tilde{a}) := \nu \tilde{F}^P(\tilde{a}) + (1 - \nu) \tilde{F}^N(\tilde{a}),
$$

(22)

$$
\tilde{F}^\nu(\tilde{a}) := \nu \tilde{F}^P(\tilde{a}) + (1 - \nu) \tilde{F}^N(\tilde{a}).
$$

(23)

Then, $\nu$ is called a possibility-necessity weight, and (22) and (23) are the mean value and the measurement of fuzziness with the possibility-necessity weight $\nu$. We note that (22) and (23) are well-defined. The possibility mean $\tilde{F}^P(\cdot)$ and the necessity mean $\tilde{F}^N(\cdot)$ are represented by the mean values (22) with the corresponding possibility-necessity weights $\nu = 1, 0$ respectively. In this paper, we focus on this type of mean value (22) for numerical computation and we apply it to a mathematical model with fuzzy numbers in the previous Sections B and C. Hence (22) satisfies Assumption M with $\omega(\alpha) = \nu + 2(1 - \nu)(1 - \alpha)$. We also discuss about the measurement of fuzziness $\tilde{F}^\nu(\cdot)$ in the same way as $\bar{E}^\nu(\cdot)$. Thus we obtain the following theorem which is convenient for numerical calculations in applications.

**Theorem 3.** Let a fuzzy number $\tilde{a} \in \mathbb{R}$ and $\nu, \lambda \in [0, 1]$. Then, the mean value $\bar{E}^\nu(\tilde{a})$ and the measurement of fuzziness $\bar{F}^\nu(\tilde{a})$ with the possibility-necessity weight $\nu$ and the pessimistic-optimistic index $\lambda$ are represented by

$$
\bar{E}^\nu(\tilde{a}) = \int_0^1 g(\tilde{a}_\alpha) (\nu + 2(1 - \nu)(1 - \alpha)) \, d\alpha,
$$

(24)

$$
\tilde{F}^\nu(\tilde{a}) = \int_0^1 (\tilde{a}_+ - \tilde{a}_-)(\nu + (1 - \nu)(1 - \alpha)) \, d\alpha,
$$

(25)

where $\lambda$-mean function $g$ is given by $g(\tilde{a}_\alpha) = \lambda \tilde{a}_- + (1 - \lambda) \tilde{a}_+$.

**III. MEAN AND VARIANCE OF FUZZY RANDOM VARIABLES**

A. Mean values of fuzzy random variables by evaluation measures

Using evaluation measures, we discuss mean values, variances and co-variances regarding fuzzy random variables. To introduce these, we have difficulty that the expectation of fuzzy random variables are given by fuzzy numbers. Let $(\Omega, \mathcal{M}, P)$ be a probability space, where $\mathcal{M}$ is a $\sigma$-field of $\Omega$ and $P$ is a non-atomic probability measure. A fuzzy-number-valued map $\tilde{X} : \Omega \mapsto \mathbb{R}$ is called a fuzzy random variable if the maps $\omega \mapsto \tilde{X}_\alpha^{-}(\omega)$ and $\omega \mapsto \tilde{X}_\alpha^{+}(\omega)$ are measurable for all $\alpha \in [0, 1]$, where $\tilde{X}_\alpha^{-}(\omega) = [\tilde{X}_\alpha^{-}(\omega), \tilde{X}_\alpha^{+}(\omega)] = \{x \in \mathbb{R} | \tilde{X}(\omega)(x) \geq \alpha\}$. A fuzzy random variable $\tilde{X}$ is called integrably bounded if both maps $\omega \mapsto \tilde{X}_\alpha^{-}(\omega)$ and $\omega \mapsto \tilde{X}_\alpha^{+}(\omega)$ are integrable for all $\alpha \in [0, 1]$. In this paper, we deal with integrably bounded fuzzy random variables. Let $\tilde{X}$ be a fuzzy random variable. The expectation $E(\tilde{X})$ of the fuzzy random variable $\tilde{X}$ is defined by a fuzzy number (6)

$$
E(\tilde{X})(x) := \sup_{\alpha \in [0,1]} \min\{\alpha, 1_{E(\tilde{X}_\alpha^{-})(x)}\}, \ x \in \mathbb{R},
$$

(26)

where $1_B$ means the characteristic function of a set $B$ and $E(\tilde{X}_\alpha^{-})$ is the $\alpha$-cut $E(\tilde{X}_\alpha^{-}) := [\int_0^{\tilde{X}_\alpha^{-}(\omega)} \, dP(\omega), \int_0^{\tilde{X}_\alpha^{+}(\omega)} \, dP(\omega)]$ $(\alpha \in [0, 1])$. From the results of Section II, under Assumption M we find that for the fuzzy random variable $\tilde{X}$, the mean of the expectation $E(\tilde{X})$ is

$$
\tilde{E}(E(\tilde{X})) = E\left(\int_0^1 g(\tilde{X}_\alpha) \, d\alpha \right). \tag{27}
$$

From Theorem 1, we obtain the following results.

**Corollary 1.** Suppose Assumption M holds. For integrable fuzzy random variables $\tilde{X}, \tilde{Y}$ and real numbers $\theta, \zeta \in \mathbb{R}$ such that $\zeta \geq 0$, the following (i) – (v) hold.

(i) $E(\tilde{E}(\tilde{X})) = \tilde{E}(E(\tilde{X}))$.

(ii) $E(\tilde{E}(\tilde{X} + 1(\theta))) = \tilde{E}(E(\tilde{X})) + \theta$.

(iii) $E(\tilde{E}(\theta \tilde{X})) = \theta \tilde{E}(E(\tilde{X}))$.

(iv) $E(\tilde{E}(\tilde{X} + \tilde{Y})) = \tilde{E}(E(\tilde{X})) + E(\tilde{E}(\tilde{Y}))$.

(v) If $\tilde{X} \triangleright \tilde{Y}$ almost surely, then $E(\tilde{E}(\tilde{X})) \triangleright E(\tilde{E}(\tilde{Y}))$ holds, where $\triangleright$ is the fuzzy max order.

For fuzzy random variables, we can calculate the mean values with the corresponding possibility-necessity weights $\nu = 1, 0$. The following corollary is trivial from Theorems 2 and 3.

**Corollary 2.** Let a fuzzy random variable $\tilde{X}$ and $\nu, \lambda \in [0, 1]$. Then, the mean value $E(\tilde{E}^\nu(\tilde{X}))$ with the possibility-necessity weight $\nu$ and the pessimistic-optimistic index $\lambda$ are represented by

$$
E(\tilde{E}^\nu(\tilde{X})) = \int_0^1 E(\tilde{E}(\tilde{X}_\alpha))(\nu + 2(1 - \nu)(1 - \alpha)) \, d\alpha. \tag{28}
$$
where \( \lambda \)-mean function \( g \) is given by \( g(\tilde{X}_a) = \lambda \tilde{X}_a + (1 - \lambda) \tilde{X}_a^+ \).

**B. Variances, co-variance and correlation by evaluation measures**

Now we introduce variances and co-variances of fuzzy random variables. From the results on the previous Section III.A, for fuzzy random variables \( \tilde{X} \) and \( \tilde{Y} \), we define variances and co-variances as follows.

\[
V^\lambda(\tilde{X}) := E \left( \frac{\int_0^1 (g(\tilde{X}_a) - E(g(\tilde{X}_a)))^2 w(\alpha) \, d\alpha}{\int_0^1 w(\alpha) \, d\alpha} \right),
\]

\[
\text{Cov}^{\lambda, \gamma}(\tilde{X}, \tilde{Y}) := E \left( \frac{\int_0^1 (g(\tilde{X}_a) - E(g(\tilde{X}_a)))(h(\tilde{Y}_a) - E(h(\tilde{Y}_a))) w(\alpha) \, d\alpha}{\int_0^1 w(\alpha) \, d\alpha} \right),
\]

where \( \lambda \) and \( \gamma \) are constants \((\lambda, \gamma \in [0, 1]) \) and \( g, h : I \mapsto \mathbb{R} \) are the \( \lambda \)-mean function and the \( \gamma \)-mean function such that \( g(x, y) := \lambda x + (1 - \lambda) y \) and \( h(x, y) := \gamma x + (1 - \gamma) y \) for \( [x, y] \in I \). The variance (29) and co-variance (30) have the following properties.

**Theorem 4.** Suppose Assumption M holds. It holds that

\[
V^\lambda(\tilde{X}) = \lambda^2 V(\tilde{X}^\lambda) + 2\lambda(1 - \lambda) \text{Cov}(\tilde{X}^\lambda, \tilde{X}^+) + (1 - \lambda)^2 V(\tilde{X}^\lambda),
\]

\[
\text{Cov}^{\lambda, \gamma}(\tilde{X}, \tilde{Y}) = \lambda \gamma \text{Cov}(\tilde{X}^\lambda, \tilde{Y}^\lambda) + \lambda(1 - \gamma) \text{Cov}(\tilde{X}^\lambda, \tilde{Y}^+) + (1 - \lambda)(1 - \gamma) \text{Cov}(\tilde{X}^+, \tilde{Y}^\lambda),
\]

where

\[
V(\tilde{X}) := E \left( \frac{\int_0^1 (\tilde{X}_a^\lambda - E(\tilde{X}_a^\lambda))^2 w(\alpha) \, d\alpha}{\int_0^1 w(\alpha) \, d\alpha} \right),
\]

\[
\text{Cov}(\tilde{X}^\lambda, \tilde{X}^+),
\]

\[
\text{Cov}(\tilde{X}^+, \tilde{Y}^\lambda),
\]

\[
\text{Cov}(\tilde{X}^\lambda, \tilde{Y}^+),
\]

\[
\text{Cov}(\tilde{X}^+, \tilde{Y}^+),
\]

\[
\text{Cov}(\tilde{X}^-, \tilde{Y}^+),
\]

Hence we obtain the following natural results about the variance \( V(\cdot) \) regarding the linearity and the inclusion.

**Theorem 5.** Suppose Assumption M holds. For fuzzy random variables \( \tilde{X}, \tilde{Y} \) and real numbers \( \theta, \zeta \in \mathbb{R} \), the following (i) – (iii) hold.

(i) \( V^\lambda(\tilde{X} + 1(\varnothing)) = V^\lambda(\tilde{X}) \).

(ii) \( V^\lambda(\zeta \tilde{X}) = \zeta^2 V^\lambda(\tilde{X}) \) if \( \zeta \geq 0 \).

\( V^\lambda(\zeta \tilde{X}) = \zeta^2 V^{1-\lambda}(\tilde{X}) \) if \( \zeta < 0 \).

(iii) \( V^\lambda(\tilde{X} + \tilde{Y}) = V^\lambda(\tilde{X}) + 2 \text{Cov}^{\lambda, \lambda}(\tilde{X}, \tilde{Y}) + V^\lambda(\tilde{Y}) \).

\( V^\lambda(\tilde{X} - \tilde{Y}) = V^\lambda(\tilde{X}) - 2 \text{Cov}^{\lambda, 1-\lambda}(\tilde{X}, \tilde{Y}) + V^{1-\lambda}(\tilde{Y}) \).

For fuzzy random variables \( \tilde{X} \) and \( \tilde{Y} \), we can define correlation as follows.

\[
\rho^{\lambda, \gamma}(\tilde{X}, \tilde{Y}) := \frac{\text{Cov}^{\lambda, \gamma}(\tilde{X}, \tilde{Y})}{\sqrt{V^\lambda(\tilde{X})} \sqrt{V^\gamma(\tilde{Y})}},
\]

where

\[
V^\lambda(\tilde{X}) := E \left( \frac{\int_0^1 (g(\tilde{X}_a) - E(g(\tilde{X}_a)))^2 w(\alpha) \, d\alpha}{\int_0^1 w(\alpha) \, d\alpha} \right),
\]

\[
V^\gamma(\tilde{Y}) := E \left( \frac{\int_0^1 (h(\tilde{Y}_a) - E(h(\tilde{Y}_a)))^2 w(\alpha) \, d\alpha}{\int_0^1 w(\alpha) \, d\alpha} \right),
\]

\[
\text{Cov}^{\lambda, \gamma}(\tilde{X}, \tilde{Y}) := E \left( \frac{\int_0^1 (g(\tilde{X}_a) - E(g(\tilde{X}_a)))(h(\tilde{Y}_a) - E(h(\tilde{Y}_a))) w(\alpha) \, d\alpha}{\int_0^1 w(\alpha) \, d\alpha} \right),
\]

and \( g, h : I \mapsto \mathbb{R} \) are the \( \lambda \)-mean function and the \( \gamma \)-mean function respectively. Then we can derive the following fundamental results.

**Theorem 6.** Suppose Assumption M holds. It holds that

\( -1 \leq \rho^{\lambda, \gamma}(\tilde{X}, \tilde{Y}) \leq 1 \).

**IV. AN EXAMPLE IN FINANCIAL MODELING**

In this section, we give an application example of financial modeling regarding portfolio.

**Example 2.** We consider a portfolio model with \( n \) assets in \( T \) periods. For an asset \( i (= 1, 2, \cdots, n) \), an initial stock price \( S^i_0 \) is a positive constant and stock prices are described by

\[
S^i_t := S^i_0 \prod_{k=1}^{t} (1 + R^i_k) \quad \text{for} \quad t = 1, 2, \cdots, T,
\]

where \( \{R^i_t\}_{t=1}^T \) is called the rate of return and it is a uniform integrable sequence of independent, identically distributed real random variables. Now we deal with a case when the rate of return \( \{R^i_t\}_{t=1}^T \) has some imprecision. Define a discrete-time stock price process \( \{\tilde{S}^i_t\}_{t=0}^T \) by fuzzy random variables

\[
\tilde{S}^i_t := S^i_0 \prod_{k=1}^{t} (1 + \tilde{R}^i_k) \quad \text{for} \quad t = 1, 2, \cdots, T,
\]

where \( \{\tilde{R}^i_t\}_{t=1}^T \) is given by triangle-type fuzzy random variables

\[
\tilde{R}^i_t(x) := \begin{cases} 0 & \text{if} \ x < R^i_t - c^i_t \\ \frac{x - R^i_t + c^i_t}{2} & \text{if} \ R^i_t - c^i_t \leq x < R^i_t \\ \frac{x - R^i_t - c^i_t}{2} & \text{if} \ R^i_t \leq x < R^i_t + c^i_t \\ c^i_t & \text{if} \ x \geq R^i_t + c^i_t \end{cases}
\]
with positive constants $c_i$. We call them fuzzy factors. From the results in the previous sections, for assets $i$ and $j$, their means $\bar{\mu}_i$, variances $(\bar{\sigma}_i)^2$ and co-variances $\bar{\sigma}_{ij}$ are calculated as follows

$$
\bar{\mu}_i := E(\bar{E}(\bar{R}_i^t)) = \mu_i + \bar{E}(\bar{\delta}_i), \quad (38)
$$

$$
(\bar{\sigma}_i)^2 := V(\bar{R}_i^t) = (\sigma_i)^2, \quad (39)
$$

$$
\bar{\sigma}_{ij} := Cov(\bar{R}_i^t, \bar{R}_j^t) = \sigma_{ij}, \quad (40)
$$

where $\mu_i = E(R_i^t)$, $\sigma_i = V(R_i^t)$, $\sigma_{ij} = Cov(R_i^t, R_j^t)$, $\bar{E}(\bar{\delta}_i) = \int_0^x g(\bar{\alpha}_i)w(\alpha)da / \int_0^1 w(\alpha)da$ and

$$
\bar{\alpha}_i(x) := \begin{cases} 
0 & \text{if } x < -c_i \\
\frac{x+c_i}{c_i} & \text{if } -c_i \leq x < 0 \\
\frac{x-c_i}{c_i} & \text{if } 0 \leq x < c_i \\
0 & \text{if } x \geq c_i 
\end{cases} \quad (41)
$$

Hence we deal with the portfolio with a trading strategy $w = (w^1, w^2, \ldots, w^n)$ such that $w^1 + w^2 + \cdots + w^n = 1$ and $w^i \geq 0 (i = 1, 2, \ldots, n)$. For the trading strategy $w = (w^1, w^2, \ldots, w^n)$, the rate of return is given by

$$
R_t := w^1 \bar{R}_1^t + w^2 \bar{R}_2^t + \cdots + w^n \bar{R}_n^t. \quad (42)
$$

Therefore, their means and variances for the portfolio are

$$
\bar{\mu}_t := \sum_{i=1}^n w^i \bar{\mu}_i = \sum_{i=1}^n w^i (\mu_i + \bar{E}(\bar{\delta}_i)), \quad (43)
$$

$$
(\bar{\sigma}_t)^2 := \sum_{i=1}^n (w^i)^2 (\sigma_i)^2 + 2 \sum_{i=1}^n \sum_{j=i+1}^n w^i w^j \sigma_{ij}. \quad (44)
$$

Hence $\bar{\mu}_t$ and $(\bar{\sigma}_t)^2$ imply the rate of return for the portfolio and the risk of the portfolio respectively. We assume that $R_t$ has the normal distribution $N(\bar{\mu}_t, (\bar{\sigma}_t)^2)$. Then the corresponding VaR (value at risk) of the portfolio is given by $v = \bar{\mu}_t - 2.326 \bar{\sigma}_t$, where $v$ implies the price which guarantees the probability $P(R_t \geq v) = 0.99$.

Now we can calculate the lower bound of VaR by the necessity mean as follows

$$
v = \bar{\mu}_t - 2.326 \bar{\sigma}_t
$$

$$
= \mu_t + \sum_{i=1}^n w^i \int_0^1 g(\bar{\alpha}_i)w(\alpha)da / \int_0^1 w(\alpha)da - 2.326 \sigma_t
$$

$$
\geq \mu_t - 2 \sum_{i=1}^n w^i \lambda - 2.326 \sigma_t, \quad (45)
$$

where $\mu_t := \sum_{i=1}^n w^i \mu_i$, $\sigma_t := (\sum_{i=1}^n (w^i)^2 (\sigma_i)^2 + 2 \sum_{i=1}^n \sum_{j=i+1}^n w^i w^j \sigma_{ij})^{1/2}$, $\lambda \in [0, 1]$ and we take the necessity mean $w(\alpha) = (1 - \alpha)$. Let the rate of return $\mu^1 = 0.06, \mu^2 = 0.1, \mu^3 = 0.07$, the constants $c_i = 0.01, c_i^2 = 0.02, c_i^3 = 0.01$ and the variance-covariance $(\sigma_i)^2 = 0.40, (\sigma_i^2)^2 = 0.20, (\sigma_i^3)^2 = 0.30, \sigma_{ij} = 0.03, \sigma_{ij} = 0.02, \sigma_{ij} = -0.06$. Then we obtain the optimal trading strategy which maximizes $(44)$: $v = -0.138799$ at $w = (0.1503, 0.4826, 0.3669)$. It is a little different from the optimal trading strategy without fuzziness: $v = -0.128903$ at $(0.1484, 0.4861, 0.3653)$. Then we note that VaT contains fuzziness: $E(\bar{F}^N(v)) = 0.00494$ and $E(\bar{F}^P(v)) = 0.01482$.

V. Conclusions

In this paper, we have discussed the followings:

- We discussed two types of uncertainty, randomness and fuzziness. The mean evaluation method as a criterion of fuzziness was derived based on measure theory. The measurement of the fuzziness was also given with possibility/necessity measures.

- For fuzzy random variables, we evaluated fuzziness by the mean induced from evaluation measures and we estimated randomness by the expectation with $\lambda$-mean functions. Formulas regarding the variances, the co-variances and the correlations was discussed for fuzzy random variables.

- As an application example in decision making, we discussed a portfolio in financial engineering.

- The results represented in this paper are applicable to fuzzy stochastic process defined by fuzzy numbers/fuzzy random variables in decision making.

References


