Receding Horizon Nash Approach to Formation Control

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Abstract—This paper presents a receding horizon Nash game approach to formation control of mobile robots. The formation control is formulated as a linear-quadratic Nash differential game through the use of graph theory. Finite horizon cost function is discussed under the open-loop information structure. An open-loop Nash equilibrium is investigated by the solutions of coupled (asymmetrical) Riccati differential equations. Based on finite horizon open-loop Nash equilibrium solution, a receding horizon approach is adopted to synthesize a state-feedback controller for the formation control. Mobile robots with double integrator dynamics are used in the formation control simulation. Simulation results are provided to justify the models and solutions.

I. INTRODUCTION

Formation control has been studied in robotics within different structures, such as behavior-based structure, leader-follower structure, and virtual leader structure. The formation control in the behavior-based structure is achieved by building up a group of formation related behaviors [1]. It is suitable for uncertain environments, but lack of a rigorous theoretic analysis. The leader-follower structure is constructed by a string of chain where each robot follows one or two robots [2]–[5]. The chain structure leads to a poor disturbance rejection property. The virtual leader structure constitutes fictitious leaders that represent the desired robot positions. Using receding horizon approaches for the virtual leader structure has been investigated, such as contractive stability constraint [6] and consensus strategy [7].

Currently, the formation control uses a common team objective as a formation objective, i.e., all the individual robot’s interests are added together to form an identical team objective [7]. However, individual robots in a team may have different interests. Simply adding all individual interests together using even weights without considering the interaction dynamics among robots does not guarantee full exploitation of individual interests. Exploring such dynamics will lead to a more reasonable formation control. Game theory is an effective tool to model such dynamics existed among multiple robots. The formation control can be modeled as a non-cooperative game where the self-enforcing Nash equilibrium can be used as the formation control strategy. Robots can adopt this mechanism to establish their strategies to interact with other team members during the process of formation keeping.

Mobile robots with double integrator dynamics can be modeled as a controllable linear system. Formation control cost functions can be casted as a linear quadratic form by using graph theory. Therefore, the formation control of mobile robots with double integrator dynamics can be modeled as a linear-quadratic Nash differential game. Under the framework of this game, the formation control problem is converted to the coupled (asymmetric) Riccati differential equation problem.

The type of coupling between coupled Riccati differential equations depends on the information structure in a game. In the practical control, the state-feedback control is particular demanding. The open-loop information structure is based on the assumption that the only information players have is their present states and the model structure. It can be interpreted as such the players simultaneously determine their strategies at the beginning of the game and using this open-loop solution for the whole period of the game [8] [9]. To synthesis a state-feedback controller under the open-loop information structure, the finite horizon open-loop Nash equilibrium can be combined with a receding horizon approach to produce a resultant receding horizon Nash control. The use of receding horizon control in differential zero-sum games has been reported in [10] [11] [12]. It works in such a way; at each step, a state is read and the first control signal in the control profile generated from the open-loop Nash equilibrium is used to control robots. At the next step, this procedure repeat again.

In the following, the formation model is formulated in section II. The finite horizon open-loop Nash equilibrium is discussed in section III. The state-feedback Nash control strategy is presented in section IV. The simulation results are provided in section V. Finally, conclusions and future work are given in section VI.

II. FORMATION MODEL

A. Robot Dynamics

A team has \( m \) robots, each of which is described by its dynamics. For robot \( i \) with \( n \) dimensional coordinates \( q_i \in \mathbb{R}^n \), the state and control vectors are \( z_i(t) = [q_i(t)^T, \dot{q}_i(t)^T]^T \in \mathbb{R}^{2n} \) and \( u_i(t) \in \mathbb{R}^n \) \( (i = 1, \ldots, m) \). The robot dynamics is:

\[
\dot{z}_i = az_i + bu_i
\]  

where \( a = \begin{bmatrix} 0 & I_{(n)} \\ 0 & 0 \end{bmatrix} \), \( b = \begin{bmatrix} 0 \\ I_{(n)} \end{bmatrix} \). The matrix \( I_{(n)} \) is the identity matrix of dimension \( n \). The state and control vectors of each robot \( i \) are defined as follows:

\[
u_i(t) \in U, z_i(t) \in \mathbb{Z}
\]
By concatenating the states of all \( m \) robots in a team into a vector \( z = [z_1^T, ..., z_m^T]^T \in \mathbb{R}^{2nm} \), the team dynamics is:

\[
\dot{z}(t) = Az(t) + \sum_{i=1}^{m} B_i u_i(t), \quad t \geq 0
\]  

(3)

where \( A = I_{(m)} \otimes a \) and \( B_i = [0, ..., 1, ..., 0]^T \otimes b \). The operator \( \otimes \) is the Kronecker product.

Let \( z_i^d = [(q_i^d)^T, (q_i^d)^T]^T \) be the desired state vector for robot \( i \). The desired state vector is then represented as \( z^d = [(z_1^d)^T, ..., (z_m^d)^T]^T \in \mathbb{R}^{2nm} \). The desired state \( z_i^d \) should also have the same dynamics with the robot dynamics (1):

\[
\dot{z}^d = az^d + bu_i^d
\]  

(4)

And the concatenating state equation is:

\[
\dot{z}^d(t) = Az^d(t) + \sum_{i=1}^{m} B_i u_i^d(t), \quad t \geq 0
\]  

(5)

### B. Formation Graph

A graph can be used to represent the formation control interconnection between robots. A vertex of the graph corresponds to a robot and the edges of the graph capture the dependence of the interconnections. Formally, a directed graph \( G = (V, E) \) consists of a set of vertices \( V = \{v_1, ..., v_m\} \), indexed by the robots in a team, and a set of edges \( E = \{(v_i, v_j) \in V \times V\} \), containing ordered pairs of distinct vertices. Assuming the graph has no loops, i.e. \( (v_i, v_j) \in E \) implies \( v_i \neq v_j \). A graph is connected if for any vertices \( (v_i, v_j) \in V \), there exists a path of edges in \( E \) from \( v_i \) to \( v_j \). An edge-weighted graph is a graph in which each edge has been assigned a weight. The edge \( (v_i, v_j) \) is associated with weight \( \omega_{ij} \geq 0 \). To control a team to keep a formation, the graph connectivity is necessary.

The incidence matrix \( D \) of a directed graph \( G \) is the \( \{0, \pm 1\} \)-matrix with rows and columns indexed by the vertices of \( V \) and edges of \( E \), respectively, such that the \( u \)th entry of \( D \) is equal to 1 if the vertex \( u \) is the head of the edge \( v \), -1 if the vertex \( u \) is the tail of the edge \( v \), and 0 otherwise. If graph \( G \) has \( m \) vertices and \( |E| \) edges, then incidence matrix \( D \) of the graph \( G \) has order \( m \times |E| \) [13].

The cohesion and separation of formation control is defined by the desired distant vector \( d_{ij}^d = z_i^d - z_j^d \) between two neighbors \( v_i \) and \( v_j \). The formation error vector is defined as \( z_j - z_j - d_{ij}^d \) for the edge \( (v_i, v_j) \). Let \( \hat{D} = D \otimes I(2m) \). From the definition of the incidence matrix, we know the whole formation error can be expressed in a matrix form:

\[
\sum_{(i,j) \in E} \omega_{ij} \parallel z_i - z_j - d_{ij}^d \parallel = (z - z^d)^T \hat{D} W \hat{D}^T (z - z^d)
\]  

(6)

where \( \hat{W} = W \otimes I(2m) \) and \( W = diag(\omega_{ij}) \) is a diagonal weight matrix with dimension \(|E|\). We use \( \parallel z - z^d \parallel^2_{\hat{D} W \hat{D}} \) for \((z - z^d)^T \hat{D} W \hat{D} (z - z^d)\). Following [13], we define the Laplacian of a graph \( G \) as:

\[
L = DWD^T
\]  

(7)

For a directed graph \( G \), the Laplacian is independent of the choice of graph orientation. The Laplacian \( L \) is symmetric and positive semi-definite.

For the real value matrices \( X, Y, U, V \) with appropriate dimensions, the Kronecker product has the following properties:

\[
(X \otimes Y)^T = (X^T \otimes Y^T)
\]

\[
(X \otimes Y)(U \otimes V) = (XU) \otimes (YV)
\]

Based on these properties, we have:

\[
\hat{L} = \hat{D} W \hat{D}^T = (D \otimes I(2m))(W \otimes I(2m))(D \otimes I(2m))^T = L \otimes I(2n)
\]  

(8)

\( \hat{L} \) is also symmetric and positive semi-definite. The team formation error is rewritten as follows:

\[
\sum_{(i,j) \in E} w_{ij} \parallel z_i - z_j - d_{ij}^d \parallel = \parallel z - z^d \parallel_L^2
\]  

(9)

### C. Formation Cost Functions

The finite horizon cost function of the formation control for robot \( i \) can be expressed as follows:

\[
J^i_{FH}(u) = g^i(T, z(T)) + \int_0^T C^i(\tau, z(\tau), u(\tau)) d\tau
\]  

(10)

\[
g^i(T, z(T)) = \sum_{(i,j) \in E} \omega_{ij} \parallel z_i(T) - z_j(T) - d_{ij}^d \parallel^2
\]

\[
C^i(\tau, z(\tau), u(\tau)) = \sum_{(i,j) \in E} \mu_{ij} \parallel z_i(\tau) - z_j(\tau) - d_{ij}^d \parallel^2
\]

\[
+ \sum_{(i,j) \in E} \parallel u_{ij}(\tau) \parallel_{R_{ij}}^2
\]

(11)

(12)

where \( T \) is the finite time horizon, and \( \mu_{ij} \geq 0, R_{ij} > 0, (i = 1, ..., m) \) are the weighting parameters. The cost function (10) can be transformed into a standard linear-quadratic form:

\[
g^i(T, z(T)) = \parallel z(T) - z^d(T) \parallel_{K_{ij}}^2
\]

\[
C^i(\tau, z(\tau), u(\tau)) = \parallel z(\tau) - z^d(\tau) \parallel_{Q_{ij}}^2
\]

\[
+ \sum_{(i,j) \in E} \parallel u_{ij}(\tau) \parallel_{R_{ij}}^2
\]

(13)

where \( K_{ij} = \hat{L}_{ij} = \hat{D} \hat{W}_{ij} \hat{D}^T, \hat{W}_{ij} = W_{ij} \otimes I(2m) \), \( W_{ij} = diag(\omega_{ij}) \), \( Q_{ii} = \hat{L}_i = \hat{D} \hat{W}_i \hat{D}^T, \hat{W}_i = W_i \otimes I(2m) \), \( W_i = diag(\mu_{ij}) \). \( K_{ij} \) and \( Q_i \) are symmetric and positive semi-definite.

The formation cost functions are used to design controllers, which can control robots to have the desired distances \( d_{ij}^d \). To track a specific trajectory \( z_i^d \), at least the leader robot \( l \) should track \( z_i^d \). Thus, the cost function of the leader robot should
include a linear-quadratic standard tracking form:

\[
g'(T, z(T)) = \|z(T) - z^d(T)\|^2_{K_{tf}'} + \|z_l(T) - z^l(T)\|^2_{K_{lf}}
\]

\[
C'(\tau, z(\tau), u(\tau)) = \|z(\tau) - z^d(\tau)\|^2_{Q_l'} + \|z_l(\tau) - z^l(\tau)\|^2_{Q_l}
\]

where \( k_{lf} = \text{diag}[\omega_i], \) \( q_l = \text{diag}[\mu_i], \) \( K_{lf}' = K_{lf} + \text{diag}[0, \ldots, k_{lf}, \ldots, 0], \) and \( Q_l' = Q_l + \text{diag}[0, \ldots, q_l, \ldots, 0]. \) \( K_{lf} \) and \( Q_l \) are also symmetric and positive semi-definite. The leader robot can use \( K_{lf}' = 0 \) and \( Q_l' = 0, \) which means the leader robot only track the reference trajectory without taking the formation error into account. In such situation, it is the follower robot who keeps the formation by following the leader with a fixed distance.

In the following, the weighting matrices in the cost functions are denoted as \( K_{lf} \) and \( Q_l \) for both leader robots or follower robots. From the state equations (3)(5) and the cost function (10), it can be seen that the formation control is a linear-quadratic tracking problem. By using error state and control vectors, the formation control is viewed as a linear-quadratic regulate problem with \( z(t) \) as the state vector and \( u(t) \) as the control vector in the following presentation.

III. Finite Horizon Open-loop Nash Differential Games

A. Nash Differential Games

Each robot in a team can be viewed as a player or decision maker of a differential game. The robot dynamic equation (3) is the state equation of the differential game with the given initial state \( z_0 \) and rewritten here:

\[
\dot{z}(t) = Az(t) + \sum_{i=1}^{m} B_i u_i(t) \quad (15)
\]

\[
= Az(t) + Bu(t)
\]

\[
z(0) = z_0, \quad t \geq 0
\]

where \( B = [B_1, B_2, \ldots, B_m] \) and \( u = [u_1^T, u_2^T, \ldots, u_m^T]^T. \)

The cost function \( J^i \) defined in (10) in the last section is known to each player. The players in the game need to minimize their cost functions in order to find their strategies, or it can be said that the robots in the team need to minimize their cost functions in order to find their controllers. If all players have the same cost function, it is a team game. In a team game, the differential game problem is mapped down to an optimal control problem. If the players have different cost functions, the optimal control theory used for a single cost function can not be used to solve the problem. Instead the Nash equilibria have to be found. A Nash equilibrium is a collection of strategies for all players in a group and is a best response strategy of each player to the other players’ strategies. No one in the group can gain higher benefits by changing its strategy while the other players keep their strategies fixed or stationary. A collection of strategies \( \bar{u}_i(t), (t \geq 0, i = 1, \ldots, m) \) constitutes a Nash equilibrium if and only if the following inequalities are satisfied for all \( u_i(t) \in U, (t \geq 0, i = 1, \ldots, m): \)

\[
J^i(\bar{u}_1, \ldots, \bar{u}_{i-1}, u_i, \bar{u}_{i+1}, \ldots, \bar{u}_m) \leq J^i(\bar{u}_1, \ldots, \bar{u}_{i-1}, u_i, \bar{u}_{i+1}, \ldots, \bar{u}_m), \quad (i = 1, \ldots, m)
\]

The cost function can be the finite horizon cost function \( J_{FH}^i \) or infinite horizon cost function \( J_{IH}^i. \)

In the open-loop information structure, all players make their decisions based on the initial state \( z(0) \). Each player computes its equilibrium strategy at the beginning of the game and no state feedback is available during the whole control period. To derive a state-feedback controller for practical uses in formation control, the open-loop Nash equilibrium solution can be combined with a receding horizon approach to synthesis a state-feedback controller: receding horizon Nash control. In this receding horizon Nash control, each robot computes its open-loop Nash equilibrium at each time instant, but is not committed to follow that equilibrium during the whole period. It only uses it to control one step. In the next step, this procedure is repeated again. This receding horizon approaches will be discussed in the section IV.

B. Linear Quadratic Open-loop Nash Equilibria

Under the open-loop information structure of a Nash game, the derivation of open-loop Nash equilibria is closely related to the problem of jointly solving \( m \) optimal control problem (see page 267 in [14]). According to the minimum principle, the necessary conditions for an open-loop Nash equilibrium \( \bar{u}_i(t), (t \geq 0, i = 1, \ldots, m) \) for the game defined by (15) and (16) are given as follows:

\[
\ddot{z}(t) = -R_i^{-1} B_i^T p_i(t), \quad i = 1, \ldots, m \quad (17)
\]

where the costate vectors \( p_i(t) \) must satisfy

\[
\dot{p}_i(t) = -Q_i z(t) - A_i^T p_i(t) \quad (18)
\]

\[
p_i(T) = K_{if} z(T) \quad i = 1, \ldots, m
\]

and the state \( z \) is a solution of:

\[
\dot{z}(t) = Az(t) - \sum_{i=1}^{m} B_i R_i^{-1} B_i^T p_i(t) \quad (19)
\]

\[
= Az(t) - \sum_{i=1}^{m} S_i p_i(t)
\]

\[
z(0) = z_0
\]

with \( S_i = B_i R_i^{-1} B_i^T. \) Due to the linearity of the aforementioned equations, there exists a linear relation between the state and the costate vectors through a square matrix \( K_i (i = 1, \ldots, m). \) The costate is linearly related with the state vector, i.e.,

\[
p_i(t) = K_i(t) z(t), \quad i = 1, \ldots, m \quad (20)
\]

The open-loop Nash equilibrium is then determined as:

\[
\bar{u}_i(t) = -R_i^{-1} B_i^T K_i(t) \Phi(t, 0) z(0) \quad (21)
\]
where $K_i(t)$ must satisfy the set of coupled Riccati differential equations:

\[
\begin{align*}
\dot{K}_i &= -A^T K_i - K_i A - Q_i + K_i \sum_{j=1}^{m} S_j K_j \quad (22) \\
K_i(T) &= K_{if}
\end{align*}
\]

$\Phi$ is the associated transition matrix, verifying:

\[
\begin{align*}
\dot{\Phi}(t,0) &= (A - \sum_{i=1}^{m} S_i K_i(t)) \Phi(t,0) \\
\Phi(0,0) &= I
\end{align*}
\]

where $A_{cl}(t) = A - \sum_{i=1}^{m} S_i K_i(t)$ is the closed-loop system matrix. It is easily verified that $z(t) = \Phi(t,0)z_0$. The closed-loop system is:

\[
\dot{z}(t) = A_{cl}(t)z(t); t \geq 0 \quad (24)
\]

The following result can be obtained directly following the above derivation: for a $m$-robot formation control defined as a finite horizon open-loop Nash differential game by (15) and (16), let there exist a solution set $(K_i, i = 1, ..., m)$ to the coupled Riccati differential equations (22). Then the formation control admits an open-loop Nash equilibrium solution given by (21) and (23).

If the open-loop Nash differential game (15)(16) admits a solution, there exists the matrix $K_i(t)$. Let $\tilde{p}_i = p_i(t) - K_i(t)z(t), (i = 1, ..., m)$. Differentiating $\tilde{p}_i$ and using the equations (22)(18):

\[
\begin{align*}
\dot{\tilde{p}}_i &= \dot{p}_i(t) - \dot{K}_i(t)z(t) - K_i(t)\dot{z}(t) \\
&= -A^T p_i(t) + A^T K_i(t)z(t) \\
&\quad -K_i(t) \sum_{j=1}^{m} S_j K_j(t)z(t) + K_i(t) \sum_{j=1}^{m} S_j p_j(t) \\
&= -A^T (p_i(t) - K_i(t)z(t)) \\
&\quad -K_i(t) \sum_{j=1}^{m} S_j (K_j(t)z(t) - p_j(t)) \\
&= -A^T \tilde{p}_i(t) + K_i(t) \sum_{j=1}^{m} S_j \tilde{p}_j(t)
\end{align*}
\]

Let $\tilde{P}(t) = [\tilde{p}_1(t)^T, ..., \tilde{p}_m(t)^T]^T$, we have:

\[
\dot{\tilde{P}}(t) = \Psi \tilde{P}(t) \quad (25)
\]

where

\[
\Psi = \begin{bmatrix}
-A^T + K_1 S_1 & K_1 S_2 & \cdots & K_1 S_m \\
K_2 S_1 & -A^T + K_2 S_2 & \cdots & K_2 S_m \\
\vdots & \vdots & & \vdots \\
K_m S_1 & K_m S_2 & \cdots & -A^T + K_m S_m
\end{bmatrix}
\]

If $\tilde{P}(T) = 0$, then $\tilde{P}(t) = 0$. So $p_i(t) = K_i(t)z(t)$. Therefore the solution is unique.

### C. Coupled Riccati differential equations

The solvability of the coupled Riccati differential equations (22) is vital to the finite horizon Nash equilibrium solution. In the following, a necessary and sufficient condition is established for the solvability of the coupled Riccati differential equations.

Based on (18) and (19), the formation control admits an open-loop Nash equilibrium solution if and only if the differential equation

\[
\begin{align*}
\frac{d}{dt} &\begin{bmatrix}
z(t) \\
p_1(t) \\
\vdots \\
p_m(t)
\end{bmatrix} \\
&= M \begin{bmatrix}
z(t) \\
p_1(t) \\
\vdots \\
p_m(t)
\end{bmatrix}
\end{align*}
\]

with boundary conditions $z(0) = z_0, p_i(T) = K_{if}z(T)$, has a unique solution where

\[
M = \begin{bmatrix}
A & -S_1 & \cdots & -S_m \\
-Q_1 & -A^T & 0 & 0 \\
\vdots & \vdots & & \vdots \\
-Q_m & 0 & 0 & -A^T
\end{bmatrix}
\]

Let $y(t) = [z(t)^T, p_1(t)^T, \ldots, p_m(t)^T]^T$ and

\[
H(T) = \begin{bmatrix}
I_{(2nm)} & \cdots & 0 \\
0 & \cdots & I_{(2nm)}
\end{bmatrix} e^{-MT} \begin{bmatrix}
I_{(2nm)} \\
K_{mf} \\
\vdots \\
K_{mf}
\end{bmatrix}
\]

It follows from the two-point boundary-value problem (27) and matrix $H(T)$ (see the proof of theorem 7.1 in [9]) that

\[
y(0) = e^{-MT} \begin{bmatrix}
I_{(2nm)} \\
K_{mf} \\
\vdots \\
K_{mf}
\end{bmatrix} H^{-1}(T)z(0) \quad (28)
\]

So the analytic solution of the closed-loop system is

\[
\begin{align*}
z(t) &= \begin{bmatrix}
I_{(2nm)} & \cdots & 0 \\
0 & \cdots & I_{(2nm)}
\end{bmatrix} e^{-MT} \begin{bmatrix}
I_{(2nm)} \\
K_{mf} \\
\vdots \\
K_{mf}
\end{bmatrix} H^{-1}(T)z(0) \\
\end{align*}
\]

According to matrix $H(T)$, the theorem 7.1 in [9] provides an approach to judge if the solution exists for two-player games. It can be easily extended to $m$-player games. Based on this theorem with $m$ players, the formation control problem has the following result: for a $m$-robot finite horizon formation control
defined as a finite horizon open-loop Nash differential game by (15) and (16), the coupled Riccati differential equations (22) has a solution for every initial state \( z_0 \) on \([0, T]\) if and only if matrix \( H(T) \) is invertible.

The matrix \( M \) consists of \((m+1) \times (m+1)\) blocks. \( e^{-(MT)} \) also has the same block structure. Denote by \( W_{ij}(T) \) as the \( ij \)th block of \( e^{-(MT)} \), we have \( H(T) = W_{ij}(T) \). The invertibility of \( H(T) \) depends on \( M \) and \( T \). It has been shown in [14] [9] that different \( T \) leads to different invertibility of \( H(T) \). In the finite receding horizon Nash control discussed in the next section, \( T \) is the length of control horizon. The selection of \( T \) in the finite receding horizon control should guarantee \( H(T) \) is invertible.

IV. STATE-FEEDBACK FORMATION CONTROL

A. Receding Horizon Nash Control

Assuming the current time instant is \( t \) and the current state vector is \( z(t) \). At each time instant, the receding horizon control uses \( z(t) \) as the initial state to find the finite horizon open-loop Nash equilibrium \( \bar{u}(0) \) based on the following cost function:

\[
J_{FH}(t, z(t), u(t)) = g^i(t + T) + \int_t^{t+T} C^i(\tau, z(\tau), u(\tau))d\tau
\]

The receding horizon control signal is defined as:

\[
u^*_i(t, z(t)) = \bar{u}_i(0) = -R_{ii}B_i^T K_i(0) z(t)
\]

As the control signal \( u^*_i(t, z(t)) \) depends on the current state \( z(t) \), the receding horizon Nash control is a state-feedback control. The existence conditions of the receding horizon Nash control is the same as those of the finite horizon open-loop Nash control, i.e. the receding horizon Nash control exists for every initial state \( z_0 \) if and only if matrix \( H(T) \) is invertible.

The receding horizon Nash control needs to check whether or not the closed-loop system is stable. The closed-loop system with the receding horizon Nash control \( u^*_i(t, z(t)) \) (31) is

\[
\dot{z}(t) = (A - \sum_{i=1}^{m} S_i K_i(0)) z(t) + A_{cl}(0) z(t)
\]

where the closed-loop system matrix \( A_{cl}(0) = A - \sum_{i=1}^{m} S_i K_i(0) \). The following result can be made about the receding horizon Nash control:

The formation control defined as a finite horizon Nash differential game (15)(30) has a receding horizon Nash control for every initial state \( z_0 \) if and only if matrix \( H(T) \) is invertible. As long as all the eigenvalues of \( A_{cl}(0) \) have negative real parts, the receding horizon Nash control is asymptotic stable.

B. Receding Horizon Nash Control Algorithm

Let \( \delta \) denote the control time interval and \( 0 < \delta < T \). At each time instant \( t \), all robots share the current state \( z(t) \). Based on \( z(t) \), each robot computes an open-loop Nash equilibrium solution \( \bar{u}(\tau) \) for the period \( t < \tau < t + \delta \). To indicate that this solution is an open-loop Nash equilibrium solution and depends on the initial state \( z(t) \), it is rewritten as \( \bar{u}(\tau, z(t)) \). The algorithm uses this solution to control robots for the period \( [t, t + \delta] \). At the next time instant \( t + \delta \), this procedure repeats. The details of the algorithm are listed as follows:

1) read the current state \( z(t) \).
2) find the open loop Nash equilibrium solution \( \bar{u}_i(\tau, z(t)) \) and its state trajectory \( \bar{z}(\tau, z(t)) \).
3) construct the receding horizon Nash control \( u^*_i(\tau, z(t)) \) based on the open loop Nash equilibrium \( \bar{u}(\tau, z(t)) \) for the period \( [t, t + \delta] \) (31).
4) use the receding horizon Nash control \( u^*_i(\tau, z(t)) \) to control robots. The resulting state trajectory \( z^*(\tau, z(t)) \) should be.

\[
z^*(\tau, z(t)) = \bar{z}(\tau, z(t)), \ \tau \in [t, t + \delta]
\]

5) update \( t \leftarrow t + \delta \).
6) loop until the control achieves a satisfying performance.

V. SIMULATIONS

The dimension of the coordinate vectors for all robots is two \( (n = 2) \). The simulation tests a triangle formation shape with four robots \( (m = 4) \). The tracking trajectories are assumed to be a circle. Assume the node 1 (circle) is the leader who tracks the reference trajectories. The leader robot’s tracking weighting matrix is assumed to be \( W_{1f} = q_1 = 5I_{2n} \).

The leader robot can only use the tracking cost function, i.e. \( W_1 = W_{1f} = 0 \). Or the leader robot can take the formation cost into account. Its formation cost weighting matrix is selected as \( W_1 = W_{1f} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix} \) for the triangle formation.

Other three robots are the followers and their cost function weighting matrices are \( W_2 = W_{1f} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \). \( W_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \).

Solutions to the coupled Riccati differential equations of the finite horizon open-loop Nash differential games can be found by using terminal values and the backward iteration. Solutions to the coupled algebraic Riccati equations of the infinite horizon open-loop Nash differential games have to be solved by analytic approaches.

The receding horizon Nash control is used for the simulations. The finite horizon length \( T = 10s \) and sample time is \( \delta = 0.1s \). Through calculation, it can be seen that \( H(1) \) is invertible and all eigenvalues of \( A_{cl}(0) \) have negative real
parts for two formation shapes. Therefore the receding horizon Nash control is asymptotic stable.

The trajectories of the four robots are shown in figure 1 for the circle tracking. The leader robot 1 use both tracking cost function and formation error cost function. The results show that all four trajectories converge to a triangle shape. The $y$ position errors between the robots and their tracking trajectories of all four robots are shown in figures 2. It can be seen that the position errors converge to zero, i.e. all robots finally move in the circle trajectories.

The $y$ position errors when the leader robot 1 only uses the tracking cost function are shown in figures 3. All these errors have the convergent property. Due to the only use of the tracking cost function, the leader robot just tracks the circle trajectory without considering the formation error. It can be seen that the $y$ position error is kept as zero in figure 3. This is due to the initial $y$ position being the same as the tracking trajectory. The leader robot only adjusts its $x$ position error and catch up with the circle.

VI. CONCLUSIONS AND FUTURE WORK

This paper proposes to exploit the mechanism of non-cooperative games to handle the formation control problem. Through building Nash equilibria, all team members can find a self-enforcing controller, which can guarantee the formation control exists and is stable in terms of their individual cost functions. The state-feedback control can be synthesized through the infinite horizon Nash control or receding horizon Nash control. The stability of the state-feedback controllers is discussed.

In the next step, we will concentrate on the constrained control under the framework of the receding horizon Nash control because the constrained control can deal with the practical problems, such as saturation constraints of the control signal.

REFERENCES