Integral Control Approach to High-Gain Observer Design

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Abstract—This paper introduces an integral control approach to high-gain observer design, and its stability is analyzed based on Lyapunov theory. It is assumed that their states are unmeasurable. The proposed high-gain observer has the integrator of the estimation error in dynamics. It improves the stability of high-gain observers, which is robust to noisy measurements, uncertainties and peaking phenomenon as well. Its stability is analyzed by the Lyapunov approach. In order to verify the effectiveness of the proposed scheme, some graphical analysis is given.

Index Terms—integral control, high-gain, robust, stability, peaking phenomenon

I. INTRODUCTION

The technique, known as high-gain observer (HGO) is to design the observer gain that makes the observer robust against model uncertainties in nonlinear functions. Hence, it works for a wide class of nonlinear systems. Furthermore, the HGO scheme guarantees that the output feedback controller recovers the performance of the state feedback controller when the observer gain is sufficiently high [11,12]. However, high gains may excite hidden dynamics and amplify measurement noise: large oscillation in the transient response and sensitivity to measurement noise. Thus, they could not be applicable to practice.

In order to overcome such problems, several authors have successfully designed sliding-mode approach to construct observers that are highly robust with respect to noise in the input of the system. However, it turned out that the corresponding stability analysis should not be directly applied when output noise is presented. Therefore, it is still a challenge for the control system community to suggest a manageable technique to analyze the stability of identification error generated by sliding-mode type observers whose structure is obtained by differential-algebra techniques[13,14].

In this paper, an integral control-type HGO design scheme for nonlinear systems is proposed. It adopts the integrator of the error dynamics for improving the performance of HGO in transient response and the robustness against noisy measurements, uncertainties and peaking phenomenon. The integral control approach to HGO design has the robust stability against peaking phenomenon. It is verified by the Lyapunov based stability analysis. It is assumed that the states of nonlinear systems are unmeasurable. The effectiveness of the proposed scheme is illustrated by a graphical analysis, the phase portrait.

The rest of this paper is organized as follows. In section II, a problem is stated and the design method of the proposed observer is expressed in section III. Section IV introduces the stability analysis based on the Lyapunov theory. In section V, Some phase portrait shows the effectiveness of the proposed method. Finally, we make some conclusion in section VI.

II. PROBLEM STATEMENT

Let the state space representation of the system can be described as follows.

\[
\dot{x} = Ax + F(x,u) \quad (1)
\]
\[
y = Cx
\]
where, \( A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}, \quad F(x,u) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f(x,u) \end{bmatrix}
\]

\( C = [1 \ 0 \ \cdots \ 0 \ 0]. \)

where, \( x = [x_1 \ x_2 \ \cdots \ x_n]^T \in \mathbb{R}^n \) is the state vector of the system, which is assumed to be unmeasurable. \( F(x,u) \) is unknown but bounded continuous nonlinear system. \( u \in \mathbb{R} \) is a control input and \( y \in \mathbb{R} \) is an output of the system respectively. It is assumed that only its measurable and the system (1) is observable.

The goal is to secure the robust stability of HGO. In order to accomplish it, the integral-type structure is adopted to modify the dynamics of HGO and its stability analysis is derived based on Lyapunov theory.

III. OBSERVER DESIGN

In this section, an integral control approach to HGO design is introduced. The proposed method has an integral-type structure in dynamics and it gives robustness to the observer against noisy measurements, uncertainties and peaking.
phenomenon. The design process starts with the modified dynamics presentation of the proposed observer system. The following presentation is proposed.

**The observer system**

\[
\begin{align*}
\dot{x} &= A\dot{x} + F(\dot{x}, u) + L(y - \hat{y}) + M\sigma \\
\dot{y} &= C\dot{x}
\end{align*}
\]  

(2) 

\[
\begin{align*}
\sigma &= y - \hat{y}
\end{align*}
\]  

(3) 

\[
\begin{align*}
\hat{y} &= C\dot{x}
\end{align*}
\]  

(4)

where, \( L = E[L_1 \ L_2 \ \cdots \ L_n]^T \) is an observer gain vector. \( M \in \mathbb{R}^{n \times n} \) is an integral gain vector. \( \sigma \) is a new state describing the integral regulation error between the system output and the observer output. \( \sigma^* \) is an optimal-nominal value. \( \sigma = \sigma^* - \sigma \) and

\[
E = \begin{bmatrix} \frac{1}{\epsilon} & 0 & \cdots & 0 \\ 0 & \frac{1}{\epsilon^2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\epsilon^n} \end{bmatrix} \in \mathbb{R}^{n \times n}
\]

The observer gain can be every value that places all eigenvalues of \( \det[I - (A - LC)] = 0 \) in negative real part. However, The developed observer expressed as (2), (3) and (4) will estimate states of the original system fast enough when the observer gain is sufficiently high as a high-gain observer.

**IV. STABILITY ANALYSIS**

Stability analysis is performed by Lyapunov stability theory and Lipchitz condition. However, In order to use such popular technique, the following two remarks are needed.

**Remark 1.** \( \lambda_{\text{min}}(N) \) and \( \lambda_{\text{max}}(N) \) are the smallest and the largest eigenvalue of \( N \), then it follows from \( N = U^T \Lambda U \) that

\[
\lambda_{\text{min}}(N)\|x\| \leq x^T N x \leq \lambda_{\text{max}}(N)\|x\|
\]

where, \( N \) is a positive definite matrix, \( U^T U = I \) and \( \Lambda \) is a diagonal matrix containing the eigenvalues of the matrix \( N \).

**Remark 2.** According to Lyapunov equation [9], there exist \( P \) and \( Q \), which satisfy that

\[
A^T P + PA = -Q, \quad B^T P = C
\]

where, \( P \) and \( Q \) are symmetric positive definite matrices.

The observer error is defined as \( e := x - \hat{x} \). Then we get the error dynamics using (1), (2), (3) and (4). An error equation can be obtained as follows

\[
\begin{align*}
\dot{e} &= (A - LC)e - M\sigma + F(x, u) - F(\hat{x}, u) \\
\dot{\sigma} &= y - \hat{y}
\end{align*}
\]  

(5) 

(6)

Let \( z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \sigma \\ e \end{bmatrix} \) as a new state vector, then the dynamic equation (5) and (6) can take the form (7), which is the proposed observation error dynamics.

\[
\begin{align*}
\dot{z} &= \begin{bmatrix} \sigma \\ e \end{bmatrix} = \begin{bmatrix} 0 & C \\ -M & A - LC \end{bmatrix} \begin{bmatrix} \sigma \\ e \end{bmatrix} + B[F(x, u) - F(\hat{x}, u)] \\
\dot{z} &= \Gamma z + B g(z, u)
\end{align*}
\]  

(7)

where, \( \Gamma = \begin{bmatrix} 0 & C \\ -M & A - LC \end{bmatrix} \), \( B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \) and 

\[
g(z, u) = F(x, u) - F(\hat{x}, u)
\]

**Theorem 1.**

For (7), consider a Lyapunov candidate as

\[
V = z^T P z
\]  

(8)

where, \( V \) : a positive definite and radially unbounded function 
\( P \) : a symmetric positive definite matrix

If there is \( k_f, Q \) and \( P \) such that \( k_f \|B\| < \frac{\lambda_{\text{max}}(Q)}{2\lambda_{\text{min}}(P)} \) then, \( \dot{V} < 0 \) which means that \( V \) is negative definite. It guarantees the asymptotic stability of \( z \) for the equilibrium point \( z = 0 \), which means \( \sigma \) and \( e \) go to 0, and \( \hat{x} = x \), where \( x \) is an original state vector. \( V \) is called a Lyapunov function. Where, \( k_f \) is a Lipchitz constant.

**Proof.**

The differentiating \( V \) yields

\[
\dot{V} = 2z^T P z + z^T P z
\]  

(9)

By substituting (8) into (9) and under **Remark 1.** and **Remark 2.**, \( \dot{V} \) is expressed as

\[
\begin{align*}
\dot{V} &= [z^T \Gamma^T + g(z, u)^T B^T] P z + z^T P [\Gamma z + B g(z, u)] \\
&= z^T [\Gamma P + P \Gamma^T] z + 2z^T P B g(z, u) \\
&= -z^T Q z + 2z^T P B g(z, u) \leq -z^T Q z + 2z^T P B k_f \|z\|
\end{align*}
\]
\[ \leq -\lambda_\text{max}(Q)\|z\|^2 + 2k_i\|B\|\lambda_\text{max}(P)\|e\|^2 \]

and \(Q\) can be readily chosen, which satisfy

\[ k_i \|B\| < \frac{\lambda_\text{max}(Q)}{2\lambda_\text{max}(P)} \]

Hence,

\[ \dot{V} \leq -\lambda_\text{max}(Q)\|e\|^2 + 2k_i\|B\|\lambda_\text{max}(P)\|e\|^2 < 0 \]

According to Lyapunov theory, \(V\) guarantees the asymptotic stability of \(z\) for the equilibrium point \(z = 0\), which means \(\dot{\sigma}\) and \(e\) go to 0, and \(\dot{x} = x\), where \(x\) is an original state vector. \(V\) is called a Lyapunov function.

V. GRAPHICAL ANALYSIS

We use a phase portrait to graphically show the robust stability of the proposed method. This section presents the effectiveness of the proposed method. The simulation example borrowed from [7] is provided. Its goal is to design an output feedback controller based on the given observer. The integral control-type high-gain observer is compared with the existed high-gain observer.

The illustrative system

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_1^3 + u \\
y &= x_1
\end{align*}
\]

which can be globally stabilized by the state feedback controller;

\[ u = -x_1 - x_1 - x_2 \]

The existing HGO based output feedback controller (OFC) and the integral control-type HGO based output feedback controller are taken as

The general HGO based OFC system

\[
\begin{align*}
u &= -x_1 - x_1 - x_2 \\
\dot{x}_1 &= \dot{x}_2 + \frac{2}{\varepsilon}(y - \dot{x}_1) \\
\dot{x}_2 &= \frac{1}{\varepsilon}(y - \dot{x}_1)
\end{align*}
\]

The proposed HGO based OFC system

\[
\begin{align*}
u &= -\dot{x}_1 - x_1 - x_2 \\
\dot{x}_1 &= \dot{x}_2 + \frac{2}{\varepsilon}(y - \dot{x}_1) \\
\dot{x}_2 &= \frac{1}{\varepsilon}(y - \dot{x}_1)
\end{align*}
\]

As \(\varepsilon\) is reduced in figs. 1, 2 and 3, the behavior of the existed high-gain observer based system gets wild and it is likely to diverge. This is because the gap between initial values becomes large by the high gain. Whereas, the behavior of the integral control-type HGO based system is much stable comparing with the existed high-gain observer based system.
regardless of \( \varepsilon \), which means that the integral control-type HGO compensates the weak stability of the existed high gain observer.

![Fig. 3 Phase portrait at \( \varepsilon = 0.0063 \)]

Finally, the existed high-gain observer based system comes to diverge in fig. 4. However, the integral control-type based system still works well.

VI. CONCLUSION

In this paper, we proposed the robust high-gain observer based output feedback controller. Lyapunov theory based convergence analysis of the proposed observer was performed. It was assumed that states of nonlinear systems are unmeasurable. The proposed observer has an integral-type structure to be acceptable perturbations. It improved the robustness and overcomes the lack of the conventional HGO, peaking phenomenon. To verify the effectiveness of the proposed observer, it was applied to OFC, and compared with the conventional HGO and SFC. At last, comparative simulation results successfully confirmed the performance of the proposed scheme.

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