An Effective Potential in SU(2) Yang-Mills Theory and Stability of the Savvidy Vacuum

Naoko Komiya\(\diamond\) and Shinichi Deguchi\(\dagger\)

\(\diamond\)Graduate Course in Quantum Science and Technology, Nihon University
\(\dagger\)Institute of Quantum Science, Nihon University

An effective potential in SU(2) Yang-Mills (YM) theory has a minimum at the point where a constant magnetic field exists. This point is known as the Savvidy vacuum [1]. As has been pointed out in Ref. [2], the Savvidy vacuum is unstable in the sense that the vacuum energy density involves an imaginary part. This fact indicates that the Savvidy vacuum is a false vacuum. In the present work, we try to find a true vacuum proposing an idea to overcome the difficulty of arising the imaginary part. We conjecture that the instability of the Savvidy vacuum is caused by ignoring effects of the four-point interactions of YM fields. This conjecture is based on a preliminary consideration on the Goldstone model with the potential \(V_G(\phi) = \lambda (|\phi|^2 - v^2)^2\). If we evaluate the effective potential around the stationary point \(\phi = 0\), the resulting vacuum energy density involves an imaginary part. Ignoring the \(\phi^4\) term in \(V_G\), we can never find a true vacuum in the Goldstone model. Learning it, we arrive at the idea that effective potentials in YM theories should be calculated with including the four-point interactions.

Let \(A_\mu^a (a = 1, 2, 3)\) be the SU(2) YM fields. Introducing the notation \(W_\mu \equiv \frac{1}{\sqrt{2}}(A_\mu^1 + i A_\mu^2)\), \(A_\mu \equiv A_\mu^3\), and auxiliary fields \(B_{\mu
u} (-B_{\nu\mu})\), \(\Psi\) and \(\Phi\), we consider the Lagrangian

\[
\mathcal{L} = -\frac{1}{4g^2} f_{\mu\nu} f^{\mu\nu} + \frac{1}{g^2} W_\mu (D_\mu D^\alpha \delta^{\alpha\nu} - 2if^{\alpha\nu}) W^\nu + k \left\{ \frac{M_1}{g} B_{\mu\nu} W_\mu W^\nu + \frac{1}{4} M_2^2 B_{\mu\nu} B^{\mu\nu} \right\} \\
+ (1 - k) \left\{ \frac{M_2}{2g} (\Psi W^*_\mu W^\mu + \Psi^* W_\mu W^\mu) - \frac{1}{2} M_2^2 \Psi^* \Psi + \frac{M_3}{g} \Phi W^*_\mu W^\mu + \frac{1}{2} M_3^2 \Phi^2 \right\} + \mathcal{L}_E,
\]

with \(D_\mu \equiv \partial_\mu - i A_\mu\) and \(f_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu\). Here \(g\), \(M_1\), \(M_2\), \(M_3\) and \(k\) are constants, and \(\mathcal{L}_E\) denotes a suitable ghost term. It is easy to see that \(\mathcal{L}\) is equivalent to the ordinary YM Lagrangian with the gauge condition \(D^\mu W_\mu = 0\). From \(\mathcal{L}\) we define the effective action

\(S = -i \ln \int D^M D^W D^B D^M E^i \exp i \int d^4x [d^2x \mathcal{L}]\),

where \(D^M, D^W, D^B\) stands for the path-integral measures of relevant ghost and antighost fields. We now focus attention on the case in which a constant magnetic field \(H\) is in the \(x^3\) direction. This condition is realized by taking \(A_2 = H x^1\), \(A_0 = A_1 = A_3 = 0\). After carrying out the integrations in \(S\), we can exactly calculate the effective potential \(V_{\text{eff}}\), defined by \(-V_{\text{eff}} \int d^4x = S\), for the configuration in which \(B \equiv B_{12}\), \(\Psi^\prime = \Psi - 2i \tilde{C} C / g M_2 (1 - k)\) and \(\Phi^\prime = \Phi + i (\tilde{C}^* C + C^* \tilde{C}) / g M_3 (1 - k)\) are taken to be constants, while \(B_{\mu\nu} = 0\) for \((\mu, \nu) \neq (1, 2)\). Here \((C, C^*)\) and \((\tilde{C}, \tilde{C}^*)\) are, respectively, the ghost fields and the antighost fields associated with \((W_\mu, W^*_\mu)\). (In deriving \(V_{\text{eff}}\), we have taken into account the path-integrals over \(B_{\mu\nu}, \Psi, \Psi^*\) and \(\Phi\).) Unitarity in the theory is guaranteed if and only if \(k = 2\). As a result of the calculation, we obtain \(V_{\text{eff}}\) written in terms of \(H, B, |\Psi^\prime|, \Phi^\prime\) and \(\Phi^\prime\).

If the condition \(B = \Psi^\prime = \Phi^\prime = 0\) is imposed, \(V_{\text{eff}}\) reduces to the earlier effective potential involving an imaginary part [2]. This result is only natural, because the above condition implies that the sum of the four-point interaction terms of YM fields and several terms in \(\mathcal{L}_E\) is ignored. When \(B, |\Psi^\prime|\) and \(\Phi^\prime\) are finite values belonging to a certain region of the \(B|\Psi^\prime|\Phi^\prime\)-space, \(V_{\text{eff}}\) involves no imaginary parts, while maintaining the asymptotic freedom of SU(2) Yang-Mills theory. Therefore we may find a true vacuum with desirable properties. To find such a vacuum, we have to search for the absolute minimum of \(V_{\text{eff}}\). This will be done in a systematic way.

References