Enhanced binding for the generalized semi-relativistic quantum field model.

Toshimitsu TAKAESU
(Kyushu University)

1 Introduction

In this article, a system of semi-relativistic particles interacting with a massive Bose field is analyzed. The dynamics of the semi-relativistic particles is determined by the free relativistic Schrödinger operator. We consider the total Hamiltonian with a generalized interaction. By imposing generalized ultraviolet cutoff conditions on the Bose field, it is seen that the total Hamiltonian of the system is a self-adjoint operator on a Hilbert space. We are interested in spectral properties of the total Hamiltonian.

In the last decade, the spectral analysis for the quantum particles system interacting with Bose fields has been successfully analyzed. In [6], the detailed survey on the recent progress in this subject is given. The main problems of spectral analysis for quantum field models can be classified into (i) ground states, (ii) resonances, (iii) scattering theory, (iv) non-relativistic limits, scaling limits, and so on. In this article, we are mainly interested in analysis of the phenomena, called enhanced binding, which is related to the ground states problems. For a self-adjoint operator $X$ with bounded from below, it is said that $X$ has the ground state if the bottom of the spectrum of $X$ is the eigenvalue of $X$. For quantum field system, it is said that there is the enhanced binding if the total Hamiltonian has the ground state even if the free Hamiltonian does not have the ground state. In this article, it is proven that under some assumptions including a binding condition, there is the enhanced binding in the interacting system. Historically, analysis of the enhanced binding for the quantum field model is initiated by Hiroshima-Spohn [7]. They analyze the non-relativistic QED model, and this model has also been analyzed in [2, 3, 4, 5]. Arai-Kawano [1] analyze the enhanced binding for the generalized quantum field model, and Hiroshima-Sasaki [8] investigate the Nelson model. Recently, the absence of the enhanced binding of the non-relativistic QED model has been investigated in [9].
2 Main Result

Let us define the state space and the total Hamiltonian for the semi-relativistic particles system interacting with a massive Bose field. The state space is given by \( \mathcal{H} = L^2(\mathbb{R}^d_N) \otimes \mathcal{F}_b(L^2(\mathbb{R}^d_k)) \) where \( \mathcal{F}_b(\mathcal{K}) \) denotes the boson Fock space over the Hilbert space \( \mathcal{K} \). We use a natural identification \( \mathcal{H} \simeq \int_{\mathbb{R}^d_N} \mathcal{F}_b(L^2(\mathbb{R}^d_k)) dx_1 \cdots dx_N \) where \( \int_X d\mu \) denotes the fibre direct integral with base space \( (X, d\mu) \). The total Hamiltonian is given by

\[
H(\kappa) = H_p \otimes I + I \otimes H_b + \kappa H_I.
\]

Here \( H_p = \sum_{j=1}^N \left( \sqrt{-\Delta_j + M^2} - M \right) \) is the relativistic Schrödinger operator with the rest mass \( M > 0 \) and \( H_b = d\Gamma_b(\omega) \) is the second quantization of \( \omega \in C(\mathbb{R}^d) \) satisfying \( \inf \omega(k) > 0 \). To assume the condition \( \inf \omega(k) > 0 \) is called the massive condition. The interaction \( H_I \) is defined by \( H_I = \sum_{j=1}^N \int_{\mathbb{R}^d_N} \phi(u_x) dx_1 \cdots dx_N \) where \( \phi(\xi) = \frac{1}{\sqrt{2}} (a(\xi) + a^*(\xi)) \) denotes the field operator for \( \xi \in L^2(\mathbb{R}^d_k) \), and \( u_x \) is a multiplication operator on \( L^2(\mathbb{R}^d_k) \) satisfying the following condition:

(A.1) (Generalized ultraviolet-cutoff condition)

\[
\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} |u_x(k)|^2 dk < \infty.
\]

Let us set \( H_0 = H_p \otimes I + I \otimes H_b \). Under the condition (A.1), the interaction \( H_I \) is relatively bounded to \( I \otimes H_b^{1/2} \). Then, \( H_I \) is relatively bounded to \( H_0 \) with infinitely small bond. Hence, from the Kato-Rellich theorem, \( H(\kappa) \) is self-adjoint on \( D(H_0) = D(H_p \otimes I) \cap D(I \otimes H_b) \) and essentially self-adjoint on any core of \( H_0 \).

Our main interest is the ground state of the total Hamiltonian \( H(\kappa) \). Here let us prepare for some notations. For an operator \( X \), \( \sigma(X) \) denotes the spectrum of \( X \) and set \( E_0(X) = \inf \sigma(X) \). \( \sigma_p(X) \) and \( \sigma_{ess}(X) \) denote the point spectrum and essential spectrum of \( X \), respectively. It is noted that, the ground state of \( H_0 \) does not exist, i.e. \( E_0(H_0) \notin \sigma_p(H_0) \), since any external potentials are not turned on \( H_p \).
To derive an potential of the particles from the interaction between the particles and the field, let us use the unitary transformation, called dressing transformation defined by

$$U(\kappa) = \exp \left( i\kappa \sum_{j=1}^{N} \int_{\mathbb{R}^d} \pi \left( \frac{u_{x_j}}{\omega} \right) dx_1 \cdots dx_N \right),$$

where $\pi(\xi) = \frac{i}{\sqrt{2}} (-a(\xi) + a^*(\xi))$ is the conjugate operator for $\xi \in L^2(\mathbb{R}^d)$. We introduce the following condition.

$$\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\nabla u_{x}(k)|^2 dk < \infty, \quad \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\Delta u_{x}(k)|^2 dk < \infty,$$

\((\nabla_{f_x}, f_y) \in \mathbb{R}, \quad x, y \in \mathbb{R}^d,\)

Then from the canonical commutation relations, we have

$$U(\kappa)^{-1}H(\kappa)U(\kappa) = (H_\rho + \kappa^2 V_{\text{eff}}) \otimes I + I \otimes H_\rho + \delta H_\rho(\kappa), \quad (1)$$

where

$$V_{\text{eff}} = -\frac{1}{2} \sum_{j,l=1}^{N} \int_{\mathbb{R}^d} \frac{u_{x_j}(k)u_{x_l}(k)}{\omega(k)} dk, \quad (2)$$

$$\delta H_\rho(\kappa) = U(\kappa)^{-1} (H_\rho \otimes I) U(\kappa) - H_\rho \otimes I. \quad (3)$$

It is proven that $\delta H_\rho(\kappa)$ is relatively bounded with respect to $H_0$ in a similar way to ( [10], Proposition 3.1 ).

Let us introduce the following assumptions :

\(\textbf{(A.3)}\) There exist constant $d > 0$ and $\tau > 0$ such that

$$\sup_{x \in \mathbb{R}^d} |u_x(k) - u_x(k')| \leq c|k - k'|^\tau.$$

\(\textbf{(A.4) (Binding Condition)}\)

There exists constant $\kappa_s > 0$ such that for $0 < \kappa < \kappa_s$,

$$\inf \sigma_{\text{ess}} \left( H_\rho + \kappa^2 V_{\text{eff}} \right) \geq E_0 \left( H_\rho + \kappa^2 V_{\text{eff}} \right) + v(\kappa),$$

where $v(\kappa) > 0$ is positive real number.
Theorem (Ground states in massive condition) Assume (A.1) - (A.4). Then $H(\kappa)$ has purely discrete spectrum in $[E_0(H(\kappa)), E_0(H(\kappa)) + m)$ for sufficiently small $\kappa > 0$. In particular the ground state of $H(\kappa)$ exist for sufficiently small $\kappa > 0$.

(Outline of the Proof) Let us set $H^U(\kappa) = U(\kappa)^{-1}H(\kappa)U(\kappa)$. By applying the methods of the finite volume approximation (refer to e.g. [1, 7]), we can construct the Hamiltonians $H_{\Lambda, \varepsilon}^U(\kappa)$ and $H_{\Lambda}^U(\kappa)$ such that (1) $H_{\Lambda, \varepsilon}^U(\kappa)$ has purely discrete spectrum in $E_0(H_{\Lambda, \varepsilon}^U(\kappa)), E_0(H_{\Lambda, \varepsilon}^U(\kappa)) + m)$ for sufficiently small $\kappa > 0$, (2) $H_{\Lambda, \varepsilon}^U(\kappa)$ and $H_{\Lambda}^U(\kappa)$ converges to $H_{\Lambda}^U(\kappa)$ and $H^U(\kappa)$, respectively, in norm resolvent sense. Then the proof is obtained.

References


