Bicubic Interpolation with Spectral Derivatives

Takeshi Enomoto

The Earth Simulator Center, JAMSTEC, Yokohama, Japan

Abstract

A simple and accurate interpolation method applicable to semi-Lagrangian advection in a spectral global atmospheric model and downscaling is presented. The derivatives required for bicubic interpolation are usually represented by finite differences. Accuracy of bicubic interpolation is found to be improved by using derivatives calculated by the spectral method. Thus, the zonal and meridional derivatives are obtained by the Fourier and Legendre transforms, respectively. The proposed method is validated with the Gaussian hill rotation tests. The semi-Lagrangian advection model with this method produces a minimal error, comparable to that of the non-interpolating semi-Lagrangian model. In order to avoid the computationally expensive Legendre transforms, semi-spectral interpolation methods using only zonal spectral derivatives are also tested. Semi-spectral interpolation is found to be as accurate as full-spectral bicubic interpolation when quintic interpolation is used in the meridional direction.

1. Introduction

The semi-Lagrangian advection is widely adopted in many forecast and climate models (Staniforth and Coté 1991). It is usually used with the semi-implicit scheme to reduce computational cost (Robert 1982). The semi-Lagrangian and semi-implicit schemes enable a longer integration. Thus Eulerian treatment of residual flows is treated by the Eulerian scheme. In this scheme, the value at the grid point closest to the departure point is used and the advection by residual flows is treated by the Eulerian method. This method reduces damping associated with the semi-Lagrangian method. This method is often applied to the vertical advection in primitive equation models.

The method proposed in this paper achieves accuracy comparable to the non-interpolating semi-Lagrangian scheme in the interpolating semi-Lagrangian scheme. Thus Eulerian treatment of residual terms is not required.

2. Bilinear and bicubic interpolations

In this section, two dimensional linear and cubic (bilinear and bicubic) interpolations are reviewed and the new method is proposed. A value \( f(\lambda, \theta) \), where \( \lambda \) and \( \theta \) are the longitude and latitude, is obtained from the values at the surrounding four points, \( f_1 = f(\lambda_1, \theta_1) \), \( f_2 = f(\lambda_1, \theta_0) \), \( f_3 = f(\lambda_0, \theta_0) \), and \( f_4 = f(\lambda_0, \theta_0) \) by bilinear interpolation \( f \) as

\[
f_i = (1-t)(1-u)f_1 + t(1-u)f_2 + tf_3 + (1-t)uf_4,
\]

where

\[
t = (\lambda - \lambda_1)/(\lambda_2 - \lambda_1),
\]

\[
u = (\theta - \theta_1)/(\theta_2 - \theta_0).
\]

A value by bicubic interpolation \( f \) also uses \( t \) and \( u \) defined in Eq. (2) and (3).

\[
f_i = \sum_{i,j=1}^{4} a_{ij} t^i u^j,
\]

An algorithm presented in Press et al. (1989) to calculate coefficients \( a_{ij} \) is used in the present study. The calculation of coefficients \( a_{ij} \) requires \( \partial f/\partial \lambda \), \( \partial f/\partial \theta \), \( \partial^2 f/\partial \lambda \partial \theta \), at the surrounding four points.

In place of commonly used finite differences, the proposed scheme employs spectral derivatives

\[
\frac{\partial f}{\partial \lambda} = \sum_m m f_m e^{im\lambda} e^{im\lambda_0},
\]

\[
\frac{\partial^2 f}{\partial \lambda \partial \theta} = \sum_m \sum_u m f_m e^{im\lambda} dP^m_0(\theta) d\theta,
\]

\[
\frac{\partial^2 f}{\partial \lambda \partial \theta} = \sum_m \sum_u m f_m e^{im\lambda} dP^m_0(\theta) d\theta,
\]

where \( P^m_0(\theta) \) is the associated Legendre function. FFT is usually employed in place of \( m \) summation in Eq. (5)–(7). A variant of this method uses meridional derivatives...
obtained by the finite difference method. The semi-spectral method has been used in a number of models including a primitive-equation global atmospheric model by Hunt (1974) and a geodynamo model by Oishi et al. (2007) to avoid the cost of Legendre transforms, and a semi-Lagrangian, grid-point, primitive-equation global atmospheric model by Bates et al. (1993) to accelerate semi-implicit solution (Moorthi and Higgins 1993).

### 3. Gaussian hill advection tests

The proposed scheme is validated with the Gaussian hill advection (Ritchie 1987). Models used in this paper are briefly described in this section. The models are summarized in Table 1.

The basic equation is

$$\frac{dq}{dt} = \frac{\partial q}{\partial r} + \frac{1}{\cos^2 \theta} \left( U \frac{\partial q}{\partial \phi} + V \cos \theta \frac{\partial q}{\partial \theta} \right) = 0, \quad (8)$$

where \((U, V) = (u, v) \cos \theta / a \).

A temporally constant solid body rotation with angular velocity \(\omega\) is assumed in form

$$u = \omega a \cos \theta \sin \theta \sin (\lambda - \lambda_0) \sin \theta \cos \theta_0, \quad (9)$$

$$v = \omega a \sin (\lambda - \lambda_0) \cos \theta_0. \quad (10)$$

The Gaussian hill is made to pass the north pole by tilting the axis of rotation (as \((\lambda_0, \theta_0) = (0^\circ, 45^\circ)\). The angular velocity \(\omega\) is set to make one rotation in twenty days.

$$\omega = \frac{2 \pi \cos (\theta_0)}{20 \times 86400}. \quad (11)$$

The shape of the hill used here is

$$q(r) = q_0 \exp \left[ - \left( \frac{r}{r_0} \right)^2 \right], \quad (12)$$

where \(q_0 = 100\) is the height of the hill at its centre, \(r\) the distance from the centre and \(L = 2500\) km is the wave length.

The eulerian model (EUL) calculates the advective terms with spectrally obtained derivatives of \(q\) in the physical space.

$$q' = q - \frac{2 \Delta t}{\cos \theta} \left( U \frac{\partial q}{\partial \phi} + V \cos \theta \frac{\partial q}{\partial \theta} \right), \quad (13)$$

where the superscripts ‘-’ and ‘+’ denote the previous and next time levels, respectively.

In the interpolating semi-Lagrangian model, values are updated by those at the previous time step interpolated to the departure points.

$$q' - q = 0, \quad (14)$$

In order to calculate \(q\), bilinear (BILIN) and bicubic (FD, SPH, FDY) interpolations are used. FD uses the centred difference method to calculate derivatives to be used in bicubic interpolation. SPH uses the derivatives calculated in the spectral space (Eq. 5-7), FDY uses the spectral and centred-difference methods for zonal and meridional derivatives, respectively. For interpolation and meridional finite differences, the meridians are extended across the poles with 180° shift in longitudes.

In order to compare proposed methods with those commonly used, higher order Lagrange interpolation is also tested (POL3, POL5, LP3, SH3, SH5). POL3 and POL5 use bicubic and quintic Lagrange interpolations, respectively. LP3 uses quasi-cubic interpolation (Ritchie et al. 1995). In the meridional direction, SH3 and SH5 use cubic and quintic Lagrange interpolations, respectively. In the zonal direction, SH3 and SH5 use cubic Hermite and Lagrange interpolations, respectively. In the Eulerian model (EUL), non negligible oscillations are caused by dispersion (Ritchie 1987). For forecast and climate models, unphysical negative values of moisture need to be artificially filled from the surrounding area or below. This is a shortcoming of Eulerian spectral models (Williamson and Rash 1994). It is also important to note that the global mean error is of naught but is composed of the positive and negative errors of ±19.3%.

The semi-Lagrangian models are much less dispersive. Accuracy varies, however, depending on the interpolation method used. The bilinear model (BILIN) is considered to be dissipative since the height of the hill is significantly reduced. A slightly positive global mean error implies that the reduced mass is not dissipated but is spread elsewhere. Since the monotonicity is guaranteed, the negative peak is very small. With bicubic interpolation, shape preservation is much improved. FD, LP3 and POL3 yield similar results with some dissipation.
Dissipation associated with FD, POL3 (Staniforth and Coté 1991) and LP3 (Ritchie et al. 1995) is considered to be acceptable in current climate and forecast models.

The error due to interpolation can be removed in the non-interpolating semi-Lagrangian model (NISL). The shape of the Gaussian hill is almost completely preserved as evidenced by small error in Fig. 2. Although the global mean error is of the same order as those of FD and POL3, NISL gives better results than the Eulerian (EUL) or interpolating semi-Lagrangian models (FD and POL3) in terms of shape preservation.

With the spectrally obtained derivatives (SPH), the interpolating semi-Lagrangian scheme is as accurate as NISL. Although the maximum is somewhat smaller than NISL, the global mean error is an order of magnitude smaller. The horizontal error distribution of SPH (Fig. 3) is similar to that of NISL (not shown). It appears that the use of spectral derivatives reintroduced a small dispersion as in NISL (Fig. 2 and Fig. 3). Quintic Lagrange interpolation (POL5) is required to yield a result comparable to NISL or SPH.

When the meridional derivatives are replaced with finite differences (FDY), the height of the Gaussian hill is somewhat lowered (purple, Fig. 2) but not as extensively as BILIN. FDY and SH3 are as accurate as FD, LP3 or POL3. Accuracy of semi-spectral interpolation can be improved by using higher order Lagrange interpolation. SH5 is as accurate as NISL, POL5 and SPH (Fig. 2 and Table 2).

Table 2. Summary of the advection tests. The first and second column are peak values at day 20. Note that the initial peak values are 95.6 and 0.0227, respectively. The third column represents the global mean of the error (%). The rightmost two columns are the global mean of positive and negative error (%), respectively.

<table>
<thead>
<tr>
<th>model</th>
<th>max</th>
<th>min</th>
<th>Δq</th>
<th>Δq</th>
<th>Δq</th>
</tr>
</thead>
<tbody>
<tr>
<td>EUL</td>
<td>91.7</td>
<td>-13.5</td>
<td>0</td>
<td>19.3</td>
<td>-19.3</td>
</tr>
<tr>
<td>BILIN</td>
<td>37.4</td>
<td>-0.00474</td>
<td>1.30</td>
<td>36.5</td>
<td>-35.2</td>
</tr>
<tr>
<td>FD</td>
<td>84.0</td>
<td>-1.10</td>
<td>0.283</td>
<td>6.54</td>
<td>-6.25</td>
</tr>
<tr>
<td>SPH</td>
<td>93.8</td>
<td>-0.0177</td>
<td>0.000765</td>
<td>1.83</td>
<td>-1.83</td>
</tr>
<tr>
<td>FDY</td>
<td>87.1</td>
<td>-0.556</td>
<td>0.331</td>
<td>5.11</td>
<td>-4.78</td>
</tr>
<tr>
<td>POL3</td>
<td>81.3</td>
<td>-0.572</td>
<td>-0.100</td>
<td>7.58</td>
<td>-7.48</td>
</tr>
<tr>
<td>POL5</td>
<td>91.4</td>
<td>-0.00772</td>
<td>0.00709</td>
<td>2.82</td>
<td>-2.81</td>
</tr>
<tr>
<td>LP3</td>
<td>83.6</td>
<td>-0.769</td>
<td>-0.281</td>
<td>7.34</td>
<td>-7.06</td>
</tr>
<tr>
<td>SH3</td>
<td>85.1</td>
<td>-0.357</td>
<td>-0.120</td>
<td>5.68</td>
<td>-5.56</td>
</tr>
<tr>
<td>SH5</td>
<td>92.1</td>
<td>-0.00810</td>
<td>-0.00351</td>
<td>2.50</td>
<td>-2.50</td>
</tr>
<tr>
<td>NISL</td>
<td>94.2</td>
<td>-0.584</td>
<td>-0.407</td>
<td>4.30</td>
<td>-4.70</td>
</tr>
</tbody>
</table>

If the spectral derivatives are used in a semi-Lagrangian atmospheric general circulation model, the model climate might be improved. In the advection of moisture, for example, the proposed schemes require much less mass restoration and negative filling. Thus accurate advection is not only favourable for dynamical
Fig. 3. Distributions of $q$ at day 5, 10, 15 (contours at the east, north, west, respectively) and its error from the initial value of SPH at day 20 in the polar stereographic projection centred at (0°, 45°N). Meridians and latitude circles are drawn every 15°. Contour intervals are 10 on day 5, 10, 15 and 0.5 on day 20.

but also for physical schemes. Small positive and negative overshootings, however, remain due to spectral truncation error with this method. Quasi-monotone scheme by Bermejo and Staniforth (1992) is applied but found to cause non negligible reduction of the height of the hill (not shown).

In the spectral models, forecast variables are represented by spectral coefficients as well as grid point values for the semi-implicit time stepping. Therefore there is no extra computational cost of obtaining the spectral derivatives for forecast variables. However, the right-hand side terms that are not treated by the semi-Lagrangian scheme require additional spectral transforms. When the computational cost matters, the semi-Lagrangian and Lagrange in the meridional (SH3 and SH5) were suggested by one of the reviewers.

Acknowledgments

The g95 fortran 95/2003 compiler was used to create binaries of the advection model. The figures were plotted with the NCAR Command Language (NCL). Constructive comments by two anonymous reviewers were very useful in improving the manuscript. In particular, comparison with cubic Lagrange interpolation (POL3) and combination of cubic Hermite in the zonal and Lagrange in the meridional (SH3 and SH5) were suggested by one of the reviewers.

References


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