The Patterns of the Open-City Growth, and the Effects of the Land Value Tax

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1. Introduction

Land value taxation has received considerable attention ever since the publication of Henry George's Progress and Poverty. It has been well established that the property tax is non-neutral because the supply of structures and improvements is not fixed, and so the imposition of the tax on those distorts land-use decisions. Meanwhile, the land tax is neutral because the supply of land is fixed and landowners do not change their land-use decisions by the imposition of the tax. In the traditional literature, many writers have advocated the adoption of land taxes on equity considerations and on the grounds that these taxes have no effects on resource allocation.

However, more recently it was demonstrated by other writers including Bentick [3], Mills [6], and Skouras [7] that taxes on land rents are neutral while taxes on current land value is non-neutral as they create an incentive for a more rapid development of land. Anderson [1] argued that the timing effects of changes in the rates of the land value tax depend on the nature of the house market, i.e., whether the market is growing or declining. Brueckner [4] also analyzed both the long-run effects of the land value tax and the distortion of the short-run gains and losses of this tax.

The purpose of this paper is to analyze the effects of the land value tax on the urban spatial growth, using a dynamic model of urban land-use.

Dynamic models of urban land-use explained various aspects of cities that static models of Alonso-Muth types cannot do. There are Arnott [2], Fujita [5], Turnbull [8], and Wheaton [9] in such representative models. While Fujita [5] and Turnbull [8] analyzed the patterns of the open-city development, Wheaton [9] showed the patterns of the closed city growth.

In order to provide a complete characterization of the motion of the urban economy and facilitate the effects of the tax on the open-city growth, we develop a open-city model similar to previous dynamic models. Many papers analyzed the timing-effects of the land value tax through the partial equilibrium analysis. But we extend previous models for analyzing the timing effects of the tax to a model of the urban spatial growth in order to investigate the effects of the tax on the urban growth.

Brief results of this paper are as follows: if development moves outwards from the CBD, the tax decreases the structural density and expands the city boundary, so that it increases the city population at each point in time. Conversely, if development moves inwards towards the CBD, the tax increases the density and shrinks the city boundary, so that it decreases the city population at each point in time.

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This paper is organized as follows: Section 2 presents a simple model of the urban spatial growth. Section 3 modifies the model with respect to tax parameters and an attempt is made to investigate the comparative static effects of changes of the tax on the open-city growth. Summary and conclusion appear in section 4.

2. A Model of Urban Spatial Growth

The city is radially symmetric and has a central point, the Central Business District (CBD). The only land use is housing. All non-residential activities occur at the central point. The city residents, all of whom are identical, earn wage income $Y(t)$ for working at the CBD and at time $t$. Radial distance to the CBD is represented by $x$, and commuting cost from a residence at distance $x$ at time $t$ equals $k(t)x$. All residents have the same utility function, $U[Z, H]$, where $Z$ is the consumption of a numeraire nonhousing good and $H$ the consumption of housing services, and the marginal utilities of both goods are positive, $U_Z > 0$ and $U_H > 0$.

Denote by $Y(t)$, $U(t)$, and $k(t)$, respectively, the income, utility, and the unit commuting cost of the households in the city at time $t$, where they are all given exogenously. The character of housing services is expressed by the lot size. Houses are built and owned by landowners. No possibility of renewal or structural change of houses is considered here.

2.1 Behavior of the Household

Based on the above assumptions, the utility function and the budget constraint at time $t$ are given as follows:

$$U(t) = U[Z, H],$$
$$Y(t) = Z + k(t)x + R(x, t)H.$$

From the above utility function and budget constraint, the bid rent per dwelling unit at distance $x$ and time $t$ can be obtained as follows:

$$R(x, t) = \max_H \left[ Y(t) - k(t)x - Z[H, U(t)] \right]/H,$$

where $Z[H, U(t)]$ is the nonhousing consumption necessary for a household to attain the equilibrium utility level $U(t)$. The solutions of this maximization (1) are represented by

$$Z(x, t) = Z[x, Y(t), U(t)],$$
$$H(x, t) = H[x, Y(t), U(t)],$$
$$R(x, t) = R[x, Y(t), U(t)].$$

If the utility function is strictly quasi-concave (we assume it does so), then

$$dZ/dx < 0, \quad dZ/dY > 0, \quad dZ/dk < 0,$$
$$dH/dx > 0, \quad dH/dY < 0, \quad dH/dk > 0, \quad dH/dU > 0,$$
$$dR/dx < 0, \quad dR/dY > 0, \quad dR/dk < 0, \quad dR/dU < 0.$$

2.2 Behavior of the Landowners

Once housing is built, it does not deteriorate and cannot be modified structurally. Demolition is infinitely expensive, so the redevelopment cannot take place. Durable houses are constructed and rented to households by absentee landowners. The house production function with a constant returns technology is represented as $Q(K, E)$ where $K$ is input of capital and
$E$ is input of land, respectively. Housing output per unit of land is $Q(K, E)/E = Q(K/E, 1) = q(h)$ where $h$ is capital per unit of land (structural density) and $q' > 0$, $q'' < 0$ by the concavity of the production function. At each distance $x$, landowners select the structural density $h$ and development time $T$ to maximize the present discounted value of profits from a unit of land. Thus, the landowners’ problem at each distance $x$ is given by

$$L^*(h, T) = \int_T^\infty R(x, t) q(h) e^{-rt} dt - nhe^{-rT}, \quad (3)$$

where agricultural rent per unit of land is zero, $r$ is the exogenous discount rate, and $n$ is the construction cost per unit of land constant over time. The expression (3) gives the value of residential land at time zero at the location $x$ as a function of the development time $T$ and the structural density $h$. Assuming that landowners have perfect foresight about the future time path of house rent, the first-order conditions for choice of the development time and the structural density are represented as

$$L_h = \int_T^\infty R(x, t) q'(h) e^{-r(t-T)} dt - n = 0, \quad (4)$$
$$L_T = - R(x, T) q(h) + rnh = 0. \quad (5)$$

The second-order conditions are given by

$$L_{hh} = \int_T^\infty R(x, t) q''(h) e^{-r(t-T)} dt < 0,$$
$$L_{TT} = - R_T(x, T) q(h) < 0,$$
$$J = L_{hh}L_{TT} - L_{ht}^2 > 0,$$

(6)

where we denote $L_T = e^{rt}L^*_T$ and $L_h = e^{rt}L^*_h$ for simplicity. The first-order conditions state that the house should be constructed until the additional discounted present value equals the additional discounted costs, and the house should be developed when the costs of postponing development equal the benefits from postponing development.

### 2.3 The Patterns of the Open-City Growth

From (4) and (5), the optimal density and development time at each location is represented as the function of $x$ by

$$h = h(x) \text{ and } T = T(x).$$

To examine how the optimal development strategy $[T(x), h(x)]$ changes with distance $x$, taking the derivatives of each side of (4) and (5) with respect to $x$ and solving them for $h_x(x)$ and $T_x(x)$, then we obtain

$$h_x(x) = \left[ L_{th}L_{tx} - L_{tt}L_{hx} \right] J^{-1}, \quad (7)$$
$$T_x(x) = \left[ L_{th}L_{hx} - L_{hh}L_{tx} \right] J^{-1}, \quad (8)$$

with

$$L_{hx} = \int_T^\infty R_x(x, t) q'(h) e^{-r(t-T)} dt < 0,$$
$$L_{tx} = - R_x(x, T) q(h) > 0,$$
$$L_{ht} = \left[ \int_T^\infty R_t(x, t) q'(h) e^{-r(t-T)} dt \right] = - R(x, T) q'(h) + rn > 0.$$  

(9)

Though the signs of $h_x$ and $T_x$ are indeterminate, we obtain the following relationships from
\[ T_x(x) = \frac{\int_T^\infty R(x, t)e^{-\sigma(t-T)}dt}{R(x, T)} \]
\[ h_x(x) = \frac{R_x(x, T)}{\int_T^\infty R(x, t)e^{-\sigma(t-T)}dt} \]

where \( \sigma \) is the elasticity of substitution between capital and land in the production of housing, and \( \varepsilon_{qh} \) is the elasticity of housing output with respect to the structural density, \( 0 < \varepsilon_{qh} < 1 \) (see Appendix 1).

By a simple calculation, the expressions (10) and (11) can be rewritten as
\[ T_x(x) = \frac{\int_T^\infty R(x, t)e^{-\sigma(t-T)}dt}{R(x, T)} \]
\[ h_x(x) = \frac{R_x(x, T)}{\int_T^\infty R(x, t)e^{-\sigma(t-T)}dt} \]

where \( \rho \) is the average growth rate in house rent, \( \delta \) is the average growth rate in the slope of house rent, and \( m \) is the instantaneous rate of growth in house rent:
\[ \rho = r - \frac{R(x, T)}{\int_T^\infty R(x, t)e^{-\sigma(t-T)}dt}, \quad m = \frac{R_T(x, T)}{R(x, T)} \]
\[ \delta = r - \frac{R_x(x, T)}{\int_T^\infty R_x(x, t)e^{-\sigma(t-T)}dt}, \quad E = (1/\varepsilon_{qh}) - 1. \]

The results shown in Wheaton [9] are explained from (12) and (13). If \( T_x(x) > 0 \) \( \sigma(r - \rho) > r - \delta \) and \( h_x(x) < 0 \) \( \rho > \delta \), then \( \rho > \delta \). In other words, \( m \) and \( \rho \) must be relatively large in order for \( T_x \) to be greater than zero and \( h_x \) to be smaller than zero (This is necessary condition, but not sufficient condition). If the growth rate in income is greater, then that in house rent becomes larger, and if the growth rates in marginal commuting cost and utility is greater, then that in house rent becomes smaller. Hence, rising income, falling marginal commuting cost, and falling utility tend to result in a more strongly decreasing population density gradient and these may induce an outward development. Conversely, if \( T_x(x) < 0 \) and \( h_x(x) > 0 \), then \( E \rho < \sigma(r - \rho) \). In the case, falling income, rising marginal commuting cost, and rising utility may induce a decreasing population density with distance and generate inward development, \( T_x(x) < 0 \). The patterns of development can be summarized from (7) and (8) as follows:

PROPOSITION 1: 1) if \( T_x(x) < 0 \), then \( h_x(x) < 0 \).
2) if \( h_x(x) > 0 \), then \( T_x(x) > 0 \).

The proof: If \( T_x < 0 \), then \( L_{Th}/L_{hh} < L_{Tx}/L_{hx} \) and \( L_{TT}/L_{hT} < L_{hT}/L_{hh} \) from (6), (8), and (9).
Together, these imply \( L_{TT}/L_{hT} < L_{Th}/L_{hx} \), so that \( L_{TT}/L_{hT} > L_{hT}/L_{Th} \) and \( h_x < 0 \) from (7).
The inverse in not satisfied.
If \( h_x > 0 \), then \( L_{Tx}/L_{hx} < L_{TT}/L_{Th} \) and \( L_{TT}/L_{hT} < L_{Th}/L_{hh} \) from (6), (7), and (9). Hence, \( L_{Th}/L_{hT} = L_{hT}/L_{Th} > 0 \) and \( T_x > 0 \). The inverse is not satisfied.

Proposition 1 reveals that there may be several locations at which development takes place simultaneously at a given time. Type 1 implies that if only development moves inward towards the CBD, then development density declines with distance. Conversely, if develop-
ment moves outwards, development density may increase or decrease with distance. Type 2 implies that development must move outwards from the CBD if only development density increases with distance. Conversely, if development density decreases with distance, development may be moving outwards or inwards. In particular, in the case of type 1, leapfrog development can occur and this leapfrog sites will be developed more densely in the future than they would be if developed at the same time as the surrounding urbanized area.

Until now, we are concerned with the optimal development strategy at a given location \( x \). \( T(x) \) represents the development time as a function of distance from the CBD. Its inverse function, \( x(t) \) represents the distance to the city boundary as a function of time. Also, the optimal density \( h[x(t)]=h(t) \) is represented as a function of time.

Optimal location \( x(t) \) and the corresponding density \( h(t) \) must satisfy (4) and (5) namely,

\[
\int_{t}^{\infty} R[x(t), \tau] q'[h(t)] e^{-\tau(t-t)} d\tau - n = 0, \quad (14) \\
R[x(t), \tau] q[h(t)] - r nh(t) = 0. \quad (15)
\]

The population of the city is obtained by integrating the population density from the CBD to the boundary at time \( t \),

\[
\int_{0}^{x(t)} \left[ 2\pi x q(h)/H(x, t) \right] dx = \int_{0}^{t} \left[ 2\pi x(q(r)/H[x(r), t]) \right] d\tau x(t) = N(t) \quad (16)
\]

Assume that all \( 2\pi \) radians are available for residential development. These equations, (14), (15), and (16) determine the open-city growth paths. Then we investigate how \( x(t), h(t), \) and \( N(t) \) change with time. Differentiating the three equations with respect to \( t \), we obtain the following system:

\[
\begin{bmatrix}
\int_{t}^{\infty} Rq'' e^{-\tau(t-t)} d\tau \\
Rq' - rn
\end{bmatrix}
\begin{bmatrix}
\int_{t}^{\infty} Rq'e^{-\tau(t-t)} d\tau \\
Rq
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
-2\pi q/h
\end{bmatrix}
\begin{bmatrix}
h \\
h(t) \\
N
\end{bmatrix}
= \begin{bmatrix}
Rq' - rn \\
-Rq
\end{bmatrix}
\begin{bmatrix}
\dot{h} \\
\dot{x}
\end{bmatrix}
+ \int_{0}^{x(t)} [2\pi x q(H^2) H dx]. \quad (17)
\]

Denoting a matrix formed with \( (3 \times 3) \) by \( \hat{A} \) and a determinant of the matrix by \( A \), the above system can be solved by Cramer's rule as follows:

\[
\dot{h} = \frac{Rq' - rn}{-Rq} \int_{t}^{\infty} Rq'e^{-\tau(t-t)} d\tau / A^{-1} \quad (17)
\]

\[
\dot{x} = \frac{\int_{t}^{\infty} Rq'' e^{-\tau(t-t)} d\tau Rq' - rn}{-Rq} / A^{-1} \quad (18)
\]

\[
N = (2\pi x q(H^2) H dx) \dot{h}. \quad (19)
\]

where the determinant of the matrix, \( A \), depends upon the sign of \( T_x(x) \) from (8), namely, \( A > 0 \) as \( T_x(x) = 0 \). These equations provide a complete characterization of the motion of the urban economy. The determinant of the numerator matrix in (17) denoted by \( B \) has the same sign as \( h_x(x) \) from (7). The determinant of the numerator matrix in (18) has a positive sign from the second-order conditions (6). In like manner as proposition 1, we conclude that if \( A < 0 \) \( (T_x < 0) \), then \( B < 0 \) \( (h_x < 0) \) from (7) and (8), so that \( \dot{h} < 0, \dot{x} < 0, \) and \( N > 0 \), while if \( B > 0 \) \( (h_x > 0) \), then \( A > 0 \) \( (T_x > 0) \) from (7) and (8), so that \( \dot{h} > 0, \dot{x} > 0, \) and \( N > 0 \). Hence, when there
are several locations at which development takes place simultaneously at a given time, the two patterns of urban development are shown in Fig. 1.

3. The Effects of the Land Value Tax on Urban Growth

In this section we investigate the effects of the land value tax on the development density and development time. In the presence of land value tax, the value calculation per unit of land is modified to account for tax payments prior and subsequent to development. With the tax rate \( a \), the value per unit of land becomes, at time \( t \) prior to development time \( T \),

\[
L(x, h, t) = \int_t^T \left( -aL(x, h, t) \right) e^{-(r-a)(t-T)} dt + \int_T^\infty \left( R(x, \tau)q(h) - aL(x, h, \tau) \right) e^{-(r-a)(t-T)} dt - nh e^{-(r-a)(T-t)}, \quad t < T.
\]

At time \( t \) beyond the development time \( T \), the value per unit of land becomes

\[
L(x, h, t) = \int_t^\infty \left[ R(x, \tau)q(h) - aL(x, h, \tau) \right] e^{-(r-a)(t-T)} dt - nh, \quad t \geq T
\]

This equation can be solved to derive an expression for \( L(x, h, t) \) (see Appendix 2)

\[
L(x, h, t) = \int_t^\infty R(x, \tau)q(h) e^{-(r-a)(t-T)} dt - \frac{(r+\alpha)}{r} nh.
\]

The value per unit of land at time \( t \) is the discounted present value of profits per unit of land. The discount rate includes the tax rate in (22).

Since at time \( t < T \) prior to development, the second and third integral in (21) are equivalent to

\[
e^{-r(T-t)} \int_T^\infty \left[ R(x, \tau)q(h) - nh \right] e^{-(r+\alpha)(t-T)} dt,
\]

the expression (20) can be rewritten as

\[
L(x, h, t) = \int_t^T \left[ -aL(x, h, \tau) \right] e^{-(r-a)(t-T)} dt + e^{-(r-a)(T-t)} \int_T^\infty \left[ R(x, \tau)q(h) - nh \right] e^{-(r+\alpha)(t-T)} dt.
\]

Solving this expression for \( L(x, h, t) \) and substituting \( t=0 \) into the derived value expression (23)
give the objective function of the land developer, who chooses \( T \) and \( h \) to maximize the present discounted value at time \( t \) (see Appendix 3),

\[
L(x, h; 0) = \int_{t}^{\infty} R(x, \tau) q(h) e^{-(r + a) \tau} d\tau - \frac{r}{r + a} nh e^{-(r + a) T}.
\] (24)

The first-order conditions for choice of \( T \) and \( h \) in the presence of the land value tax become

\[
L_h = \int_{t}^{\infty} R(x, \tau) q(h) e^{-(r + a) (t - \tau)} d\tau - \frac{r}{r + a} nh = 0, \tag{25}
\]
\[
L_T = -R(x, T) q(h) + nh = 0. \tag{26}
\]

In the presence of the land value tax, the optimal location \( x(t) \) and the corresponding density \( h(t) \) must satisfy (25) and (26), namely,

\[
L_h[h(t), x(t)] = \int_{t}^{\infty} R[x(t), \tau] q[h(t)] e^{-(r + a) (t - \tau)} d\tau - \frac{r}{r + a} nh = 0, \tag{27}
\]
\[
L_T[h(t), x(t)] = R[x(t), t] q[h(t)] - nh = 0. \tag{28}
\]

In the presence of the land value tax, the population in the city becomes

\[
L_N[h(t), x(t), N(t)] = \int_{x(t)}^{0} - \left[ 2 \sqrt{x q(h)} / H(x, t) \right] dx + H(t) = 0. \tag{29}
\]

The open-city growth paths, \( h(t), x(t) \), and \( N(t) \) in the presence of the tax are obtained by solving (27)-(29). In order to analyze the effects of the land value tax on the open-city growth, differentiating (27)-(29) with respect to \( a \) and solving them, we obtain

\[
\dot{h}(t) = \left[ -L_{T h} A_{h h} \right]^{-1} A_{h h}^{-1}, \tag{30}
\]
\[
\dot{x}(t) = \left[ L_{T h} A_{h h} \right]^{-1}, \tag{31}
\]
\[
\dot{N}(t) = \left[ -L_{N x} A_{h h} A_{T h} \right]^{-1}, \tag{32}
\]

where

\[
A = \begin{vmatrix} L_{h h} & L_{h x} & L_{h N} \\ L_{T h} & L_{T x} & L_{T N} \\ L_{N h} & L_{N x} & L_{N N} \end{vmatrix} > 0 \text{ if } \dot{x}(t) > 0,
\]

with

\[
L_{h h} = \int_{t}^{\infty} R[x(t), \tau] q'[h(t)] e^{-(r + a) (t - \tau)} d\tau < 0,
\]
\[
L_{h x} = \int_{t}^{\infty} R_x[x(t), \tau] q'[h(t)] e^{-(r + a) (t - \tau)} d\tau < 0,
\]
\[
L_{T h} = R[x(t), t] q'[h(t)] - nh < 0,
\]
\[
L_{T x} = R_x[x(t), t] q[h(t)] < 0,
\]
\[
L_{N h} = -2 \sqrt{x q[x(t)] / H[x(t), t]} < 0,
\]
\[
L_{N x} = -\int_{t}^{\infty} (t - \tau) R[x(t), \tau] q'[h(t)] e^{-(r + a) (t - \tau)} d\tau + \frac{r}{r + a^2} n < 0,
\]
\[
L_{N N} = 1, L_{N h} = 0, L_{N T} = 0, L_{N h} = 0, L_{T x} = 0, \text{ and } L_{N h} = 0. \tag{33}
\]

From (30)-(33), effects of changes in the tax rate on the open-city growth paths are as follows:

**PROPOSITION 2:** if \( \dot{x}(t) > 0 \), then \( \frac{d h(t)}{d a} < 0, \frac{d x(t)}{d a} > 0 \), and \( \frac{d N(t)}{d a} > 0 \).

if \( \dot{x}(t) < 0 \), then \( \frac{d h(t)}{d a} > 0, \frac{d x(t)}{d a} < 0 \), and \( \frac{d N(t)}{d a} < 0 \).
Proposition 2 results from (30)-(33). It does not examine the comparative dynamic effects of a sudden change in a tax parameter at a particular moment of time. The comparative static analysis of Proposition 2 compares the open-city growth paths from otherwise identical urban areas that have different values for a particular exogenous parameter. The effects of the tax on the patterns of urban development are shown in Fig. 2. If development moves outwards from the CBD, the land value tax decreases the structural density and expands the city boundary, so that it may increase the city population at each point in time. Conversely, if development moves inwards towards the CBD, the tax increases the density and shrinks the city boundary so that it may decrease the population at each point in time.

4. Summary and Conclusion

This paper has analyzed two principal questions treated in the previous literature. First, this paper observed aspects of cities which are different from those observed in the static open city. Second, on the issues of nonneutrality of the land value tax, this paper analyzed effects of the tax in the context of the model of urban spatial growth. If development moves outwards from the CBD, the tax decreases the structural density and expands the city boundary, so that it increases the city population at each point in time. Conversely, if development moves inwards towards the CBD, the tax increases the density and shrinks the city boundary, so that it decreases the city population at each point in time.

A direction for future research could be the application of the model developed in this paper to a model with multiple land-use in order to analyze effects of the land value tax on almost all spatial aspects.

References

Appendix

A.1. Proof of (10)

$T_x(x)$ has the same sign as $[L_{hh}L_{hx} - L_{hh}L_{TX}]$ from (8).

$T_x > 0 \iff L_{hx} > L_{hh}L_{TX}$

$\iff (rn - Rq)(\int_T^\infty R_x q e^{-r(t-T)} dt) > < \int_T^\infty R_x q e^{-r(t-T)} dt [-R_x q]$

Dividing both terms by $R(x, T)$, and using (5), $[R(x, T)q] = rh$, we have

$[\frac{r}{q} - q^2] \int_T^\infty R_x q e^{-r(t-T)} dt > < \int_T^\infty R_x q e^{-r(t-T)} dt [-R_x q] / R$

Dividing both terms by $qq''$ and rearranging,

$\sigma \int_T^\infty R_x e^{-r(t-T)} dt < > \frac{1}{\sigma} \int_T^\infty R_x e^{-r(t-T)} dt / R$

where $\sigma = (hq^2 - qq')/qq''$.

$h_x(x)$ has the same sign as $[L_{hh}L_{TX} - L_{TT}L_{hx}]$ from (7)

$h_x > 0 \iff -(rn - Rq)(R_x q) > < (-R_T q) (\int_T^\infty R_x q e^{-r(t-T)} dt)$

Dividing both terms by $qq'$, and using (5), $[R(x, T)q] = rh$,

$\frac{q}{hq'} - 1 (-R_x) < > (-R_T / R) \int_T^\infty R_x e^{-r(t-T)} dt$

$\frac{1}{qe_{qh}} - 1 \frac{R_x}{\int_T^\infty R_x e^{-r(t-T)} dt} < > (R_T / R)$

where $0 < e_{qh} = hq'/q < 1$.

A.2. Proof of (22).

In order to derive $L(h, x, t)$ for $t > T$, (21) is differentiated with respect to $t$, yielding

$L_t(h, x, t) = - [R(x, t)q(h) - rh - aL(h, x, t)]$

$+ r \int_T^\infty [R(x, \tau)q(h) - rh - aL(x, h, \tau)] e^{-r(t-\tau)} d\tau.$
Simplifying and rearranging gives the differential equation,

\[ L_t(h, x, t) - (r + a) L(h, x, t) = - \left[ R(x, t) q(h) - r nh \right]. \]

This differential equation can be solved by multiplying through by the term \( e^{-\left( r + a \right) t} \) giving

\[ \frac{d}{dt} \left[ L(h, x, t) e^{-\left( r + a \right) t} \right] = - \left[ R(x, t) q(h) - r nh \right] e^{-\left( r + a \right) t}, \]

which can be integrated and simplified to give

\[ L(h, x, t) = \int_t^\infty \left[ R(x, \tau) q(h) - r nh \right] e^{-\left( r + a \right)(\tau-t)} d\tau. \]

A.3. Proof of (24)

In order to derive \( L(h, x, t) \) for \( t < T \), (23) is differentiated with respect to \( t \), yielding

\[ L_t(h, x, t) = a L(h, x, t) + r \int_t^T \left[ - a L(h, x, t) \right] e^{-\left( r + a \right) \tau} d\tau + \frac{r e^{-\left( r + a \right) \tau}}{T} \int_T^\infty \left[ R(x, \tau) q(h) - r nh \right] e^{-\left( r + a \right)(\tau-T)} d\tau \]

which gives the differential equation,

\[ L_t(h, x, t) - (r + a) L(h, x, t) = 0 \]

This equation has the solution,

\[ L(h, x, t) e^{-\left( r + a \right) t} = c \]

where \( c \) is a constant. To find \( c \), the initial condition

\[ L(h, x, T) = \int_T^\infty \left[ R(x, \tau) q(h) - r nh \right] e^{-\left( r + a \right)(\tau-T)} d\tau. \]

which follows from (23), must hold. Using this condition, a constant \( c \) is

\[ c = e^{-\left( r + a \right) T} \int_T^\infty \left[ R(x, \tau) q(h) - r nh \right] e^{-\left( r + a \right)(\tau-T)} d\tau. \]

Hence, substituting \( c \) and rearranging gives the value expression

\[ L(h, x, t) = e^{-\left( r + a \right)(T-t)} \int_T^\infty \left[ R(x, \tau) q(h) - r nh \right] e^{-\left( r + a \right)(\tau-T)} d\tau. \]

This expression evaluated at \( t = 0 \) gives the objective function of land developer, (24).