Theoretical and Applied Input-Output Analysis : A New Synthesis

Part I : Structure and Structural Changes in Input-Output Systems

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Abstract

This synopsis represents an overview of some recent developments, explorations and extensions of input-output analysis undertaken during last decade in the Regional Economics Application Laboratory, University of Illinois at Urbana-Champaign, USA and the Bar-Ilan Laboratory of Environmental Information and Dynamics, Bar-Ilan University, Israel. This paper is the first of three review papers ; in Part I, the focus will be on *structure and structural changes in input-output systems* and the following issues will be considered :

I. Temporal multipliers, temporal increment and temporal Leontief inverse : fields of influence of changes.

II. Economic Hierarchical Landscapes of backward and forward linkages : Key Sector Analysis, fields of influence and Multiplier Product Matrix.

III. Economic structural changes in the input-output systems : Barycentric Calculus of triple decompositions of gross output changes.

IV. Structural Q-analysis of the Leontief inverse.

In Parts II and III, attention will be directed to multiregional input-output analysis and the transition from comparative statics to dynamics.

Introduction

In the last decade, there have been some profound changes in the way input-output analysis has been used and in the type of research that has been conducted with the input-output tables and associated models. One of the most important developments has been the creation and elaboration of integrated models in which the input-output system is placed as one of several components ; the growth in application of applied general equilibrium modeling has generated new opportunities to explore the role of interdependencies in newer, more comprehensive versions of economy-wide modeling. Furthermore, new approaches to the understanding and interpretation of economic structure have generated a wave of new work that has focused on imaginative interpretations of the ways in which economies have changed or could possibly change. In this first of a series of papers, attention will be focused on issues of structure and structural changes in input-output systems ; subsequent papers will explore the spatial dimension in more detail and will consider more formally dynamics of change, including new methods for projecting input-output coefficients and the relationship between input-output analysis and nonlinear dynamics.

In the input-output literature, there has been considerable debate about the appropriateness of using input-output systems in the context of forecasting, especially when the forecast period

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extends beyond a period in which an assumption of the stability of the underlying input coefficients can be justified. However, it should be noted that even this period is difficult to define, although Carter (1970) suggested that the moving average nature of coefficient change dampened the tendencies for significant structural changes in periods of less than five years. An alternative approach initiated by Leontief (1977), the development of dynamic input-output models, has been fraught with difficulties, not all of which can be traced to the identification of appropriate and reliable empirical information on capital stocks, marginal capital coefficients and the distinctions between capital replacement and expansion.

In the Part I of this synopsis, an alternative vision is illustrated; the procedure exploits the notion of discrete time changes and corresponding temporal multipliers and proposes a temporal Leontief inverse discrete dynamics in lieu of the complexities underlying the formal structure of dynamic inverses associated with Leontief (1977). The proposed temporal Leontief inverse formulation is comprehensive since it is possible to trace the impact of each change in each time period to the final state of the economy. Thus, in comparison to the usual dynamic Leontief inverse, the temporal Leontief inverse does offer some greater prospect for empirical implementation.

One of the main conceptual notions that underlies much of the recent work that will be reviewed in this paper is the concept of *fields of influence* of changes (Sonis and Hewings, 1988, 1989). This concept was elaborated with the aim of providing a general tool for the measurement of the analytical impacts of changes in the direct coefficients' matrix of an input-output or social accounting system on the associated Leontief inverse. Interest in the problem of coefficient changes is not a recent phenomenon (see Sohn, 1986 and Sonis and Hewings, 1991a, for a review of some earlier work); however, the major theoretical developments that provided the basis for the present initiatives were those of Sherman and Morrison (1949, 1950), Simonovits (1975), Bullard and Sebald (1977, 1988), and Hewings and Romanos (1981). The concept of fields of influence of changes can be connected with the concept of *transfer of economic influence* as expressed analytically in structural path analysis elaborated by Lantner (1972, 1974), Gazon (1976), Defourny and Maree (1977, 1978) and several collaborators (see, Crama, Defourny and Gazon (1983) and Defourny and Thorbecke (1984).

From the foundation provided by the fields of influence, it has been possible to revisit ideas such as key sector analysis; the application of the multiplier product matrix that is derived from the products of the corresponding column and row multipliers of the Leontief inverse provides a new theoretical tool that presents key sector analysis of backward and forward linkages in a new form. For example, the multiplier product matrix possesses a maximum entropy property and this property enables the creation of a macro-economic photograph (*economic landscape*) of the structure of an economy, providing a basis for the comparison of structures of different economies or the same economy over time.

One of attractive applications of the field of influence concept is the issue of economic structural changes in the input-output system caused by simultaneous change in technology and in the composition of final demand. This application is focused on the use of triple decompositions of the components of gross output in the input-output system, the use of fields of influence analytical formalism for the analysis of the fine structure of changes in the economy and the use of the analytical and graphical technique of Barycentric Calculus for the presentation of these economic structures and their changes.

Further, in this review, an attempt is made to connect the conceptual framework of Q-analysis, based on combinatorial topology, with the essence of qualitative graph-theoretical methods of input-output scheme, referred to as *qualitative input-output analysis*. (See, for example, Holub, Schanabl, Tappeiner, 1985, Schnabl, 1993; Bellet, Lallich and Vincent, 1989; Torre, 1993). In this context a new graphical tool—the complication curve—appears, providing a new way for the explanation of economic complexity through the process of economic complication.

The exposition of material in this synopsis is conceptual. Empirical applications of the

theoretical considerations are cited in applications at the end of each section.

I. Temporal multipliers, temporal increment and temporal Leontief inverse : fields of influence of changes

1. Temporal Multipliers and Temporal Increments

Let $A = \| a_{ij} \|$ be an $n \times n$ matrix of direct input coefficients, $E = \| e_{ij} \|$ be a matrix of the incremental changes in the direct input coefficients; $B = (I - A)^{-1} = \| b_{ij} \|$ and $B(E) = (I - A - E)^{-1} = \| b_{ij}(E) \|$ be the Leontief inverses before and after changes. It is easy to see that :

$$\begin{aligned} B(E) &= (I - A - E)^{-1} = [(I - A)(I - BE)]^{-1} = (I - BE)^{-1}B \\ B(E) &= (I - A - E)^{-1} = [(I - EB)(I - A)]^{-1} = B(I - EB)^{-1} \end{aligned} \quad (1)$$

The matrices $M_L = M_L(B, E) = (I - BE)^{-1}$, $M_R = M_R(B, E) = (I - EB)^{-1}$ are called the *left and right temporal multipliers*. Obviously :

$$B(E) = M_L B = B M_R; \quad M_L = B(E)(I - A); \quad M_R = (I - A)B(E). \quad (2)$$

The multiplicative representations (2) of the Leontief inverse $B(E)$ can be converted into the following additive representations. It is easy to see that :

$$B(E) = M_L B = B + (M_L - I)B$$

and

$$B(E) = B M_R = B + B(M_R - I).$$

Using the relations :

$$M_L(I - BE) = (I - BE)M_L = I; \quad M_R(I - EB) = (I - EB)M_R = I$$

one obtains :

$$M_L - I = M_L B E = B E M_L; \quad M_R - I = M_R E B = E B M_R \quad (3)$$

Therefore :

$$B(E) = B + D(B, E) \quad (4)$$

where :

$$D(B, E) = B(E) - B = (M_L - I)B = M_L B E B = B E B M_R \quad (5)$$

The term $D(B, E)$ can be called the *temporal increment*.

2. Multiplicative and additive forms of the Temporal Leontief inverse

A definition of the temporal Leontief inverse can be introduced in the following form : consider a sequence of time-periods $t_0, t_1, \dots, t_T$ such that in the initial period t_0 there exists a matrix $A_0 = \| a_{ij}^0 \|$ of direct input coefficients and an associated Leontief inverse $B_0 = (I - A_0)^{-1}$; in each time-period t_s there are matrices of changes in direct coefficients $E_s = \| e_{ij}^s \|$ such that the matrices $A_s = \| a_{ij}^s \|$ of technological coefficients and Leontief inverses $B_s = (I - A_s)^{-1} = \| b_{ij}^s \|$ have a form :

$$\begin{aligned} A_s &= A_{s-1} + E_s = A_0 + E_1 + E_2 + \dots + E_s, \quad s = 1, 2, \dots, T \\ B_s &= (I - A_{s-1} - E_s)^{-1} = (I - A_0 - E_1 - E_2 - \dots - E_s)^{-1} \end{aligned} \quad (6)$$

The sequential application of (2) provides the following multiplicative decomposition of the temporal Leontief inverse, with the help of the temporal multipliers :

$$\begin{aligned} B_s &= M_L^s B_{s-1} = \\ &= M_L^s M_L^{s-1} B_{s-2} = \\ &\vdots \\ &= M_L^s M_L^{s-1} \dots M_L^2 M_L^1 B_0, \end{aligned} \quad (7)$$

where temporal multipliers, M_t^s , are :

$$M_t^s = (I - B_{s-1}E_s)^{-1}, \quad s=1, 2, \dots, T. \quad (8)$$

In turn, the sequential application of (4) yields the additive decomposition of the temporal Leontief inverse with the help of the temporal increments :

$$\begin{aligned} B_s &= B_{s-1} + D_s = \\ &= B_{s-2} + D_{s-1} + D_s = \\ &\quad \vdots \\ &= B_1 + D_2 + \dots + D_{s-1} + D_s = \\ &= B_0 + D_1 + D_2 + \dots + D_{s-1} + D_s \end{aligned} \quad (9)$$

where the temporal increments, D_s , are :

$$D_s = D(B_{s-1}, E_s) = B_s - B_{s-1} = (M_t^s - I)B_{s-1}, \quad s=1, 2, \dots, T. \quad (10)$$

Using (8), one can obtain from (9) the following expression :

$$\begin{aligned} B_s &= I + \underbrace{(B_0 - I)}_{B_0 + (M_t^1 - I)B_0 + (M_t^2 - 1)M_t^1 B_0 + \dots + (M_t^s - I)M_t^{s-1} \dots M_2^1 M_1^1 B_0} \\ &= \underbrace{B_0 + (M_t^1 - I)B_0 + (M_t^2 - 1)M_t^1 B_0 + \dots + (M_t^s - I)M_t^{s-1} \dots M_2^1 M_1^1 B_0}_{B_1 + (M_t^2 - 1)M_t^1 B_0 + \dots + (M_t^s - I)M_t^{s-1} \dots M_2^1 M_1^1 B_0} \\ &= \underbrace{B_1 + (M_t^2 - 1)M_t^1 B_0 + \dots + (M_t^s - I)M_t^{s-1} \dots M_2^1 M_1^1 B_0}_{\vdots} \\ &= \quad \quad \quad B_{s-1} \quad \quad \quad + (M_t^s - I)M_t^{s-1} \dots M_2^1 M_1^1 B_0 \end{aligned} \quad (11)$$

Such a representation implies that the temporal Leontief inverse depends on the temporal *evolutionary tail* of the technological changes and this dependence is highly non-linear. The expressions (8) and (10) can serve as a basis of temporal analysis of the evolving input-output system. For example, if f_s is a final demand vector in the s^{th} time-period then the corresponding gross output vector $g_s = B_s f_s$ can be decomposed into a sum of the effects of first time-period, second time-period, ect., s^{th} time-period increments, as follows :

$$\begin{aligned} g_s = B_s f_s &= \quad \quad \quad f_s + \\ &\quad \quad \quad (B_0 - I)f_s + \\ &\quad \quad \quad (M_t^1 - I)B_0 f_s + \\ &\quad \quad \quad (M_t^2 - 1)M_t^1 B_0 f_s + \\ &\quad \quad \quad \vdots \\ &\quad \quad \quad (M_t^s - I)M_t^{s-1} \dots M_2^1 M_1^1 B_0 f_s \end{aligned} \quad (12)$$

3. Fine structure of temporal increments $D(B, E)$

The following sections deal with methods designed to trace the impact of each change in the individual direct inputs in each time period to the final state of economy. The analytical apparatus used required the identification of the fine structure of temporal multipliers and increments.

The fine structure of the increment, $D(B, E)$, can be ascertained with the help of the following important decomposition (Sonis and Hewings, 1989) :

$$D(B, E) = \frac{1}{Q(E)} \left[\sum_{k=1}^n \sum'_{\substack{ir < ir+1 \\ jr = js}} F(i_1, i_2, \dots, i_k; j_1, j_2, \dots, j_k) e_{j_1 i_1} e_{j_2 i_2} \dots e_{j_k i_k} \right] \quad (13)$$

Here

$$Q(E) = \frac{\det B}{\det B(E)} = 1 - \sum_{i_1 j_1} b_{j_1 i_1} e_{i_1 j_1} + \sum_{k=1}^n (-1)^k \sum'_{\substack{ir < ir+1 \\ jr \neq js}} B_{0r}(j_1, j_2, \dots, j_k; i_1, i_2, \dots, i_k) e_{i_1 j_1} e_{i_2 j_2} \dots e_{i_k j_k}; \quad (14)$$

and $B_{0r}(j_1, j_2, \dots, j_k; i_1, i_2, \dots, i_k)$ is a determinant of order k that includes the components of the

Leontief inverse B from the ordered set of columns $i_1, i_2, \dots, i_k$ and rows $j_1, j_2, \dots, j_k$; further, in the sum Σ' , the products of changes $e_{i_1 j_1} e_{i_2 j_2} \dots e_{i_k j_k}$ that differ only by the order of multiplication, are counted only once.

Furthermore, the matrix field of influence $F(i_1, i_2, \dots, i_k; j_1, j_2, \dots, j_k)$ of the incremental changes $e_{i_1 j_1} e_{i_2 j_2} \dots e_{i_k j_k}$ includes the components

$$f_{ij}(i_1, i_2, \dots, i_k; j_1, j_2, \dots, j_k) = (-1)^k [B_{0r}(i_1, i_2, \dots, i_k; i; j_1, j_2, \dots, j_k, j) - b_{ij} B_{0r}(i_1, i_2, \dots, i_k; j_1, j_2, \dots, j_k)] =$$

$$= (-1)^k \begin{vmatrix} b_{i_1 j_1} & b_{i_1 j_2} & \dots & b_{i_1 j_k} & b_{i_1 j} \\ b_{i_2 j_1} & b_{i_2 j_2} & \dots & b_{i_2 j_k} & b_{i_2 j} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ b_{i_k j_1} & b_{i_k j_2} & \dots & b_{i_k j_k} & b_{i_k j} \\ b_{ij_1} & b_{ij_2} & \dots & b_{ij_k} & 0 \end{vmatrix}, \quad i, j = 1, 2, \dots, n \quad (15)$$

Equation (15) provides the basis for the following description of the structure of fields of influence:

1. The first order field of influence $F(i_1; j_1)$ of the increment $e_{j_1 i_1}$ is the matrix generated by a multiplication of the j_1^{th} column of the Leontief inverse B with its i_1^{th} row:

$$F(i_1; j_1) = \begin{pmatrix} b_{1j_1} \\ b_{2j_1} \\ \vdots \\ b_{nj_1} \end{pmatrix} (b_{i_1 1} b_{i_1 2} \dots b_{i_1 n}) \quad (16)$$

This means that if change, e , occurs only in one place, (i_1, j_1) , then the Leontief inverse $B(e)$ has a form:

$$B(e) = B + \frac{e}{1 - b_{j_1 i_1}} A(j_1; i_1) \quad (17)$$

or, in the original Sherman-Morrison (1950) coordinate form:

$$b_{ij}(e) = b_{ij} + \frac{b_{ij_1} b_{i_1 j} e}{1 - b_{j_1 i_1} e} \quad (18)$$

2. In the case when the changes occur in the i_1^{th} row of the matrix A , $(e_{i_1 1}, e_{i_1 2}, \dots, e_{i_1 n})$, then:

$$B(E) = B + \frac{\sum_{s=1}^n e_{i_1 s} F(s; i_1)}{1 - \sum_{s=1}^n b_{s i_1} e_{i_1 s}} \quad (19)$$

or, in the Sherman-Morrison (1949) coordinate form:

$$b_{ij}(E) = b_{ij} + \frac{b_{i i_1} \sum_{s=1}^n b_{s j} e_{i_1 s}}{1 - \sum_{s=1}^n b_{s i_1} e_{i_1 s}} \quad (20)$$

3. The second order synergetic interaction between two incremental changes $e_{j_1 i_1}$ and $e_{j_2 i_2}$ is reduced to the following linear combination of four first order fields of influence:

$$F(i_1, i_2; j_1, j_2) = -b_{i_1 j_1} F(i_2; j_2) + b_{i_1 j_2} F(i_2; j_1) + b_{i_2 j_1} F(i_1; j_2) - b_{i_2 j_2} F(i_1; j_1) \quad (21)$$

4. The following recurrent formula provides the possibility to present the field of influence of order k through the use of fields of influence of lesser orders $1, 2, \dots, k-1$:

$$F(i_1, \dots, i_k; j_1, \dots, j_k) = \frac{1}{k-1} \sum_{s=1}^k \sum_{r=1}^k (-1)^{r+s+1} b_{i_s j_r} F(i_1, \dots, i_{s-1}, i_{s+1}, \dots, i_k; j_1, \dots, j_{r-1}, j_{r+1}, \dots, j_k) \quad (22)$$

4. Fine structure of temporal multipliers

The fine structure of temporal multipliers, M_L and M_R , can be easily obtained from the properties of the fine structure of the temporal increment. Let us consider the left temporal multiplier, M_L ; a similar form would characterize the right temporal multiplier.

Equations (2), (4), (6) and (7) imply that

$$\begin{aligned} M_L(B, E) &= B(E)(I - A) = [B + D(B, E)](I - A) = I + D(B, E)(I - A) = \\ &= I + \frac{1}{Q(E)} \left[\sum_{k=1}^n \sum'_{i_r < i_{r+1} \atop j_r - j_s} F(i_1, \dots, i_k; j_1, \dots, j_k) e \right] S(j_1 i_1) e_{j_2 i_2} \dots e_{j_k i_k} (I - A) = \\ &= I + \frac{1}{Q(E)} \left[\sum_{k=1}^n \sum'_{i_r < i_{r+1} \atop j_r - j_s} G_L(i_1, \dots, i_k; j_1, \dots, j_k) e_{j_1 i_1} e_{j_2 i_2} \dots e_{j_k i_k} \right] \end{aligned} \quad (23)$$

where

$$Q(E) = \frac{\det B}{\det B(E)} = \det(I - BE) = \det(I - EB) \quad (24)$$

and

$$G_L(i_1, \dots, i_k; j_1, \dots, j_k) = F(i_1, \dots, i_k; j_1, \dots, j_k)(I - A) \quad (25)$$

are the multiplier fields of influence.

Using (25) immediately implies that:

$$G_L(i_1; j_1) = F(i_1; j_1)(I - A) = \begin{bmatrix} b_{1j_1} \\ b_{2j_1} \\ \vdots \\ b_{nj_1} \end{bmatrix} (b_{i_1 1} b_{i_1 2} \dots b_{i_1 n})(I - A) = \begin{bmatrix} b_{1j_1} \\ b_{2j_1} \\ \vdots \\ b_{nj_1} \end{bmatrix} (0, \dots, 1, 0, \dots, 0) \quad (26)$$

This matrix includes only one non-zero i_1^{th} column $\begin{bmatrix} b_{1j_1} \\ b_{2j_1} \\ \vdots \\ b_{nj_1} \end{bmatrix}$. Further, the components $g_{ij}(i_1, i_2, \dots, i_k; j_1, j_2, \dots, j_k)$ of the multiplier field of influence $G_L(i_1, i_2, \dots, i_k; j_1, \dots, j_k)$ can be calculated with the help of formula:

$$g_{ij}(i_1, i_2, \dots, i_k; j_1, j_2, \dots, j_k) = \sum_{s=1}^k f_{is}(i_1, i_2, \dots, i_k; j_1, j_2, \dots, j_k) c_{sj}$$

where $f_{is}(i_1, i_2, \dots, i_k; j_1, j_2, \dots, j_k)$ are the components of the field of influence, $F(i_1, \dots, i_k; j_1, \dots, j_k)$, and c_{sj} are the components of the matrix $I - A$. Therefore,

$$g_{ij}(i_1, i_2, \dots, i_k; j_1, j_2, \dots, j_k) = (-1)^{k+1} \sum_{s=1}^k d_{isj} B_{0r}(i_1, \dots, i_{s-1}, i, i_{s+1}, \dots, i_k; j_1, \dots, j_k) \quad (27)$$

where d_{isj} are Kroneker's symbols. Thus, the multiplier field of influence $G_L(i_1, \dots, i_k; j_1, \dots, j_k)$ includes only k non-zero $i_1, i_2, \dots, i_k$ columns. Moreover, the following recurrent formula holds:

$$G_L(i_1, \dots, i_k; j_1, \dots, j_k) = \frac{1}{k-1} \sum_{s=1}^k \sum_{r=1}^k (-1)^{r+s+1} b_{isjr} (G_L(i_1, \dots, i_{s-1}, i_{s+1}, \dots, i_k; j_1, \dots, j_{r-1}, j_{r+1}, \dots, j_k)) \quad (28)$$

5. Cross-structure of the direct fields of influence

Assume that the Hawkins-Simon condition holds, i.e., all principal minors of the matrix $I - A$ are positive; then all the components b_{ij} of the Leontief inverse $(I - A)^{-1}$ are non-negative. Therefore, all components $f_{ij}[(i_1, j_1)] = b_{i_1 i_1} b_{j_1 j_1}$ of the matrix of the direct field of influence $F[(i_1, j_1)]$ are non-negative as well. Moreover, the maximal and minimal components of the direct field of influence are the products of maximal and, correspondingly, minimal components of the i_1^{th} row and

j_1^{th} column of the Leonitef inverse :

$$\begin{aligned} \max_{ij} f_{ij}[(i, j_1)] &= (\max_i b_{ii_1})(\max_j b_{j_1j}) \\ \min_{ij} f_{ij}[(i, j_1)] &= (\min_i b_{ii_1})(\min_j b_{j_1j}) \end{aligned} \quad (29)$$

Further, all columns (rows) of the matrix $F(i_1, j_1)$ are proportional :

$$\begin{aligned} \frac{f_{i1}}{f_{s1}} = \frac{f_{i2}}{f_{s2}} = \dots = \frac{f_{ij}}{f_{sj}} = \dots = \frac{f_{in}}{f_{sn}} = \frac{b_{ii_1}}{b_{si_1}} \\ \frac{f_{1j}}{f_{1k}} = \frac{f_{2j}}{f_{2k}} = \dots = \frac{f_{ij}}{f_{ik}} = \dots = \frac{f_{nj}}{f_{nk}} = \frac{b_{j_1j}}{b_{j_1k}} \end{aligned} \quad (30)$$

Rearranging the components of the i_1^{th} row and j_1^{th} column of the Leontief inverse according to size, provides the following :

$$\begin{aligned} b_{s_1 i_1} \geq b_{s_1 i_1} \geq \dots \geq b_{s_n i_1} \quad (b_{s_1 i_1} = \max_i b_{ii_1}; \quad b_{s_n i_1} = \min_i b_{ii_1}); \\ b_{j_1 r_1} \geq b_{j_1 r_2} \geq \dots \geq b_{j_1 r_n} \quad (b_{j_1 r_1} = \max_j b_{j_1 j}; \quad b_{j_1 r_n} = \min_j b_{j_1 j}). \end{aligned}$$

These proportionality properties imply that the biggest components of the direct fields of influence $F[(i, j_1)]$ are included in the cross (s_1, r_1) generated by the s_1^{th} row and r_1^{th} column in such a way that for each column (row) of the matrix $F[(i, j_1)]$ the largest element lies in the s_1^{th} row (r_1^{th} column). Further, if the cross (s_1, r_1) is excluded from the matrix $F[(i, j_1)]$, then the next cross (s_2, r_2) will include the largest remaining elements; the same property holds for the succeeding crosses (s_3, r_3) , ..., (s_n, r_n) . An analogous picture is true for the minimal components; their hierarchy is given by the inverse sequence of crosses

$$(s_n, r_n), (s_{n-1}, r_{n-1}), \dots, (s_1, r_1).$$

This cross-structure of the direct fields of influence is very essential for the construction of the basic elements of the key sector analysis.

6. Fields of influence and Structural Path Analysis

The field of influence approach supplies an easy way to obtain the matrix form of the structural path analysis elaborated two decades ago (see, Thorbecke, 1991). Replace, in (23) and (24) the matrix B by the unit matrix I , and the matrix E by the matrix A ; then :

$$B(I-A)^{-1} = I + \frac{1}{\det(I-A)} \left[\sum_{k=1}^n \sum'_{\substack{i_r < i_{r+1} \\ j_r - j_s}} G_i^k(i_1, \dots, i_k; j_1, \dots, j_k) a_{j_1 i_1} a_{j_2 i_2} \dots a_{j_k i_k} \right] \quad (31)$$

where matrices $G_i^k(i_1, \dots, i_k; j_1, \dots, j_k)$ are the multiplier fields of influence of the unit matrix I and can be computed with the help of (20) and (21). It is obvious that the fields of influence $G_i^k(i; j)$ of the first order for the unit matrix are equal to the matrix P_{ij} with only one non-zero unit component in the j^{th} row and i^{th} column. Therefore :

$$\sum_{i_1 j_1} G_i^1(i_1; j_1) a_{j_1 i_1} = A$$

and

$$B = (I-A)^{-1} = I + \frac{1}{\det(I-A)} \left[A + \sum_{k=2}^n \sum'_{\substack{i_r < i_{r+1} \\ j_r - j_s}} G_i^k(i_1, \dots, i_k; j_1, \dots, j_k) a_{j_1 i_1} a_{j_2 i_2} \dots a_{j_k i_k} \right] \quad (32)$$

Further, equation (28) implies that the multiplier fields of influence $G_i^k(i_1, \dots, i_k; j_1, \dots, j_k)$ are linear combinations of the matrix of the form P_{ji} with the coefficients $\frac{1}{(k-1)!}, -\frac{1}{(k-1)!}, 0$.

7. Summary Remarks

The concept of a field of influence has proven to be a general, useful tool whose application extends across a variety of techniques within input-output and social accounting systems. In particular, it offers the potential for uncovering the set of analytically important elements and thus provides the basis for concentrating resources in updating input-output tables. Since many nations are now abandoning five-year, full census instruments, carefully focused efforts on sample survey information will be required. The field of influence technique offers an important contribution in this area. The new areas of application —the temporal Leontief inverse and the link with structural path analysis offer considerable promise for the future, especially in uncovering ways in which system changes are manifested throughout an economy. The analysis also has proven of value in linkages with other decomposition techniques and analyses of structural change; these issues are addressed in sections II and III of this paper.

Applications : Hewings and Jensen, 1991 ; Hewings, Sonis and Jensen, 1988a, 1988b ; Sonis and Hewings, 1989, 1991, 1992, 1995 ; Sonis, Hewings and Jensen, 1991 ; Sonis, Hewings and Lee, 1994.

II. Economic hierarchical landscapes of backward and forward linkages : key sector analysis, fields of influence and the multiplier product matrix

1. Key sector analysis and multiplier product matrix (MPM)

The notion of backward and forward linkages and the corresponding concept of key sectors is associated with the work of both Rasmussen (1956) and Hirschman (1958). The major thrust of the economic concept has been towards the identification of sectors whose linkages are such that they create an above-average impact on the rest of economy. The analytical techniques are based on the calculation of the column and row multipliers of the Leontief Matrix $B = \| b_{ij} \|$

$$B_{\cdot i} = \sum_{i=1}^n b_{ii}, B_{j \cdot} = \sum_{j=1}^n b_{ji} \quad (33)$$

and the global intensity of the Leontief inverse :

$$V = \sum_{i,j=1}^n b_{ij} \quad (34)$$

Rasmussen (1956) proposed two types of indices :

1. Power of dispersion for the backward linkages :

$$BL_j = \frac{1}{n} \sum_{i=1}^n b_{ij} / \frac{1}{n} \sum_{i,j=1}^n b_{ij} = nB_{\cdot j} / b \quad (35)$$

and

2. Sensitivity of dispersion for forward linkages :

$$FL_i = \frac{1}{n} \sum_{j=1}^n b_{ij} / \frac{1}{n} \sum_{i,j=1}^n b_{ij} = nB_{i \cdot} / V \quad (36)$$

Obviously

$$BL_j = nB_{\cdot j} / V ; FL_i = nB_{i \cdot} / V \quad (37)$$

Thus, the definition of backward and forward linkages provided by (35) and (36) implies that the rank-size ordering of these indices coincides with the rank-size hierarchies of the column and row multipliers.

Let us consider the following matrix —the multiplier product matrix— MPM :

$$M = \| m_{ij} \| = \frac{1}{V} \| B_{i \cdot} B_{j \cdot} \| = \frac{1}{V} \begin{bmatrix} B_{1 \cdot} \\ B_{2 \cdot} \\ \vdots \\ B_{n \cdot} \end{bmatrix} (B_{\cdot 1} B_{\cdot 2} \dots B_{\cdot n}) \quad (38)$$

It is important to underline that the column and row multipliers of MPM are the same as those of the Leontief inverse. Furthermore, MPM has the same cross-structure as the direct fields of influence: the largest element of MPM is the product of the largest column and row multipliers, and, hence, takes on a form similar to that proposed in section I.5. One can reorganize the locations of rows and columns of MPM in such a way that the descending sequence of the center of each cross appears on the main diagonal; thus a descending *economic landscape* will be apparent (see figure 1). This rearrangement also reveals the descending rank-size hierarchies of Hirschman-Rasmussen indices. Inspection of that part of the landscape with indices >1 (the usual criterion for specification of key sectors enables the identification of key sectors.

It is important to stress that the construction of economic landscapes for different regions or for the same region at different time periods creates the possibility for the establishment of a taxonomy of these economies on the basis of visual representation of the similarities and differences in the linkage structures of their economies.

2. MPM as the matrix of intensities of direct fields of influence

Here we represent the important connection between the MPM and the intensities of point-wise changes in the matrix of direct inputs. Since the elements of the Leontief inverse and, therefore, the elements of each direct field of influence are positive (non-negative) numbers, one can define the intensity of the direct field of influence $F[(i, j)] = \|f_{ij}(i, j)\|$ as

$$Int F[(i, j)] = \sum_{ij} f_{ij}[(i, j)] = B_{\cdot i} B_{j \cdot} \quad (39)$$

Figure 1 A Descending Economic Landscape Associated with the MPM.

The next step is the construction of the first order Intensity matrix whose components are the normalized intensities of the direct (first order) fields of influence:

$$INT_1 = \frac{1}{V} \| \text{Int } F(i, j) \| \quad (40)$$

Formulae (37), (38) imply that

$$INT_1 = \frac{1}{V} \| B_{i.} B_{j.} \| = \frac{1}{V} \begin{bmatrix} B_{1.} \\ B_{2.} \\ \vdots \\ B_{n.} \end{bmatrix} (B_{.1} B_{.2} \dots B_{.n}) = M \quad (41)$$

Therefore, the first order Intensity Matrix, INT_1 , coincides with the MPM and, thus, can be decomposed into the hierarchy of crosses. The location of the first cross is defined by the place of the direct field of influence with maximal intensity measured by the product of largest row and column multipliers. The elements of rows and columns of the first order Total Intensity Matrix are proportional (but smaller) to these of the first cross. Excluding the first cross from INT_1 one can derive next cross with analogous features, and the process can continue until all rows and columns are exhausted.

Thus, key sector analysis, based on the static description of input-output economies, describes also the directions of some of the more important changes in these economies.

3. Second order (synergetic) fields of influence and their intensities

Using (16), the following is true for the components of the matrix of the second order field of influence $F[(i_1, j_1), (i_2, j_2)] = \| f_{ij}[(i_1, j_1), (i_2, j_2)] \|$:

$$f_{ij}[(i_1, j_1), (i_2, j_2)] = \begin{vmatrix} b_{i_1 j_1} & b_{i_2 j_2} & b_{i_1 j} \\ b_{i_2 j_1} & b_{i_2 j_2} & b_{i_2 j} \\ b_{i j_1} & b_{i j_2} & 0 \end{vmatrix} \quad (42)$$

The intensity of the individual second order field of influence can be defined as a sum of all components of the matrix $F[(i_1, j_1), (i_2, j_2)]$:

$$\text{Int } A[(i_1, j_1), (i_2, j_2)] = \sum_{ij} f_{ij}[(i_1, j_1), (i_2, j_2)] \quad (43)$$

Using (42) and rules for the summation of determinants, one obtains:

$$\text{Int } F[(i_1, j_1), (i_2, j_2)] = \begin{vmatrix} b_{i_1 j_1} & b_{i_2 j_2} & B_{i_1.} \\ b_{i_2 j_1} & b_{i_2 j_2} & B_{i_2.} \\ B_{.j_1} & B_{.j_2} & 0 \end{vmatrix} \quad (44)$$

where $B_{i_1.}$, $B_{i_2.}$, $B_{.j_1}$, $B_{.j_2}$ are the corresponding row and column multipliers of the Leontief inverse B . Therefore,

$$\begin{aligned} \text{Int } F[(i_1, j_1), (i_2, j_2)] &= b_{i_1 j_2} B_{i_2.} B_{.j_1} + b_{i_2 j_1} B_{i_1.} B_{.j_2} - (b_{i_1 j_1} B_{i_2.} B_{.j_2} + b_{i_2 j_2} B_{i_1.} B_{.j_1}) = \\ &= \text{Int}^+ F[(i_1, j_1), (i_2, j_2)] - \text{Int}^- F[(i_1, j_1), (i_2, j_2)] \end{aligned} \quad (45)$$

where $\text{Int}^+ F[(i_1, j_1), (i_2, j_2)]$ and $\text{Int}^- F[(i_1, j_1), (i_2, j_2)]$ are the positive and negative parts of the Intensity $\text{Int } F[(i_1, j_1), (i_2, j_2)]$.

Consider, further, for each cell, (i_1, j_1) , the following sums:

$$\begin{aligned} \text{Int } F[(i_1, j_1), (\bullet, \bullet)] &= \sum_{i_2 j_2} \text{Int } F[(i_1, j_1), (i_2, j_2)]; \\ \text{Int}^+ F[(i_1, j_1), (\bullet, \bullet)] &= \sum_{i_2 j_2} \text{Int}^+ F[(i_1, j_1), (i_2, j_2)]; \\ \text{Int}^- F[(i_1, j_1), (\bullet, \bullet)] &= \sum_{i_2 j_2} \text{Int}^- F[(i_1, j_1), (i_2, j_2)]. \end{aligned} \quad (46)$$

Using the formula (44), the following is true:

$$Int F[(i, j), (\bullet, \bullet)] = \begin{vmatrix} b_{ij} & B_{i\cdot} & B_{i\cdot} \\ B_{\cdot j} & V & V \\ B_{\cdot j} & V & 0 \end{vmatrix} = 2VB_{i\cdot}B_{\cdot j} - (V^2b_{ij} + VB_{i\cdot}B_{\cdot j}) \quad (47)$$

Further, (44) implies:

$$\begin{aligned} Int^+ F[(i, j), (\bullet, \bullet)] &= 2VB_{i\cdot}B_{\cdot j} \\ Int^- F[(i, j), (\bullet, \bullet)] &= V^2b_{ij} + VB_{i\cdot}B_{\cdot j}, \end{aligned} \quad (48)$$

and

$$Int F[(i, j), (\bullet, \bullet)] = Int^+ F[(i, j), (\bullet, \bullet)] - Int^- F[(i, j), (\bullet, \bullet)] \quad (49)$$

Therefore, the following total second order Intensity Matrix can be defined:

$$INT_2 = \frac{1}{V^2} \| Int F[(i, j), (\bullet, \bullet)] \| \quad (50)$$

and its positive and negative parts:

$$INT_2^+ = \frac{1}{V^2} \| Int^+ F[(i, j), (\bullet, \bullet)] \|; \quad INT_2^- = \frac{1}{V^2} \| Int^- F[(i, j), (\bullet, \bullet)] \| \quad (51)$$

It is easy to obtain that:

$$INT_2^+ = 2INT_1; \quad INT_2^- = B + INT_1; \quad (52)$$

and

$$INT_2 = INT_2^+ - INT_2^- = INT_1 - B. \quad (53)$$

Thus, the matrix INT_2^+ has a cross-structure, while for the matrices INT_2^- and INT_2 , this cross-structure is superimposed on the structure of the Leontief inverse, B .

Equation (53) can be rewritten as a fundamental decomposition of the Leontief inverse into the difference between the MPM presenting the intensities of direct fields of influence and the intensities of synergetic fields of influence:

$$B = M - INT_2 \quad (54)$$

The importance of this formula can be underlined by the fact that MPM, within the class of all matrices with the same column and row multipliers, provides the maximal value of the Shannon entropy (Sonis, 1968, Wilson, 1970). Therefore, MPM may be considered as a standard representing the most homogeneous distribution of the components of the column and row multipliers of the Leontief inverse. The synergetic intensities of changes, INT_2 , thus represent the sectoral specifics in the transfer of economic influence.

3. Summary

Even though many methods have been developed to measure structure, comparative analysis, especially visual, has proven elusive. In applications of the MPM methodology to selected Chinese metropolitan regions, it became clear that there were significant differences in their structures, differences that were poorly defined when reference was made only to relative distributions of sectoral activity. Obviously, the methodology provides no explanatory power, but serves as a first stage in differentiating structures. In other applications, it was possible to visualize the hollowing-out process characterizing the Chicago region over the period 1970-2018 without recourse to further, more sophisticated analyses. Subsequently, the process of explication could move ahead but within a framework provided by the first-cut visualization afforded by mpm.

Applications: Hewings, Guo, Guilhoto and Sonis, 1994; Sonis, Guo and Hewings, 1996; Sonis and Oosterhaven, 1996; Hewings, Israilevich, Sonis and Schindler, 1996.

III. Economic structural changes in the input-output systems: Barycentric calculus of triple decompositions of gross output changes

An important component of the modern process of industrialization is the change in the nature of interdependence in production summarized in input-output and social accounting tables. Analysis of the evolution of interindustry relations has now become, once more, a focal point of interest of economic analysts. The traditional approach, proposed by Chenery (1953), Chenery and Watanabe (1958) and Chenery and Clark (1959) has been extended further in various subsequent studies (see Carter, 1970; Long Jr., 1970; Ohkawa and Rosovsky, 1973; Song, 1977; Matthews *et al.*, 1982; Harrigan *et al.*, 1980; Syrquin, 1988; Deutsch and Syrquin, 1989 among others).

The main purpose of this section is to explore the methodologies that will assist in understanding the economic structural changes in an economy caused by simultaneous technological changes and changes in final demand. Within the Leontief input-output framework, the problem focuses on the decomposition of changes in gross sectoral output into the portion directly and indirectly caused by changes in the value and mix of final demand and the portion caused by changes in the technological coefficients themselves. In this regard, the empirical evidence provided by Feldman, McClain and Palmer (1987) for the United States revealed that for nearly 80 percent of 400 industries in the period 1963-1978, final demand changes accounted for the majority of gross output changes. Unfortunately, these results are of the restricted importance because of the manner in which the authors treated the synergetic effects of the simultaneous changes in technology and final demand, attributing half of the synergetic effects to changes related to technology and the other half to the changes associated with final demand. It is important to underline that such an attribution distorts the overall picture of the structure of gross output because the synergetic effects are not small: in some sectors they can account for as much as 25 percent of the output. Therefore, attention is focused on the triple decomposition of the components of the gross output, separating the synergetic interactions from the pure effects of changes in technology and in final demand. The triple decomposition of the outputs in each sector also provides the opportunity to use the analytical and graphical technique of the Möbius Barycentric Calculus in the form of a triangular diagram of changes in the components of gross outputs (Sonis, 1981). The use of such a diagram enables the elaboration of a taxonomy of economic structural changes and the description of the evolutionary changes in economies.

Furthermore, the application of the of fields of influence approach reveals the fine structure of such triple decompositions in greater detail, considering the changes separately of technological coefficients and their clusters, those associated with final demand as well as their different combinations. Complementary approaches are provided in Dewhurst (1993), and Rose and Casler (1996).

1. The analysis of the Feldman *et al.* (1987) approach

In their work, Feldman, McClain and Palmer (1987) extended the Chenery and Clark (1959) double decomposition of the gross output by introducing the following procedure of the analysis of influence of changes in technical coefficients of the input-output model and in the components of final demand: let A_{old} and A_{new} be the matrices of direct inputs, $B_{old} = (1 - A_{old})^{-1}$, $B_{new} = (1 - A_{new})^{-1}$ be the Leontief inverses, and f_{old} and f_{new} be the vectors of final demand before and after change, respectively. Then:

$$\begin{aligned} X_{old} &= (1 - A_{old})^{-1} f_{old} = B_{old} f_{old}, \\ X_{new} &= (1 - A_{new})^{-1} f_{new} = B_{new} f_{new} \end{aligned}$$

are the vectors of gross output before and after change. Further, let:

$$\Delta X = X_{new} - X_{old}; \Delta B = B_{new} - B_{old}; \Delta f = f_{new} - f_{old}.$$

(It is assumed, further, that the matrices of direct input coefficients are $n \times n$ matrices, the vectors of final demand are $n \times 1$ vectors and the matrices of "bridge coefficients" are $n \times n$ unit matrices).

Feldman *et al.* (1987) proposed the following decomposition of the change in gross output

$$\Delta X = X_{new} - X_{old} = B_{new}f_{new} - B_{old}f_{old} \quad (55)$$

into two components:

$$1/2[B_{new}(f_{new} - f_{old}) + B_{old}(f_{new} - f_{old})] \quad (56)$$

representing the contribution of final demand change to change in gross output, and

$$1/2[(B_{new} - B_{old})f_{old} + (B_{new} - B_{old})f_{new}] \quad (57)$$

representing the contribution of technical coefficient change to gross output change.

It is easy to see that the change in gross output (55) is:

$$\Delta X = B_{old}\Delta f + \Delta B\Delta f + \Delta Bf_{old} \quad (58)$$

According to the remark of the referee of the Feldman *et al.* paper (1987) the component (56) can be presented as:

$$B_{old}\Delta f + 1/2\Delta B\Delta f \quad (59)$$

while the other component (57) is

$$\Delta Bf_{old} + 1/2\Delta B\Delta f \quad (60)$$

Therefore, it is evident that the Feldman *et al.* (1987) decomposition of the change in gross output ΔX into two components (59) and (60) cannot separate the changes caused by change in final demand and in technological coefficients. In contrast, the triple decomposition (58) includes the components $B_{old}\Delta f$ representing the influence of change in final demand under the conditions of unchanged technology; the component ΔBf_{old} representing the influence of changes in technology under the conditions of unchanged final demand; and the component $\Delta B\Delta f$ representing the synergetic effects of the interaction between simultaneous changes in technology and final demand.

The consideration of these three components of change in gross output can be elaborated along with detailed analysis of the structure of changes ΔB and Δf through the conceptual framework of Möbius Barycentric Calculus (Möbius, 1827; Sonis, 1981) and the concept of fields of influence.

2. Triangular diagram (triangram) of relative significance of components of gross output changes

The purpose of this section is to present the use of the elements of the Möbius Barycentric Calculus (see Sonis, 1981) for the analysis of changes in gross output. The formula (58) gives the decomposition of the gross output change Δx_i for each i^{th} sector into three components: Δx_i^f , Δx_i^B , Δx_i^{Bf} , representing the changes caused by change in final demand, f , technological coefficients, B , and their synergetic interaction Bf :

$$\Delta x_i = \Delta x_i^f + \Delta x_i^B + \Delta x_i^{Bf} \quad (61)$$

Assuming that $\Delta x_i \neq 0$, introduce the relative changes:

$$p_i = \Delta x_i^f / \Delta x_i; \quad q_i = \Delta x_i^B / \Delta x_i; \quad r_i = \Delta x_i^{Bf} / \Delta x_i \quad (62)$$

with a property

$$p_i + q_i + r_i = 1 \quad (63)$$

The triangular diagram (triangram) is a graphical tool for the presentation of the statistical space of all decompositions of type (62). The triangram is based on the equilateral triangle (the Möbius triangle) with unit sides. The components of the equality (63) are the barycentric coordi-

nates of some point within a plane. Their geometrical presentation is based on the three coordinate axes $-\infty < p < +\infty$, $-\infty < q < +\infty$, $-\infty < r < +\infty$, generating the Möbius triangle (see figure 2). For each point inside the triangle, P , or outside the triangle, Q , its barycentric coordinates are generated by projecting it parallel to the axes. Figure 3 represents the domains on the plane corresponding to different combinations of signs of the components p , q , r for positive or negative (in brackets) change in gross outputs. These domains represent fourteen possible qualitative categories of change. The triagram can be used for the clustering of sectors according these qualitative categories of change in gross outputs. It can be also used for the representation of the evolution of changes in time and space.

3. Fields of influence approach to the analysis of changes in gross output

Fields of influence provide the fine structure of decomposition of the change, ΔX , in gross output in greater detail. Using (58) and (61), one obtains:

$$\begin{aligned} \Delta X = & B_{old} \Delta f + \frac{1}{Q(E)} \sum_{k=1}^n (-1)^k \sum_{\substack{i_r \neq i_s \\ j_r \neq j_s}} F \begin{pmatrix} i_1 & i_2 & \dots & i_k \\ j_1 & j_2 & \dots & j_k \end{pmatrix} f_{old}(e_{j_1 i_1} e_{j_2 i_2} \dots e_{j_k i_k}) + \\ & + \frac{1}{Q(E)} \sum_{k=1}^n (-1)^k \sum_{\substack{i_r \neq i_s \\ j_r \neq j_s}} F \begin{pmatrix} i_1 & i_2 & \dots & i_k \\ j_1 & j_2 & \dots & j_k \end{pmatrix} \Delta f(e_{j_1 i_1} e_{j_2 i_2} \dots e_{j_k i_k}) \end{aligned} \quad (64)$$

The sum of the first and third components in (64) is equal to $B_{new} \Delta f$ representing the portion of change in gross output attributed to change in final demand weighted by the Leontief inverse after the change. Thus:

$$\Delta X = B_{new} \Delta f + \frac{1}{Q(E)} \sum_{k=1}^n (-1)^k \sum_{\substack{i_r \neq i_s \\ j_r \neq j_s}} F \begin{pmatrix} i_1 & i_2 & \dots & i_k \\ j_1 & j_2 & \dots & j_k \end{pmatrix} f_{old} e_{j_1 i_1} e_{j_2 i_2} \dots e_{j_k i_k} \quad (65)$$

In the decomposition (64), the first component $B_{old} \Delta f$ represents the portion of change in the gross output attributed to changing final demand weighted by the Leontief inverse before change. Further analysis of this component can be accomplished if one represents the vector, Δf , of changes in final demand in the form:

Figure 2 The Möbius Triangle.

Figure 3 Positive and Negative relative Changes in Gross.

Classification	Sign Δp_i	Sign Δq_i	Sign Δr_i	Qualitative description of changes
1	+	+	+	Growth in influence of final demand and technology; decline in synergetic effect
2	+	-	-	Growth in influence of final demand; decline in technology and synergetic effect
3	+	-	+	Growth in influence of final demand and interaction; decline in technology
4	-	-	+	Growth in synergetic effect; decline in final demand and technology
5	-	+	+	Growth in technology and synergetic effect; decline in final demand
6	-	+	-	Growth in technology, decline in final demand and synergetic influence

$$\Delta f = \begin{bmatrix} \Delta f_1 \\ \Delta f_2 \\ \vdots \\ \Delta f_n \end{bmatrix} = \sum_{l=1}^n \Delta f_l h_l \quad (66)$$

where each vector h_l represents the unit change of final demand in the l^{th} sector:

$$h_l = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad l=1, 2, \dots, n \quad (67)$$

Then

$$B_{old}\Delta f = \sum_{l=1}^n \Delta f_l B_{old} h_l = \sum_{l=1}^n \Delta f_l (B_{old})_l = \sum_{l=1}^n \Delta f_l \begin{bmatrix} b_{1l} \\ b_{2l} \\ \vdots \\ b_{nl} \end{bmatrix} \quad (68)$$

where the vector $(B_{old})_l = \begin{bmatrix} b_{1l} \\ b_{2l} \\ \vdots \\ b_{nl} \end{bmatrix}$ represents the l^{th} vector-column of the Leontief inverse B_{old} .

The second component of the decomposition (64) represents the contributions of each individual field of influence weighted by the final demand f_{old} before change. If one presents f_{old} in the form :

$$f_{old} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} = \sum_{l=1}^n f_l h_l \quad (69)$$

then the second component in (64) will obtain the form :

$$\frac{1}{Q(E)} \sum_{k=1}^n (-1)^k \sum_{\substack{i_r \neq i_s \\ j_r \neq j_s}} \sum_{l=1}^n F_l \begin{bmatrix} i_1 & i_2 & \dots & i_k \\ j_1 & j_2 & \dots & j_k \end{bmatrix} f_l e_{j_1 i_1} e_{j_2 i_2} \dots e_{j_k i_k} \quad (70)$$

where the expression $F_l \begin{bmatrix} i_1 & i_2 & \dots & i_k \\ j_1 & j_2 & \dots & j_k \end{bmatrix}$ represents l^{th} vector-column of the matrix field of influence F_l $\begin{bmatrix} i_1 & i_2 & \dots & i_k \\ j_1 & j_2 & \dots & j_k \end{bmatrix}$.

Analogously, the third component in (64) is the portion of the gross output change attributed to the impacts of all individual fields of influence weighted by the changes in final demand. In greater detail this componens has a form :

$$\frac{1}{Q(E)} \sum_{k=1}^n (-1)^k \sum_{\substack{i_r \neq i_s \\ j_r \neq j_s}} \sum_{l=1}^n F_l \begin{bmatrix} i_1 & i_2 & \dots & i_k \\ j_1 & j_2 & \dots & j_k \end{bmatrix} \Delta f_l e_{j_1 i_1} e_{j_2 i_2} \dots e_{j_k i_k} \quad (71)$$

4. Simultaneous marginal changes in technological coefficients and in final demand

The information about the economic structure of the input-output system which is given by equations (64)-(71) can be decoded further if one concentrates attention on the partial sets of changes in technological coefficients and/or the subsets of components of final demand. Especially important here is the case of simultaneous marginal changes in the individual components.

Consider the change e in the cell (i_l, j_l) of the matrix of direct inputs A and the change π in the final demand of sector l . Then

$$\begin{aligned} \Delta X(i_l, j_l; l) &= \pi (B_{old})_l + \frac{e}{1 - b_{j_l i_l}} F \begin{bmatrix} i_l \\ j_l \end{bmatrix} f_{old} + \frac{e\pi}{1 - b_{j_l i_l}} F(i_l, j_l) = \\ &= \pi (B_{old})_l + \frac{e}{1 - b_{j_l i_l}} (B_{old})_{j_l} (B_{old})_{i_l} f_{old} + \frac{e\pi}{1 - b_{j_l i_l}} F_l(i_l, j_l) \end{aligned} \quad (72)$$

The further penetration into the economic structure can be achieved with the help of the triangram representing the triple decompositions of the components $\Delta X(i_l, j_l; l)$ of the gross output for all possible i_l, j_l and l .

5. Summary

Even though few economies have been blessed with a long time series of input-output tables, there are enough cases where the series is extensive enough to warrant a careful examination of structural changes. Again, as in the case of the MPM application, visualization of changes has been a major driving force in the application of the Möbius approach. In particular, the possibility

for examining the evolution of changes provides a unique opportunity to gain insights into the paths of change. In addition, comparative analysis is equally possible with the attendant advantage of visual realization of changes in the relative contributions of the components. If further components are added (for example, the components of final demand are shown individually), the picture becomes less transparent.

Applications: Sonis, 1981 ; Sonis, Guilhoto, Hewings and Martins, 1995 ; Sonis, Hewings and Bronstein, 1994.

IV. Structural Q-analysis

In this section, attention is directed to the application of the ideas of Atkin's Structural Q-analysis for the analysis of economic structure of input-output systems. In this interpretation, structural Q-analysis is viewed as a part of the newly developed qualitative input-output analysis (QIOA). A central concern of QIOA is structure and structural change: the regional and multi-regional fundamental economic structures appear in a form of hierarchies of interacting economic subsystems. Their structural changes need new tools for illustration and visualization. The QIOA include, at present, recently elaborated methods such as minimal flow analysis, economic filières analysis, MPM, feedback loop flow analysis and structural Q-analysis. Until now, these new branches of the QIOA have been developed independently without much interaction by different economic research groups in Germany, France, USA and Israel. For example, qualitative input-output analysis began in German speaking world with the work of Czayka (1972), Schnabl and Holub (1979), Holub, Schnabl and Tappeiner (1985), and were followed by further work of Holub and Tappeiner (1987), and Schnabl (1992, 1994). In France, some other applications of graph-theoretical methods for structuring production (so called causal channel method) have been brought forward by Bellet, Lallich and Vincent (1989) and the concept of filières introduction by De Bresson and Torre (199x). Over the period 1988-1995, the work of Sonis and Hewings brought forward the field of influence approach and its qualitative interpretation in the multiplier product matrix, as described in earlier sections of this paper. In addition to MPM, feedback loop analysis was explored and, more recently, structural Q-analysis. Recently, it became clear that the conceptual unification of these branches of the QIOA is possible on the basis of the ideas of combinatorial topology and graph-theoretical methods leading to a visualization of the economic structures and their changes.

In the following section, the operational procedure of structural Q-analysis based on ideas of algebraic topology is provided and, thereafter, its application to the analysis of the Leontief Inverse (this procedure is applicable to each matrix of economic interactions).

1. Slicing procedure

Consider the Leontief Inverse matrix $B = \| b_{ij} \|$ and let $(i_1, j_1), (i_2, j_2), \dots, (i_m, j_m)$ be a fixed set of pairs of economic sectors entering the input-output system. Let $b_{i_1 j_1}, b_{i_2 j_2}, \dots, b_{i_m j_m}$ be the corresponding components of the matrix B . The *slicing procedure* results in a choice of a new matrix B_s whose only non-zero components are $b_{i_1 j_1}, b_{i_2 j_2}, \dots, b_{i_m j_m}$ while all other components are zeroes. This slicing procedure, referred to under the name of *variable filter* approach is the basic element of the minimal flow analysis (see, Holub, Schnabl, Tappeiner, 1985 ; Schnabl, 1993).

The matrix I_s with unit entries in place of non-zero components of the matrix B_s is called the *incidence matrix* associated with the slicing procedure. Obviously, 2^{n^2} different slicing procedures exist for each $n \times n$ influence matrix B . The simplest slicing procedure consists of the choice of the slicing parameter μ , and the exclusion from the matrix B of all components b_{ij} such that $b_{ij} < \mu$. The choice of a definite slicing parameter depends on the investigator's view-point on the economic structure of interaction matrix.

2. Simplicial families for forward linkages

At the beginning of the procedure of Q-analysis, attention will be focused on the forward linkages. Consider a slicing procedure defined with the help of the set of components $f_{i_1j_1}, f_{i_2j_2}, \dots, f_{i_mj_m}$. This procedure defines the sliced matrix F_s and the corresponding incidence matrix I_s . The net $j_1, j_2, \dots, j_m$ of the corresponding economic sectors serves as a set of vertices of a many-dimensional polyhedron generating the *partial forward linkages backcloth*.

The procedure for the construction and partition of this polyhedron into a set of simplexes can be defined in the following way: for each fixed economic sector, $i_k, k=1, 2, \dots, m$, consider the set of all different economic sectors $j_{r_0}, j_{r_1}, \dots, j_{r_{q_k}}$ corresponding to the non-zero $f_{i_kj_{r_0}}, f_{i_kj_{r_1}}, \dots, f_{i_kj_{r_{q_k}}}$ associated with the inputs of the sector i_k into the economic sectors $j_{r_0}, j_{r_1}, j_{r_{q_k}}$. The simplex, $S_{q_k}^f(i_k) = S_{i_k}$ associated with the sector i_k , is a minimal convex polyhedron in q_k -dimensional space with $q+1$ vertices $j_{r_0}, j_{r_1}, \dots, j_{r_{q_k}}$. (Each such simplex represents part of some economic *filière*,—the notion elaborated by Torre, 1993).

The set of simplexes, $S_{q_1}^f(i_1), S_{q_2}^f(i_2), \dots, S_{q_m}^f(i_m)$, associated with all economic sectors $i_k, k=1, 2, m$, is called the *forward linkages simplicial family*, generating the polyhedron with vertices $j_1, j_2, \dots, j_m$, and its partition—the *simplicial complex* $K(S)$.

3. q-nearness and q-connectedness

Two simplexes $S(i_k)$ and $S(i_s)$ are *q-near* in the simplicial family *iff* they share at least $q+1$ vertices. Thus, two sectors i_k and i_s are *q-near* *iff* there are at least $q+1$ economic sectors accepting the inputs from the sectors i_k and i_s . If all vertices of a simplex $S(i_s)$ are the vertices of a simplex $S(i_k)$ then the simplex $S(i_s)$ is a face of the simplex $S(i_k)$.

Two simplexes $S(i_k)$ and $S(i_s)$ are *q-connected* by a chain of simplexes of length r *iff* there is a sequence of r pair-wise *q-near* simplexes $S(i_k), S(i_p), \dots, S(i_q), S(i_s)$. The relationship of *q-connectedness* generates the partition of the simplicial family $K(S)$ into *q-connected* components.

The enumeration of all *q-connected* components for each dimension $q \geq 0$ is the essence of the Q-analysis of the simplicial family.

4. Procedure and meaning of the forward linkages input-output Q-analysis

Following Atkin (1974, 1981) the operational basis for Q-analysis is given by a *shared face matrix* SF of the form:

$$SF = I_s I_s^T - U \quad (73)$$

where I_s is the incidence matrix corresponding to the chosen slicing procedure, I_s^T is its transpose and U is the matrix with unit entries. The components of the matrix SF provide the amounts of mutual vertices for each pair of sectors $i_k, i_s, k, s=1, 2, \dots, m$. In other words, the components of the shared face matrix SF are the dimensions of the maximal mutual faces for each pair of simplexes $S(i_k)$, and $S(i_s)$.

The Atkin operational algorithm for Q-analysis includes the following iterative steps for each dimension $q, q=0, 1, \dots, N$, where N is the maximal dimension of simplexes from the simplicial complex:

Identify the economic sectors and their corresponding simplexes whose dimensions are equal to or larger than q ; these dimensions are on the main diagonal of the shared face matrix SF .

Identify all distinct *q-connected* components —*q-chains*— of the set of simplexes constructed in the previous step: two *q-dimensional* simplexes $S_q(i_k)$ and $S_q(i_s)$ belong to the same *q-chain* if the corresponding rows i_k and i_s of the shared face matrix SF include at least one column with entries larger than or equal to q ; the number of distinct *q-chains* is denoted as Q_q . The vector

$$Q = \{Q_N, Q_{N-1}, \dots, Q_0\} \quad (74)$$

is called the *structural vector* of the simplicial complex $K(S)$ and the maximal q -value N is a dimension of this complex.

5. The conjugate simplicial complex and the backward linkages input-output Q-analysis

So far, the one-sided view of the forward linkages Q-analysis of the interaction matrix has been presented. The same fixed slicing procedure can be used for the generation of the scheme of backward linkages Q-analysis. Consider the transpose interaction matrix F^T and the corresponding conjugate slicing matrix F_s^T with the corresponding conjugate incidence matrix I_s^T . This matrix can be used for the construction of the conjugate simplicial complex $K(S^T)$. As a result, the set of simplexes can be constructed corresponding to the set of economic sectors $j_1, j_2, \dots, j_m$: each simplex $S(j_k)$ represents the set of economic sectors sending the inputs into the sector j_k .

Moreover, the backward linkages input-output Q-analysis can be performed analogously with the help of the *conjugate shared face matrix*:

$$S^T F = I_s^T I_s - U \quad (75)$$

6. Chains of Structural Complication of simplicial families and the Rank-Size ordering method

Consider two slicing procedures S_1 and S_2 and their corresponding simplicial families associated with the simplicial complexes $K(S_1)$ and $K(S_2)$. The simplicial complex $K(S_2)$ is called the *structural complication* of the simplicial complex $K(S_1)$ and noted $K(S_1) \leq K(S_2)$ if each simplex $S_p^1(i_k)$ from $K(S_1)$ is a face of some simplex $S_q^2(i_k)$ from $K(S_2)$. This means that the incidence matrix I_{S_2} includes all non-zero (unit) components from the incidence matrix I_{S_1} . The set of m simplicial complexes $K(S_1), K(S_2), \dots, K(S_m)$ is called the *chain of structural complication* if for each pair of complexes $K(S_s)$ and $K(S_r)$, one of them is the structural complication of the other.

Obviously, the chain of structural complication is defined with the help of the set of corresponding incidence matrices such that for each pair of incidence matrices one of them includes all the units from the other. This means also that the chain of structural complication is generated by the sequence of extending sets of the components of the interaction matrix B .

One of the important methods for the generation of the chains of structural complication is the Rank-Size ordering method. The rank-size ordering method is based on the construction of the sequence of all components of the matrix $B = \|b_{ij}\|$ ordered by size in such a way that the largest components is at the top of a decreasing-by-size sequence of components. It is possible to consider only the qualitative rank-size sequence instead of the absolute value of each component. Consider the sequence of slicing procedures and the corresponding set of sliced matrices, the first of which includes only the largest components of the matrix B , the second matrix includes the two largest components, and so forth. In such a way, one obtains the chain of structural complication associated with the relative size on the matrix components. The Q-analysis of each element of the chain of structural complication reveals some hidden features of the inter-sectoral interactions.

The geometrical form of the structural complication, associated with pictures of simplices, is not possible in multi-dimensional spaces. Therefore, use is made instead of Q -maps, elaborated for the needs of migration analysis (see, Sonis, 1988; Portugali and Sonis, 1991). The Q -map represents a simplex with the help of a planar graph: the vertices of the simplex associated with i^{th} economic sector are represented with the help of the points on the plane, and the oriented arcs are connecting the point of the sector i with all other points (see explanation to the Figure 4). The Q -map provides a convenient graphical representation of the multi-dimensional simplices, and also a simple picture of the mutual faces for each pair of simplices. Furthermore, Q -maps reveal a straight forward picture of the process of structural complication of input-output system. Figure

Figure 4. Q-Maps of Structural Complication.

4 depicts the simultaneous gradual enlargement of each backward linkage simplices for the Leontief inverse matrix. It is important to stress that the process of structural complication can be described with the help of complication curves, depicting the process of sequential enlargement of simplices in the process of structural complication (see Figure 5). The complication curve describes the sequential addition of economic sectors to the simplex accordingly to the rank-size step of structural complication. Moreover, the structural complication leads to the new approach of explanation of economic complexity through the process of complication.

7. Summary

Qualitative input-output analysis represents an important field of inquiry along the paths of interpretation of structural change in economies. The approaches should not be seen as ones that

m	$S^b(AGR)$	$S^b(MNF)$	$S^b(CNS)$	$S^b(ELT)$	$S^b(TRS)$	$S^b(SRV)$
1						
6						
12						
18						
21						
30						

EXPLANATION

Q-map for the simplex $S_3^b(AGR) = \{AGR, MNF, ELT, SRV\}$

Figure 5 Complication Curves of Structural Complication.

replace some of the more formal, quantitative analytical techniques but as valued partners in the search for patterns of change and regularities across time and space. The complication curve of Q-analysis provides a very important graphical way of describing the different ways in which sectors are linked in the economy in which they are located. Together with the MPM approach and the information provided on the size of multipliers, the analyst is thus armed with valued, but different insights into how changes in economic activity are transformed across the economy.

Applications: Sonis, 1988; Portugali and Sonis, 1991; Sonis, Hewings and Bronstein, 1994.

Part I Summary

The selection of methods reviewed in this paper draw their initial inspiration from concerns about coefficient change and error and sensitivity analysis. The path of analytical development has moved across a varied methodological landscape that has managed to combine very formal methods with derivatives that enable visual comparison. Experience suggests that the search for one optimal technique may prove fruitless since the variety of uses for input-output analysis transcend any analyst's capability to reduce this variety to one all-encompassing method. Furthermore, the attractions of different methods lie in the important new insights that can be gleaned about the way economies function and change.

In the next paper, attention will be drawn to spatial analysis, examining the way in which

systems of economies may be viewed in a variety of input-output representations. In addition, attention will be devoted to demoeconomic analysis and issues of trade and trade structure. The final paper in this trilogy will examine new directions in comparative analysis and dynamics. A summary of the contents of the other papers is provided in an Appendix.

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Appendix :

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