A Survey of Analyses of the Transboundary Pollution Problem: Symmetric and Asymmetric Dynamic Models

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Abstract. The region which discharges pollution from the production processes is not necessarily coincident with the region which suffers from this pollution. This kind of problem is often called a transboundary pollution problem (TBP problem). Many articles which deal with TBP problems have been written since 1990. In this paper, we will survey these preceding articles by contrasting the symmetric two-region model with the asymmetric one, where in our model, “asymmetric two-region” means that region 1 derives less satisfaction from consuming the good but cares more about the pollution stock and the welfare of the coming generation than does region 2. In the symmetric model, the steady state of the pollution stock is always less under cooperative pollution control than in the non-cooperative case in which a Markov-perfect strategy is supposed, but the converse is possible in the asymmetric model. In the analyses of TBP model, we usually employ Hamilton-Jacobi-Bellman equation (H-J-B equation). Therefore, based on the “principle of optimality” in dynamic programming, we have established the H-J-B equation as a general formula applicable to TBP problems.

1. Introduction

There are many articles which deal with economic-environmental systems in a dynamic context. Keeler et al. (1971) is one of the pioneer works. However, it is only recently that attention has been directed to such systems from an inter-regional viewpoint. We can easily find relevant works in the 1990s; see, for example, Clemhout and Wan (1991), Kaitala et al. (1991), Dockner and van Long (1993), Martin et al. (1993), Tahvonen (1996), and Zagonari (1998). We call this kind of problem a transboundary pollution problem (TBP problem for short). They formulate their models as differential games in which two players self-enforceably control their pollution emissions under a feedback information structure in order to maximize the discounted stream of welfare; and they analyze the models with the aid of the

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Hamilton-Jacobi-Bellman equation (H-J-B equation for short) from dynamic programming. Most of those who have worked in this field have confined themselves to the special linear-quadratic functional form and, in their simulations, have employed only a very restricted set of alternative parameter values.

In section 2, we survey how the H-J-B equation is derived from what is called the "principle of optimality" and make the equation applicable to models of TBP. In section 3, following the above-mentioned articles, we apply the reduced H-J-B equation to the TBP model known as the linear-quadratic game for two regions. There are two cases, according as the two regions are symmetric or asymmetric. To identify an essential difference between symmetric and asymmetric two-region models, we consider the cooperative outcome of the game and the non-cooperative one in each case. Furthermore, we adopt two approaches to the non-cooperative case, depending on whether the two governments impose a linear relation between their strategies and the global pollution stock. Section 4 contains concluding remarks.

2. Feedback strategies and the principle of optimality

Since TBP problems involve more than one decision maker and since we are interested in some long-term effects of accumulated pollution, TBP models are often formulated as differential games. It is usually assumed that the participants in a game (we call them players) are subject to some information structures. When players choose their actions at time $t$, they utilize the available information. Basar and Olsder (1995, p. 231) proposed the following five information structures;

(a) open-loop type $\ : x(0), t$

(b) closed-loop perfect state type $\ : x(s)(s \in [0, T]), t$

(c) $\epsilon$-delayed closed-loop perfect state type $\ : x(s)(s \in [0, t-\epsilon]), t$

(d) memoryless perfect state type $\ : x(0), x(t), t$

(e) feedback (perfect state) type $\ : x(t), t,$

where at the right side of information type, we list the information available when players choose their action at time $t$ and $x(s)$ means the value of state variable $x$ at time $s$. However, in most analyses of the TBP model, information structures (a) and (e) are supposed and, of these, (e) is the more familiar (see Clemhout and Wan (1994, p. 804)). We also focus on information structures, (a) and (e).

We now formulate an $n$-player dynamic game as follows :

\[
\max_{v(t)} \int_{t_0}^{T} g^i(t, x(t), v(t)) dt \quad i=1, \ldots, n
\]
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\[ s. t. \quad \dot{x}_j(t) = f_j(t, x(t), v(t)) \quad j = 1, \ldots, m \]
\[ x_i(t_0) = x_{i0} \]
\[ v_i : \text{the control variable of player } i \quad v := (v_1, \ldots, v_n) \]
\[ x_j : \text{the } j\text{-th state variable} \quad x := (x_1, \ldots, x_m), \]
where \( a(t) \) means that variable \( a \) is evaluated at time \( t \). It is assumed that \( t_0, T, \) and \( x_{i0} \) are given constants, that functions \( g^i \) and \( f^j \) are continuously differentiable, and that \( v_i \) is piecewise continuous.

A strategy for a player is a complete plan of action; it specifies a feasible action for the player in every contingency in which the player might be called on to act.\(^1\) Player \( i \)'s strategy \((\gamma_i(\cdot))\) determines his choice of control variables under an information structure. We now introduce three assumptions. First, each player has one control variable. Second, all players choose their strategies simultaneously, taking other players' strategies into account. Third, all players acknowledge the same constraint, namely, the time path of the state variables.

We call a strategy under open-loop information an "open-loop strategy". This strategy is a contingency plan which, at the beginning of the game, instructs a player how to achieve optimality and we represent it as follows:

\[ v_i(t) = \gamma_i(t). \]

Thus, we can analyze this case by using Pontryagin's maximum principle. On the other hand, we call a strategy under feedback information a "feedback strategy". This strategy is a contingency plan which, at every time point, instructs player how to achieve optimality, depending on the values of the state variables, and this strategy is represented by

\[ v_i(t) = \gamma_i(t, x(t)). \]

In this case, it seems to be difficult to employ the maximum principle for analysis, because each control variable depends on the values of the state variables (see Inoue (1997, pp. 270-272)).

Kamien and Schwartz (1991, p. 276) said that "The nature of a feedback strategy is in the spirit of the principle of optimality of dynamic programming." Following Leonard and van Long (1992, chap. 5), we derive the H-J-B equation from Bellman's principle of optimality. For the purpose of grasping the relationship between a feedback strategy and the principle of optimality, we consider the following dynamic programming model:

\[ [D-DP] \max_{(v(t), t \leq t \leq T)} V = \sum_{t=1}^{T} g(t, x(t), v(t)) \]
\[ s. t. \quad x(t+1) = h(t, x(t), v(t)) \quad t = 1, \ldots, T \]
\[ x(1) = x_i, \quad x(T+1) = \bar{x} \]

, where \( T \) (the endpoint), \( x_i \), and \( \bar{x} \) are given constants.

Using this formulation, the principle of optimality is written as follows:

\(^1\)See Gibbons (1992, p. 117).
A necessary and sufficient condition for \((v^*(1), v^*(2), \ldots, v^*(T))\) to be an optimal solution for problem [D-DP] is that, for any \(t \in \{1, \ldots, T\}\), \((v^*(t), v^*(t+1), \ldots, v^*(T))\) is optimal for the problem

\[
\max_{(v(t), t \leq \tau \leq T)} \sum_{\tau=t}^{T} g(\tau, x(\tau), v(\tau)) \quad \text{s. t.} \quad x(\tau+1) = h(\tau, x(\tau), v(\tau)) \quad \tau = t, t+1, \ldots, T
\]

We are immediately aware that a solution path which satisfies the principle is optimal in feedback strategies.\(^2\) We now consider the continuous time version of problem [D-DP]. This is given by

\[
[C-DP] \quad \max_{(v(t), t \leq \tau \leq T)} \int_{0}^{T} g(t, x(t), v(t)) dt + q(T, x(T)), \quad T = \min\{t : l(t, x(t)) = 0\}
\]

\[
\text{s. t.} \quad \dot{x}(t) = f^\gamma(t, x(t), v(t)), \quad x_0 = x_0
\]

where \(v(t)\) : control vector, \(v(t) = \gamma(t, v(t)) \in S, \quad \gamma \in \Gamma\)

\(S\) : the range of values for strategy \(\gamma\)

\(\Gamma\) : strategy set

\(l\) : real valued function,

where \(T\) and \(x_0\) are given constants.

Define a value function as

\[
V(t, x(t)) = \max_{(v(t), t \leq \tau \leq T)} \left[ \int_{t}^{T} g(\tau, x(\tau), v(\tau)) d\tau + q(T, x(T)) \right],
\]

which satisfies the boundary condition,

\[
V(T, x) = q(T, x) \quad \text{along} \quad l(T, x) = 0.
\]  \(2-1\)

The function \(V\) is supposed to be continuously differentiable. The value function may be re writtem as

\[
V(t, x(t)) = \max_{(v(t), t \leq \tau \leq T)} \left[ \int_{t}^{t+\Delta t} g(\tau, x(\tau), v(\tau)) d\tau + \int_{t+\Delta t}^{T} g(\tau, x(\tau), v(\tau)) d\tau + q(T, x(T)) \right].
\]

Using the principle of optimality, we have

\[
V(t, x(t)) = \max_{(v(t), t \leq \tau \leq t+\Delta t)} \left[ \int_{t}^{t+\Delta t} g(\tau, x(\tau), v(\tau)) d\tau + \max_{(v(t), t+\Delta t \leq \tau \leq T)} \left[ \int_{t+\Delta t}^{\tau} g(\tau, x(\tau), v(\tau)) d\tau + q(T, x(T)) \right] \right]
\]

\[
= \max_{(v(t), t \leq \tau \leq t+\Delta t)} \left[ \int_{t}^{t+\Delta t} g(\tau, x(\tau), v(\tau)) d\tau + V(t+\Delta t, x(t+\Delta t)) \right].
\]

For sufficiently small \(\Delta t\), we have

\[
V(t, x(t)) = \max_v [g(t, x(t), v(t)) \cdot \Delta t + V(t+\Delta t, x(t+\Delta t))] + o(\Delta t), \quad (2-2)
\]

where \(\lim_{\Delta t \to 0} \frac{o(\Delta t)}{\Delta t} = 0\).

\(^2\)Any optimal result in feedback strategies satisfies the property of “subgame perfection”.

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From the differentiability of $V$,

$$V(t+\Delta t, x(t+\Delta t)) = V(t, x(t)) + \frac{\partial V(t, x)}{\partial t} \Delta t + \frac{\partial V(t, x)}{\partial x} \Delta x + o(\Delta x).$$

Combining the above equation and (2-2), we obtain the H-J-B equation

$$0 = \max_{v} \left[ g(t, x(t), v(t)) + \frac{\partial V(t, x)}{\partial t} + \frac{\partial V(t, x)}{\partial x} \cdot f(t, x(t), v(t)) \right]. \quad (2-3)$$

**Theorem 2-1**: If there exists a continuously differentiable value function $V(t, x(t))$ which satisfies the boundary condition (2-1) and the H-J-B equation (2-3), $v^*(t, x(t)) = v^*(t)$ derived from the H-J-B equation is a optimal strategy for the problem [C-DP].

Basar and Olsder (1995, p. 243, Theorem 5.3) call this a “sufficiency result”. We prove this theorem in the framework of our model.

**Proof** We consider the strategy derived from (2-3), $v^* \in \Gamma$, and any given strategy, $v \in \Gamma$. Let us denote by $x^*(x)$ the trajectory of the state variable which corresponds to strategy $v^*(\gamma)$, and let us denote the terminal by $T^*(T)$. Then, from (2-3),

$$g(t, x, v) + \frac{\partial V(t, x)}{\partial x} \cdot f(t, x, v) + \frac{\partial V(t, x)}{\partial t} \leq 0 \quad (1)$$

and

$$g(t, x^*, v^*) + \frac{\partial V(t, x^*)}{\partial x} \cdot f(t, x^*, v^*) + \frac{\partial V(t, x^*)}{\partial t} = 0, \quad (2)$$

where $v^*(\gamma)$ is replaced by the relevant control variable $v^*(v)$. Since (1) reduces to

$$g(t, x, v) + \frac{dV(t, x(t))}{dt} \leq 0,$$

we have

$$\int_0^T g(t, x, v) dt + V(T, x(T)) - V(0, x_0) \leq 0. \quad (3)$$

In the same way, the following equation is derived from (2).

$$\int_0^T g(t, x^*, v^*) dt + V(T^*, x^*(T^*)) - V(0, x_0) = 0. \quad (4)$$

In view of (2-1), (3), and (4),

$$\int_0^T g(t, x, v) dt + q(T, x(T)) \leq \int_0^{T^*} g(t, x^*, v^*) dt + q(T^*, x^*(T^*)).$$

Accordingly, it has been shown that $v^*$ is an optimal solution for problem [C-DP]. In other words, $v^*$ is an optimal strategy for problem [C-DP]. Q. E. D.

Most of the TBP models in which we are interested are of the type [AP], which is characterized by “infinite horizon”, “continuous time”, “discounted stream of welfare”, and “autonomous system”.

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\[ \bar{V}(t_0, b) = \max_{\bar{v}} \left[ \int_{t_0}^{\infty} e^{-\tau} \cdot u(x(t), v(t)) d\tau \right] \]
\[ \text{s. t.} \quad \dot{x}_j = f(x(t), v(t)), \quad j = 1, \ldots, m \]  ... (2-4)

constraint

\[ \psi(x(t), v(t)) \geq 0 \quad l = 1, \ldots, L' \]  ... (2-5)

initial condition

\[ x_i(t_0) = b_j \quad j = 1, \ldots, m \]  ... (2-6)

terminal behavior of state variables

\[ \lim_{t \to \infty} x_j(t) = \bar{x}_j \quad j = 1, \ldots, m' \] \[ \lim_{t \to \infty} x_j(t) \geq \bar{x}_j \quad j = m' + 1, \ldots, m' + m'' \]  ... (2-6)

Utilizing the characteristic of an infinite horizon problem, problem \([\text{AP}]\) reduces to

\[ \bar{V}(t_0, b) = e^{-\tau t_0} \cdot \max_{\bar{v}} \left[ \int_{t_0}^{\tau} e^{-\tau} \cdot u(x(t), v(t)) d\tau \right] = e^{-\tau t_0} \cdot \bar{V}(0, b) \]
\[ \text{s. t.} \quad (2-4), (2-5), (2-6), \text{and } x_j(0) = b_j (j = 1, \ldots, m). \]

Let introduce the current-value-return function \( W(t_0, b) (= e^{\tau t_0} \cdot \bar{V}(t_0, b)) \). Then, from (2-7), we immediately have

\[ W(t_0, b) = e^{-\tau t_0} \cdot \bar{V}(t_0, b) = \bar{V}(0, b). \]

Thus \( W(t_0, b) \) is independent of \( t_0 \), allowing to denote \( W(t_0, b) \) by \( W(b) \). Accordingly,

\[ \bar{V}(t_0, b) = e^{-\tau t_0} \cdot \bar{V}(0, b) = e^{-\tau t_0} \cdot W(b). \]

In the same way, for any \( t \) and \( x(t) (\equiv x) \),

\[ \bar{V}(t, x) = e^{-\tau t} \cdot \bar{V}(0, x) = e^{-\tau t} \cdot W(x). \]  ... (2-8)

Using (2-8), the H-J-B equation (2-3) reduces to

\[ 0 = \max_{\bar{v}} \left[ e^{-\tau t} \cdot u(x, v) + \frac{\partial \bar{V}(t, x)}{\partial t} + \frac{\partial \bar{V}(t, x)}{\partial x} \cdot f(x, v) \right]. \]

Thus we have obtained the following H-J-B equation which is applicable to models of TBP.

\[ \frac{\partial}{\partial x} W(x) = \max_{\bar{v}} \left[ u(x, v) + \frac{dW(x)}{dx} \cdot f(x, v) \right] \]

3. An application of the H-J-B equation to TBP problems

In this section, we study the question “How does the H-J-B equation apply to TBP problems?” We especially focus on the difference between symmetric and asymmetric two-region models. For this purpose, we follow Dockner and van Long (1993) in the symmetric case and Zagonari (1998) in the asymmetric case.
We consider a two-region model in which each region produces a single consumption good. Pollution is emitted by the production process and each of the two governments controls its pollution emission to maximize welfare. Depending on the relationship of the two regions, there are two cases of pollution control. First is the case in which both regions cooperatively control their pollution emissions. Second is the case where each region self-enforcingly (non-cooperatively) controls its own pollution emission. Moreover, we divide the latter case into two. In one sub-case each region adopts a linear Markov-perfect strategy; and in the other sub-case neither region imposes linearity on its strategies. We shall refer to the second sub-case as involving non-linear Markov-perfect strategies.

We provide analyses for a symmetric two-region model and an asymmetric two-region model. The following three main results emerge.

(1) In the symmetric model, the steady-state of the pollution stock is always less under cooperative pollution control than in the non-cooperative case in which a linear Markov-perfect strategy is supposed. On the other hand, the converse is possible in the asymmetric model.

(2) Whether the model is symmetric or asymmetric, the solution curve with a linear Markov-perfect strategy corresponds to a limiting case of the solution curve with a non-linear Markov-perfect strategy.

(3) In the symmetric model, if the discount rate is sufficiently small, the steady-state of the pollution stock under non-cooperative pollution control in which a non-linear Markov-perfect strategy is supposed is close enough to that under cooperative pollution control, but the former is higher than the latter. However, in the asymmetric model, it is possible that the former is lower than the latter.

In 3-1, we introduce notation and formulate the time path of the pollution stock and welfare functions. Then, in 3-2 and 3-3, respectively, we analyze the cooperative and non-cooperative cases. Depending on the supposed strategies, 3-3 is divided into 3-3-1 and 3-3-2. In the former, we deal with the case of a linear Markov-perfect strategy by using the guessing method; and in the latter, we deal with the non-linear case by using an auxiliary equation.

3-1 Notation
We use the following notation in this section (i=1, 2).

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3 In the guessing method, we guess the functional form of value functions to be quadratic with respect to the state variable (pollution stock, here).
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\[ N_i : \text{the number of identical consumers in region } i \]
\[ Q_i(t) : \text{the amount of the single consumption good in region } i \text{ at time } t \]
\[ E_i(t) : \text{the flow of pollution emitted by region } i \text{ at time } t \]
\[ P(t) : \text{the stock of pollution of region } i \text{ at time } t \]
\[ k : \text{natural purification rate } [\text{we assume } k > 0] \]
\[ \delta_i : \text{the discount rate of region } i \ [\text{we assume } \delta_i > 0] \]
\[ \mu : \text{the relative bargaining power of region 1 } [\text{we assume } 0 \leq \mu \leq 1] \]
\[ F_i(\cdot) : \text{the emission-consumption trade-off function of region } i \]
\[ \text{[It is assumed that } F_i(\cdot) \text{ is strictly concave and that } F_i(0) = 0] \]
\[ u_i(\cdot) : \text{the consumer utility function of region } i \]
\[ \text{[} u_i(\cdot) \text{ is assumed to be strictly concave]} \]
\[ c_i(\cdot) : \text{the cost function of region } i \text{ to the polluted environment} \]
\[ \text{[} c_i(\cdot) \text{ is assumed to be strictly convex]} \]

For example, \( Q_i(t) \) means the value \( Q_i \) at time \( t \). If necessary, we attach \( t \) to the relevant variable from now on.

The economy with which we are concerned is as follows. There are two regions. \( N_i \) identical consumers live in region \( i \) \((i = 1, 2)\). The consumers of region \( i \) produce a single consumption good. However, the production of one unit of the consumption good in region \( i \) generate the pollution \( E_i \), and that region's emission-consumption trade-off is expressed as \( Q_i = F_i(E_i) \).

Given natural purification rate \( k \), the time path of the pollution stock is supposed to be

\[ \dot{P}(t) = E_1(t) + E_2(t) - kP(t). \quad (3-1-1) \]

For the purpose of easy calculation, we specify the functional form of utility functions and cost functions as

\[ u_i(Q_i/N_i) = A_i \cdot E_i - \frac{1}{2} \cdot E_i^2 \]

and

\[ c_i(P) = \frac{s_i}{2} \cdot P^2, \]

where \( A_i \) and \( s_i \) are given positive constants \((i = 1, 2)\). Under such a specification, we have to suppose that \( A_i - E_i > 0 \) to ensure the positive marginal utility of \( E_i \). Moreover, we define the net benefit to the representative consumer of region \( i \) at time \( t \) as

\[ u_i(F_i(E_i)/N_i) - c_i(P), \]

namely,

\[ A_i \cdot E_i - \frac{1}{2} \cdot E_i^2 - \frac{s_i}{2} \cdot P^2. \]

In our model, “asymmetry” means that region 1 derives less satisfaction from consuming the good but cares more about the pollution stock and the welfare of the coming generation than does region 2. Concretely, the difference between the symmetric and the asymmetric
models appears in the parameter values. It is assumed that

symmetric model : \( A_1 = A_2, \ s_1 = s_2, \) and \( \delta_1 = \delta_2 \)

asymmetric model : \( A_1 < A_2, \ s_1 > 0, \ s_2 = 0, \) and \( \delta_1 < \delta_2 \).

In the analysis of the symmetric model, if not necessary, we suppress the subscript of parameters.

### 3-2 Cooperative pollution control

We now consider the case in which the governments of the two regions cooperatively control their pollution emissions \( (E_i, i=1, 2) \) to maximize the discounted stream of the sum of two regions' welfare functions. This maximization problem \([\text{CPC}]\) is formulated as

\[
\max_{E_1, E_2} \int_0^\infty e^{-\lambda t} \left\{ \left( A_1 E_1 - \frac{1}{2} E_1^2 - \frac{s_1}{2} P^2 \right) + \left( A_2 E_2 - \frac{1}{2} E_2^2 - \frac{s_2}{2} P^2 \right) \right\} dt
\]

\[
\text{s. t.} \quad \dot{P} = E_1(t) + E_2(t) - kP(t)
\]

\[
P(0) = P_0, \text{ where } P_0 \text{ is a given positive constant}
\]

\[
\delta = \mu \delta_1 + (1 - \mu) \delta_2.
\]

The current-value Hamiltonian of problem \([\text{CPC}]\) is given by:

\[
H(E_1, E_2, P, \lambda) = A_1 E_1 - \frac{1}{2} E_1^2 - \frac{s_1}{2} P^2 + A_2 E_2 - \frac{1}{2} E_2^2 - \frac{s_2}{2} P^2 + \lambda (E_1 + E_2 - kP),
\]

where \( \lambda \) is the Hamiltonian multiplier (the co-state variable to state variable \( P \)). We can interpret problem \([\text{CPC}]\) as one of private maximization. Accordingly, we can obtain the necessary conditions for maximization by appealing to Pontryagin’s maximum principle:

\[
\frac{\partial H}{\partial E_1} = A_1 - E_1 + \lambda = 0 \tag{3-2-1}
\]

\[
\frac{\partial H}{\partial E_2} = A_2 - E_2 + \lambda = 0 \tag{3-2-2}
\]

\[
\dot{\lambda} = \dot{\delta} \cdot \lambda - \frac{\partial H}{\partial P} = \dot{\delta} \cdot \lambda - ((s_1 + s_2)P - \lambda k) = (s_1 + s_2)P + (\dot{\delta} + k)\lambda \tag{3-2-3}
\]

The transversality condition is

\[
\lim_{t \to \infty} \lambda(t) \cdot P(t) \cdot e^{-\lambda t} = 0.
\]

From equations (3-2-1) and (3-2-2), the differential equation (3-1-1) reduces to

\[
\dot{P} = (A_1 + A_2) + 2\lambda - kP.
\]

Accordingly, the dynamic system which the solution to \([\text{CPC}]\) must satisfy is

\[
\begin{align*}
\dot{P} &= (A_1 + A_2) + 2\lambda - kP \\
\dot{\lambda} &= (s_1 + s_2)P + (\dot{\delta} + k)\lambda.
\end{align*}
\tag{3-2-4}
\]

If \( \dot{P} = 0 \) and \( \dot{\lambda} = 0 \), the system reduces to

\[
\begin{align*}
(A_1 + A_2) + 2\lambda - kP &= 0 \\
(s_1 + s_2)P + (\dot{\delta} + k)\lambda &= 0.
\end{align*}
\]

Thus, solving, the steady-state of the pollution stock \( P^c \) is
Replacing $P_C$ by $P^{CS}$ in the symmetric model and by $P^{CA}$ in the asymmetric model, we have Theorem 3-1.

**Theorem 3-1**: (Dockner and van Long (1993, Proposition 1) and Zagonari (1998, Proposition 2): If two regions cooperatively control their pollution emission, the pollution stock can attain a unique steady-state under the symmetric two-region model and under the asymmetric two-region model.

We have discussed the implications of cooperative pollution control. However, it is very unlikely that many regions will cooperatively reduce their pollution emissions at the cost of economic growth. Thus it is difficult to achieve cooperative pollution control. Therefore, in 3-3, we study the non-cooperative case.

### 3-3 Non-cooperative pollution control

Suppose then that the governments of the two regions non-cooperatively enforce their pollution controls. In this case, the problem is formulated as a 2-player differential game:

$$[\text{NCPC}]$$

$$\max_{E_i} \int_0^\infty e^{-\beta t} \cdot \left( A E_i - \frac{1}{2} E_i^2 - \frac{s}{2} P^2 \right) dt \quad i = 1, 2$$

s. t. \hspace{1cm} \dot{P} = E_1(t) + E_2(t) - kP(t)

$$P(0) = P_0,$$ where $P_0$ is a given constant

In this game, the payoff function of region $i$ is the discounted stream of that region's utility and region $i$ control its own pollution emission to maximize the payoff function under a feedback information structure. As discussed in the top of this section, it is assumed that both regions adopt Markov-perfect strategies, which implies a feedback strategy. And the strategy set of our differential game is defined as

$$S_{it}^{MP} = \{ E_i(t, P(t)) | E_i(t, P(t)) \text{ is continuous w. r. t. } t \text{ and Lipschitz continuous w. r. t. } P(t) \}.$$

As we mentioned in Theorem 2-1, if there exists a value function which satisfies the H-J-B equation of this game, the equation generates an optimal feedback solution for problem [NCPC]. Let us assume that there is a current-value function for region $i$, $W_i(P)$, which is continuously differentiable with respect to $P$ and satisfies the equation,

$$\delta_t W_i(P) = \max_{E_i} \left[ A E_i - \frac{1}{2} E_i^2 - \frac{s}{2} P^2 + W'_i(P) \cdot (E_i + E_2 - kP) \right] \quad i = 1, 2.$$ (3-3-1)

To find the solution for $E_i$, $i = 1, 2$ from (3-3-1), we conduct two methods. One is the guessing
method in which the solution for $E_i$ is assumed to be a linear function of $P$ and the other is the auxiliary equation approach in which linearity is not necessarily assumed. We call the first strategy the linear Markov-perfect strategy and the second the non-linear Markov-perfect strategy.

**Linear Markov-perfect strategy (guessing method)**

If a linear Markov-perfect strategy is assumed, we obtain the following theorem.

**Theorem 3-2**: If two regions adopt the linear Markov-perfect strategy, there exists a set of linear Markov-perfect strategies,

- Under the symmetric model:
  \[
  E_i^{LS}(P(t)) = (A - \beta^*) - \alpha^* P(t) \quad i=1,2
  \]

- Under the asymmetric model:
  \[
  E_i^{LA}(P(t)) = (A_i - \beta_i^*) - \alpha_i^* P(t) \quad i=1,2,
  \]

where, in the latter case, the condition $\delta \geq \frac{A_2 \delta_1}{A_1 k} - k$ is needed to ensure that $E_i^{LA}(P^{LA}) \geq 0$.

The above Markov-perfect equilibrium is asymptotically stable and results in the following steady-state of the pollution stock under the symmetric model (S1) and under the asymmetric model (AS1):

\[
\begin{align*}
P^{LS} &= \frac{2A(\delta + k + \alpha)}{(2\alpha + k)(\delta + k + 3\alpha)} \quad \text{under the symmetric model} \\
P^{LA} &= \frac{(A_i + A_2)(\delta + k)}{\delta_i + k}(\delta + k) \quad \text{under the asymmetric model}
\end{align*}
\]

And the values taken by the two value functions at time 0, $W_i(P_0)$ and $W_2(P_0)$, are

\[
W_i(P_0) = -\frac{1}{2} \alpha_i P_0^2 - \beta_i P_0 - \gamma_i \quad i=1,2.
\]

As for parameters, in the symmetric case,

\[
\begin{align*}
\alpha^* &= \frac{1}{3} \left( -\left(k + \frac{\delta}{2}\right) + \sqrt{\left(k + \frac{\delta}{2}\right)^2 + 3s} \right) > 0, \\
\beta^* &= \frac{2A\alpha^*}{\delta + k + 3\alpha^*}, \quad \gamma^* = -\frac{(A - \beta^*)(A - 3\beta^*)}{2\delta}
\end{align*}
\]

and in the asymmetric case,

\[
\begin{align*}
\alpha_i^* &= \frac{-\delta_i + 2k + \sqrt{(\delta_i + 2k)^2 + 4s_1}}{2} > 0, \\
\beta_i^* &= \frac{A_i + A_2}{\delta_i + \alpha_i^* + k} \alpha_i^*, \quad \gamma_i^* = -\frac{A_i^2 + (\beta_i^*)^2 - 2(A_i + A_2)\beta_i^*}{2\delta_i}, \\
\alpha_i^* &= \beta_i^* = 0, \quad \text{and} \quad \gamma_i^* = -\frac{A_i^2}{2\delta_i}
\end{align*}
\]

**Proof** We assume that the functional form of the value function of game [NCPC] is

\[
W_i(P) = -\frac{1}{2} \alpha_i P^2 - \beta_i P - \gamma_i, \quad \text{where} \quad W'(P) \leq 0.
\]

Combining (1) and the necessary conditions for the maximization problem on the right hand side of (3-3-1), we obtain
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\[ E_i = A_i + W'(P) = A_i - (\alpha_i P + \beta_i) \quad i = 1, 2. \]  \hfill (2)

Substituting (1) and (2) into (3-3-1), the constant terms and the coefficients of the terms \( P \) and \( P^2 \) must to satisfy the system

\[
- \gamma \delta_i = \frac{1}{2} A_i^2 - \frac{1}{2} \beta_i^2 - (A_i + A_2) \beta_i + (\beta_1 + \beta_2) \beta_i \\
\text{(constant terms)}
\]

\[
- \beta_i \delta_i = - \alpha_i \beta_i - (A_i + A_2) \alpha_i + (\beta_1 + \beta_2) \alpha_i + (\alpha_1 + \alpha_2) \beta_i + \beta_i k \\
\text{(coefficients of the } P \text{-terms)}
\]

\[
- \frac{1}{2} \alpha_i \delta_i = - \frac{s_i}{2} - \frac{1}{2} \alpha_i^2 + (\alpha_1 + \alpha_2) \alpha_i + \alpha_i k \\
\text{(coefficients of the } P \text{-terms)}.
\]

In the symmetric model, the system reduces to

\[
- \gamma \delta = \frac{1}{2} A^2 + \frac{3}{2} A^2 - 2 A \beta, \\
- \beta \delta = 3 \alpha \beta - 2 A \alpha + \beta k, \text{ and} \\
- \frac{1}{2} \alpha \delta = - \frac{s}{2} + \frac{3}{2} A^2 + \alpha k.
\]

Accordingly, we obtain (S2). However, we also obtain a second solution for \( \alpha \), namely,

\[
\alpha = \frac{1}{3} \left( - \left( k + \frac{\delta}{2} \right) - \sqrt{\left( k + \frac{\delta}{2} \right)^2 + 3 s} \right) < 0.
\]

On the other hand, in the asymmetric model, the system reduces to

\[
- \gamma \delta_i = \frac{1}{2} A_i^2 + \frac{1}{2} \beta_i^2 - (A_i + A_2 + \beta_i) \beta_i, \\
- \beta_i \delta_i = - (A_i + A_2) \alpha_i + (\beta_1 + \beta_2) \alpha_i + \beta_i(\alpha_i + k), \\
- \frac{1}{2} \alpha_i \delta_i = - \frac{s_i}{2} + \frac{1}{2} \alpha_i^2 + \alpha_i(\alpha_i + k), \\
- \gamma \delta_2 = \frac{1}{2} A_2^2 + \frac{1}{2} \beta_2^2 - (A_1 + A_2 + \beta_i) \beta_2, \\
- \beta_2 \delta_2 = - (A_1 + A_2) \alpha_2 + (\beta_1 + \beta_2) \alpha_2 + \beta_2(\alpha_2 + k), \text{ and}
\]

\[
- \frac{1}{2} \alpha_2 \delta_2 = \frac{1}{2} \alpha_2^2 + \alpha_2(\alpha_2 + k).
\]

By easy calculation, we obtain (AS2). However, as in the case of \( \alpha \) in the symmetric model, there is a second solution for \( \alpha_i \), namely,

\[
\alpha_i = - (\delta_i + 2 k) - \sqrt{(\delta_i + 2 k)^2 + 4 s_i} < 0.
\]

From (2), the optimal time path of \( P \) is written as

\[
P = E_i(P) + E_2(P) - k P = (A_i + A_2) - (\beta_1 + \beta_2) - (\alpha_i + \alpha_2 + k) P.
\]

Along the optimal path, the steady-state stock of pollution \( P^i \) is given by

\[
P^i = \frac{A_1 + A_2 - (\beta_1^* + \beta_2^*)}{\alpha_1^* + \alpha_2^* + k}.
\]

Thus \( P^{iS} \) and \( P^{iA} \) are derived as (S1) and (AS1), respectively. In both the symmetric and asymmetric models,
provided that we choose positive solution for \( \alpha_i \).

Accordingly, \( P^i \) is asymptotically stable. In addition, it is easy to verify that

\[
\frac{dP^i}{dt} = - (\alpha^*_i + \alpha^{**}_i + k) < 0
\]

Here, we compare \( P^{\text{LS}} \) with \( P^{\text{CS}} \) and \( P^{\text{LA}} \) with \( P^{\text{CA}} \). Since

\[
P^{\text{LS}} - P^{\text{CS}} = \frac{2A(\delta + k + \alpha)}{(2\alpha + k)(\delta + k + 3\alpha)} - \frac{2A(\delta + k)}{4s + k(\delta + k)},
\]

it is always the case that \( P^{\text{CS}} < P^{\text{LS}} \). On the other hand, since

\[
P^{\text{CA}} - P^{\text{LA}} = \frac{(A_1 + A_2)(\delta + k)}{2s + k(\delta + k)} - \frac{(A_1 + A_2)(\delta_i + k)}{s_i + k(\delta_i + k)},
\]

the inequality \( P^{\text{CA}} > P^{\text{LA}} \) holds, if and only if \( \mu < \frac{\delta_i - 2\delta_i - k}{\delta_i - \delta_i} \). This is an interesting result, because it reveals the possibility that the steady-state value of the pollution stock is smaller when two regions behave self-enforcingly than when they behave cooperatively.

**Non-linear Markov-perfect strategy (auxiliary equation approach)**

We first deal with the symmetric model. In this model, we denote the optimal solution to the maximization problem on the right hand side of (3-3-1) by \( E(P) \), which is given by

\[
E(P) = A + W'(P).
\]

Substituting (3-3-2) into (3-3-1),

\[
\delta \cdot W(P) = AE - \frac{1}{2}E^2 - \frac{s}{2}P^2 + (E - A) \cdot (2E - kP).
\]

Let us temporarily assume that \( \delta = 0 \). Then, (3-3-3) reduces to

\[
0 = AE - \frac{1}{2}E^2 - \frac{s}{2}P^2 + (E - A) \cdot (2E - kP)
\]

and, from the above equation, the optimal solution is obtained as

\[
E = \frac{1}{3}(A + kP) \pm \sqrt{\frac{(A + kP)^2}{3} + \frac{sP^2}{3} - \frac{2AkP}{3}}.
\]

Guided by the results for \( \delta = 0 \), we guess the decision rule for the case \( \delta > 0 \) to be

\[
E = \frac{1}{3}(A + kP) + h(P).
\]
Substituting (3-3-5) into (3-3-3) and rearranging,
\[ \delta W'(P) = -\frac{1}{6} A^2 - \frac{1}{6} k^2 P^2 - \frac{s}{2} P^2 + \frac{2}{3} A k P + \frac{3}{2} (h(P))^2. \]

Differentiating the above equation with respect to \( P \), we obtain
\[ \delta W'(P) = -\frac{1}{3} k^2 P - s P + \frac{2}{3} A k + 3 h(P) h'(P). \] (3-3-6)

We call (3-3-6) an “auxiliary equation”. Let us define \( F = \frac{\delta k + k^2 + 3s}{3} \) and \( C = \frac{2A(k+\delta)}{3} \).

Then, the auxiliary equation (3-3-6) reduces to
\[ h'(P) = \frac{\delta h(P) + FX}{3h(P)}. \] (3-3-7)

Returning to (3-3-4), we can easily verify that, if \( \delta = 0 \), then
\[ h(P) = \pm \sqrt{\frac{(A + kP)^2}{9} + \frac{sp^2}{3} - \frac{2AkP}{3}} \]
are the solutions to (3-3-7).

Let us define \( X = P - \frac{C}{F} \). Then \( h(\cdot) \) is a function of \( X \) and (3-3-7) may be rewritten as
\[ \frac{dh(X)}{dX} = \frac{\delta h(X) + FX}{3h(X)}. \]

Thus, by the defining \( Z = \frac{h}{X} \),
\[ \frac{dZ}{dX} X = h - \frac{h}{X} = \frac{\delta h + FX}{3h} - Z = \frac{\delta h + FX - 3hZ}{3h} = \frac{\delta Z + F - 3Z^2}{3Z}. \]

In other words,
\[ \frac{3Z}{F + \delta Z - 3Z^2} \cdot \frac{dZ}{dX} = \frac{1}{X}. \] (3-3-8)

Integrating (3-3-8),
\[ \int \frac{3Z}{F + \delta Z - 3Z^2} \cdot dZ = \int \frac{1}{X} \cdot dX. \]

This reduces to
\[ \int \left( \frac{\xi_1}{Z - Z_a} + \frac{\xi_2}{Z - Z_b} \right) dZ = \int \frac{1}{X} \cdot dX, \] (3-3-9)

where \( \xi_1 = -\xi_a - 1, \xi_2 = \frac{-Z_b}{Z_b - Z_a}, \) and \( Z_a \) and \( Z_b \) are the solutions of \( 3Z^2 - \delta Z - F = 0 \).

Denoting by \( \chi \) the constant of integration, (3-3-9) reduces to
\[ \xi_1 \log |Z - Z_a| + \xi_2 \log |Z - Z_b| - \log |X| = \chi. \]

Recalling the definition of \( Z \) and the equality \( \xi_1 + \xi_2 = -1 \),
\[ K = |h - XZ_a|^\xi_1 |h - XZ_b|^\xi_2, \]

\textsuperscript{4}Dockner and van Long (1993, p. 27) only refer to a negative value of \( h(P) \). They might consider the lower emission level preferable. However, we have to note that \( E \) is negative if \( P > \frac{2Ak}{s} \) in the case \( \delta = 0 \).
where $K = e^*$. Substituting $h = E + \frac{A}{3} - \frac{kP}{3}$ and $X = P - \frac{C}{F}$ into the above equation,

$$K = \left| E - \left( \left( Z_a + \frac{k}{3} \right) P + \frac{A}{3} - Z_a \frac{C}{F} \right) \right|^n \cdot \left| E - \left( \left( Z_b + \frac{k}{3} \right) P + \frac{A}{3} - Z_b \frac{C}{F} \right) \right|^n,$$

where

$$F = \frac{\delta k + k^2 + 3s}{3}, \quad C = \frac{2A(k + \delta)}{3}, \quad Z_a = \frac{\delta}{6} + \sqrt{\frac{\delta^2}{36} + \frac{F}{3}} > 0,$$

$$Z_b = \frac{\delta}{6} - \sqrt{\frac{\delta^2}{36} + \frac{F}{3}} < 0, \quad \xi_a = -\xi - 1, \quad \xi_b = -\frac{Z_b}{Z_b - Z_a}, \quad \text{and} \quad K = e^*.$$

Denote two singular solutions of (3-3-10) by $E_a$ and $E_b$. Then, $E_a$ and $E_b$ are given by

$$E_a = a(P) = \left( Z_a + \frac{k}{3} \right) P + \frac{A}{3} - Z_a \frac{C}{F}$$

and

$$E_b = b(P) = \left( Z_b + \frac{k}{3} \right) P + \frac{A}{3} - Z_b \frac{C}{F}.$$

In Figure 1, there are infinitely many solution curves, $E(P)$, and we notice that more complete solutions are obtained by adopting the auxiliary equation approach. It is easily verified that one of the singular solutions, $E_a = b(P)$, exactly corresponds to the optimal solution under the linear Markov-perfect strategy, $E^{LS}t(P(t)) = (A - \beta^*) - \alpha^*P(t))$.

Utilizing the above results, we obtain the following theorem.

**Theorem 3-3**: If symmetric two regions adopt the non-linear Markov-perfect strategy, any steady-state value of the pollution stock $P^{\text{NS}}$ which satisfies

\[ P = \frac{2A}{k} \]

\[ E = \frac{A + kP}{3} \]

\[ \dot{P} = 0 \]

\[ P = \frac{C}{F} \]

\[ \frac{1}{3} A + \frac{k}{3} \frac{C}{F} \]

\[ \frac{1}{3} A + Z_a \frac{C}{F} \]

\[ \frac{1}{3} A + Z_b \frac{C}{F} \]

Figure 1 (s is sufficiently large)

*Most authors who have derived such an equation have forgotten the absolute-value signs.*
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\[ P \equiv \frac{4A_2 + 4Ak}{2k\delta + k^2 + 4s} < P_{NS} < \frac{2A}{k} \]

is asymptotically stable.

**Proof** Since \( \dot{P} = 2E(P) - kP \), \( P_{NS} \) must satisfy

\[ E(P_{NS}) = \frac{kP_{NS}}{2}. \]  \hspace{1cm} (1)

A sufficient condition for \( P_{NS} \) to be asymptotically stable is

\[ \frac{d\dot{P}}{dP} = 2E'(P_{NS}) - k < 0. \]

Considering (3-3-5), (1), and (3-3-7), the above condition reduces to

\[ \frac{\delta kP_{NS} - 2\delta A + 6P_{NS}F - 6C}{3kP_{NS} - 6A} < \frac{k}{6}. \]  \hspace{1cm} (2)

From the supposition that \( A > E \), the denominator of (2) is negative. Thus (2) reduces to

\[ P_{NS} > \frac{12\delta A + 36C - 6Ak}{6\delta k + 36F - 3k^2}. \]

From the definitions of \( C \) and \( F \), the above inequality reduces to

\[ P_{NS} > \frac{4\delta A + 2Ak}{2\delta k + k^2 + 4s}. \]

Moreover, from the assumption that \( E < A \),

\[ P_{NS} = 2E(P_{NS}) < \frac{2A}{k}. \]  \hspace{1cm} (2)

Q. E. D.

If the discount rate \( \delta \) is sufficiently small, \( \dot{P} \) will be as close as need be to \( P_{CS}(= \frac{2Ak}{k^2 + 4s}) \).

Accordingly, Theorem 3-3 suggests that \( P_{NS} \) can almost attain the level \( P_{CS} \) provided that \( \delta \) is small enough.

Finally, we focus on the asymmetric model. In this model, optimal solutions must satisfy

\[ E_i(P) = A_i + W'_i(P) \] (3-3-11)

and

\[ E_2(P) = A_2 + W'_2(P) = A_2. \]

Hence (3-3-1) reduces to

\[ \delta_1 \cdot W_1(P) = \frac{1}{2} A_1^2 - \frac{1}{2} s_1 P^2 - \frac{1}{2} (W'_1(P))^2 + W'_1(P)(A_1 + A_2 + W'_1(P) - kP) \] (3-3-12)

and

\[ \delta_2 \cdot W_2(P) = \frac{1}{2} A_2^2. \]

Differentiating (3-3-12) with respect to \( P \), we obtain the auxiliary equation

\[ Y'(P) = \frac{(\delta_1 + k)Y(P) + s_1 P}{A_1 + A_2 + Y(P) - kP}, \]

where \( Y(P) \equiv W'_1(P) \). Let us define \( \eta, \theta, \bar{Y}, \) and \( \bar{P} \) as follows:
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\[ \eta = -\frac{(A_1 + A_2) s_1}{s_1 + k(\delta + k)}, \quad \theta = \frac{(A_1 + A_2)(\delta + k)}{s_1 + k(\delta + k)}(= P^{LA}), \]

\[ \bar{Y} = Y - \eta, \quad \text{and} \quad \bar{P} = P - \theta. \]

From the auxiliary equation and these definitions,

\[ \frac{d\bar{Y}}{d\bar{P}} = \frac{dY}{dP} = \frac{(\delta + k)\bar{Y} + s_1 \bar{P}}{\bar{Y} - k\bar{P}} = \frac{\delta + k + s_1(\frac{\bar{P}}{\bar{Y}})}{1 - k(\frac{\bar{P}}{\bar{Y}})}. \]

Let us further define \( D = \frac{\bar{P}}{\bar{Y}} \), so that

\[ \frac{d\bar{Y}}{d\bar{P}} = \frac{d(\frac{\bar{P}}{D})}{d\bar{P}} = \frac{D - \bar{P} \frac{dD}{d\bar{P}}}{D^2}. \]

From the above two equations, we obtain

\[ \frac{1}{D} - \frac{k}{1 - (\delta + 2k)D - s_1 D^2} \frac{dD}{d\bar{P}} = \frac{1}{\bar{P}}. \quad (3-3-13) \]

Let us denote the solutions to the equation \( s_1 x^2 + (\delta + 2k) x - 1 = 0 \) by \( D_a \) and \( D_b \) and then define

\[ \gamma_a = 1 - \gamma_b \quad \text{and} \quad \gamma_b = -\frac{D_a + \frac{k}{s_1}}{D_b - D_a} \]

so that (3-3-13) reduces to

\[ s_1 \left( \frac{1}{D} - \left( \frac{\gamma_a}{D - D_a} + \frac{\gamma_b}{D - D_b} \right) \right) \frac{dD}{d\bar{P}} = \frac{1}{\bar{P}}. \]

Integrating the above equation,

\[ \int \left( \frac{1}{D} - \left( \frac{\gamma_a}{D - D_a} + \frac{\gamma_b}{D - D_b} \right) \right) dD = \frac{1}{s_1} \int \frac{1}{\bar{P}} d\bar{P}. \]

For any constant of integration \( \kappa \),

\[ \log |D| - \gamma_a \log |D - D_a| - \gamma_b \log |D - D_b| - \frac{1}{s_1} \log |\bar{P}| = \kappa. \]

Recalling the definition of \( D \) and the equation \( \gamma_a + \gamma_b = 1 \),

\[ K \cdot |\bar{P}|^{\frac{1}{s_1} - 1} = |\bar{P} - \bar{Y} D_a|^{-\gamma_a} |\bar{P} - \bar{Y} D_b|^{-\gamma_b}, \quad (K = e^\kappa). \]

From now on we confine ourselves to the case \( s_1 = 1 \). From the definition of \( \bar{P} \) and \( \bar{Y} \),

\[ K = |P - \theta - (Y - \eta) D_a|^{-\gamma_a} |P - \theta - (Y - \eta) D_b|^{-\gamma_b}, \quad (3-3-14) \]

where

\[ \eta = -\frac{(A_1 + A_2) s_1}{s_1 + k(\delta + k)}, \quad \theta = \frac{(A_1 + A_2)(\delta + k)}{s_1 + k(\delta + k)}(= P^{LA}), \]

\[ Zagonari (1998) \text{ implicitly assumes that } B = 1 \text{ (} s_1 = 1 \text{ in our model); otherwise, the equation which the general solution of his (26) (Zagonari (1998, p. 60) has to satisfy would be} \]

\[ E(E - E_a)^{-\gamma_a}(E - E_b)^{-\gamma_b} B = CS^\frac{1}{s_1}. \]

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Let us denote the two singular solutions of (3–3–14) by \( Y_a \) and \( Y_b \). Then, \( Y_a \) and \( Y_b \) are given by

\[
Y_a = a(P) = \frac{P - \theta + \eta \gamma_a}{D_a},
\]

and

\[
Y_b = b(P) = \frac{P - \theta + \eta \gamma_b}{D_b}.
\]

We can easily verify that an optimal solution \( A_1 + Y_a \) is the same as the optimal solution under the linear Markov-perfect strategy \( E^LA_t(P(t)) = (A_1 - \beta^*_1) - \alpha^*_1 P(t) \).

We now consider the steady-state of the pollution stock. In the asymmetric model under a non-linear Markov-perfect strategy, the optimal time path of the pollution stock is given by \( \dot{P} = A_1 + A_2 + Y(P) - kP \). Thus if we denote the pollution stock which satisfies \( \dot{P} = 0 \) by \( P^{N^A} \), \( \dot{P} = 0 \) reduces to

\[
P^{N^A} = Y(P^{N^A}) = -(A_1 + A_2) + kP^{N^A}.
\]

Moreover, we can verify the following five assertions.

1. Lines \( Y_a = a(P) \), \( Y_b = b(P) \), and \( Y^{N^A} = Y(P^{N^A}) \) pass across the point \((\theta, \eta)\).
2. The intercept of line \( Y^{N^A} = Y(P^{N^A}) \) is greater than that of line \( Y_b = b(P) \).
3. The intercept of line \( Y_a = a(P) \) takes a negative value.
4. The steady-state line \( Y^{N^A} = Y(P^{N^A}) \) passes through the point \( \left(\frac{A_2}{k}, -A_1\right) \).
5. The intercept of line \( Y^{N^A} = Y(P^{N^A}) \) is smaller than \(-A_1\).

Utilizing these facts, we can depict solution curves as in Figure 2.

The assumption that \( E_t \geq 0 \) implies that \( Y \geq -A_1 \). It follows that if \( \eta > -A_1 \) then there exist solution curves reachable to the steady-state line in zone X. And we can verify that curve X is the solution curve which supports the smallest steady-state of the pollution stock provided that \( \eta > -A_1 \). Moreover

\[
\frac{d\dot{P}}{dP} = Y'(P) - k = \frac{\delta_1 Y + (s_1 + k^2)P - (A_1 + A_2)k}{A_1 + A_2 + Y - kP},
\]

the denominator of which is greater than zero in zone X and the numerator of which approaches negative values as \((P, Y)\) tends to \( \left(\frac{A_2}{k}, -A_1\right) \). Hence, the steady-state \( P^{N^A} = \frac{A_2}{k} \) is locally stable. Thus, we have the following theorem.
Theorem 3-4: If two asymmetric regions adopt non-linear Markov-perfect strategies under the condition that \( \delta_i > \frac{A_2}{A_1 k} - k \) (i.e. \( \eta > -A_1 \)), there is a solution curve which starts from \( Y(0) < 0 \) and ends at the locally stable steady-state stock of pollution, \( P_{NA} = \frac{A_2}{k} \).

Strictly speaking, \( Y(0) \) satisfies \(-A_1 < Y(0) < \) the intercept of line \( Y_a = a(P) \).

It may be verified that for \( P^{CA} > \frac{A_2}{k} \) it is necessary and sufficient that
\[
\mu < \frac{A_1 k (\delta_2 + k) - 2A_2}{A_1 k (\delta_2 - \delta_1)},
\]
for which the condition \( \delta_i > \frac{2A_2}{A_1 k} - k \) suffices.

4. Concluding remarks

In this paper, we have formulated TBP problems as 2-player dynamic games under a feedback information structure, derived a tool for analyzing such dynamic games, namely the H–J–B equation, from Bellman’s principle of optimality, and applied this equation to symmetric and asymmetric models. From the results of section 3, we arrive at the following two conclusions.

First, we compare our symmetric model with our asymmetric model. In the symmetric model, the level of the steady-state stock of pollution under non-cooperative pollution control in which a non-linear strategy is adopted is close enough to that under cooperative pollution control on condition that the discount rate is sufficiently low, but the former can never be less than the latter. On the other hand, in the asymmetric model, it is possible that the steady-state stock of pollution under non-cooperative pollution control is lower than that under cooperative pollution control. For example, in the case of assuming a non-linear
strategy, a sufficient condition for the achievement of the above situation is a sufficiently high discount rate in the eco-development oriented region (region 1).

Second, we examine whether the governments of the two regions should impose linearity on their strategy in the case of non-cooperative pollution control. The non-linear Markov-perfect strategy is preferable in that we can find more solution curves under the non-linear strategy than under the linear one; and, what is more, one of the former outcomes is characterized by a lower steady-state stock of pollution than is the latter outcome.

As already noted, most earlier analyses of TBP problems have been confined to the linear-quadratic functional form and to a restricted set of parameter values. We propose to carry out simulations for a greatly enlarged set of parameter values and even for alternative functional forms. These simulations will be based on Kimura (1998)'s "intelligent simulation method". In this way, we hope to arrive at qualitative conclusions of much greater scope than have hitherto been available.

References

A Survey of Analyses of the Transboundary Pollution Problem: Symmetric and Asymmetric Dynamic Models