A Note on the Comparison of Two Rows (Columns) of the Inverse of an M-matrix; An Elaboration of Metzler’s Method

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1. Introduction

Metzler (1951) investigated the effects of taxes and subsidies on prices in an input-output system (henceforth, Metzler’s problem, for short). Later, Atsumi (1981) extended and elaborated Metzler (1951). However, a close scrutiny of Metzler’s problem reveals that the mathematical essence of the problem lies in the comparison of two rows of the inverse of an M-matrix (defined later). To this point, the former approached by the manipulation on determinants, while the latter by his new elementary method based directly on the fact that the Leontief inverse equals the sum of an identity matrix and the product of the input-output coefficients matrix and the Leontief inverse. Speaking from another point of view, the plentiful contents of Atsumi’s mathematical result (Atsumi (1981; THEOREM, pp. 33-34)) rely widely on the special form of Leontief matrix (an identity matrix – a nonnegative square matrix) and Leontief matrix is a special species of M-matrices. On the other hand, Metzler’s analytical method is of wider applicability than Atsumi’s. Therefore, it would be of some value to inspect to what extent Atsumi’s theorem remains valid for M-matrices in general, along the line of Metzler’s thought.
2. Preliminaries

For the inspection stated in the previous section, we need the following notation.

\( \mathbb{N} \) : the set of indices; \( \mathbb{N} = \{1, 2, \cdots, n\} \)

\( \mathbb{N}(k) \) : a subset obtained from \( \mathbb{N} \) by deleting an index \( k \)

\( A \) : an \( n \times n \) matrix

\( a^j \) : the \( j \)-th column of \( A \)

\( a_i \) : the \( i \)-th row of \( A \)

\( A^{-1} \) : the inverse of \( A \)

\( a^{ij} \) : the \((i, j)\)-th element of \( A^{-1} \)

\( I \) : an \( n \times n \) identity matrix

\( e_v, (e^v) \) : the \( v \)-th row (column) of \( I \)

\( R_{ij}(c) \) : a matrix obtained from \( I \) by replacing the \((i, j)\)-th element \( (\delta_{ij}, i \neq j) \) with a scalar \( c \)

\( a^j(s)(a_i(s)) \) : a subvector obtained from \( a^j(a_i) \) by deleting the \( s \)-th element

\( A \begin{pmatrix} i \\ j \end{pmatrix} \) : a submatrix obtained from \( A \) by deleting \( a_j \) and \( a^j \)

\( A \begin{pmatrix} r & s \\ t & v \end{pmatrix} \) : a submatrix obtained from \( A \) by deleting \( a_r, a_s, a^r \) and \( a^s \)

\( I \) : an \( n \times 1 \) vector consisting of \( n \) ones

To begin with, we demonstrate a formula which is often utilized later.

Let \( A \) and \( c \) be a nonsingular real matrix of order \( n \) and scalar respectively. Then,

\[
R_{ij}(-c) \cdot A^{-1} = \left( A \cdot R_{ij} (c) \right)^{-1} = \left( \det A \right)^{-1} \cdot \text{adj} \left( A \cdot R_{ij} (c) \right),
\]

(1)

where \( \text{adj}(A \cdot R_{ij}(c)) \) denotes the adjugate matrix of \( A \cdot R_{ij}(c) \) and \( \det A \) signifies the determinant of \( A \).

In view of (1), we see that

\[
e_v \left( A \cdot R_{ij} (c) \right)^{-1} = e_v \left( R_{ij} (-c) \cdot A^{-1} \right) = \begin{cases} 
( A^{-1})_v & v \neq i \\
( A^{-1})_i - c ( A^{-1})_j & v = i
\end{cases},
\]

where \( (A^{-1})_v \) is the \( v \)-th row of \( A^{-1} \).

This equation, coupled with (1) and Cramer’s rule, yields
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\[
\left( \left( \mathbf{A} \cdot \mathbf{R}_{ij}(c) \right)^{-1} \right)_{ij} = a^{iv} - c a^{iv} = \left( \det \mathbf{A} \right)^{-1} \cdot (-1)^{iv} \cdot \det \left( \mathbf{A} \cdot \mathbf{R}_{ij}(c) \right)^{v}_{i}, \quad (2)
\]

\(v = 1, 2, \ldots, n\)

where \(\mathbf{A} \cdot \mathbf{R}_{ij}(c)^{v}_{i}\) signifies a submatrix obtained from \(\mathbf{A} \cdot \mathbf{R}_{ij}(c)\) by deleting the \(v\)-th row and the \(i\)-th column.

Next, we define an M-matrix.

**Definition:** A real square matrix \(\mathbf{A}\) is said to be an M-matrix, if it is a matrix with offdiagonal elements nonpositive and its all principal minors are positive.

3. Analysis

Preliminary matters out of way. We can now state and prove our theorem.

**Theorem:** Let \(\mathbf{A}\) be an \(n \times n\) M-matrix. Then the following assertions hold true.

(i) Notice that there exists an \(n \times 1\) positive vector \(\mathbf{x}\) such that \(\mathbf{Ax} > 0\). Then,

(i-1) if \(x_i \geq x_k\), then \(a^{ii} > a^{ik}\)

(i-2) there exists an index \(s\) satisfying

\[a^{ss} > a^{is} \quad \text{for all } i \in \mathbb{N}\]

(i-3) in particular, if the vector \(\mathbf{x}\) is proportionate to \(\mathbf{1}\), then

\[\text{for any } i \in \mathbb{N}, \quad a^{ii} > a^{ij} \quad (j = 1, 2, \ldots, n)\]

(ii) If there exist a positive number \(d\) and an index \(k\) such that

\[\mathbf{A}l = de^k\quad (3)\]

then

(ii-1) \((\mathbf{A}^{-1})^k = d^{-1} \cdot \mathbf{1}\)

(ii-2) for any \(i \in \mathbb{N}\) and any \(j \in \mathbb{N}\) \((k)\),

\[a^{ij} \geq a^{bk}\quad (4)\]

(ii-3) in addition, loose inequalities (4) are strengthened to be strict ones if and only if

\[\mathbf{A} \left( \begin{array}{c} k \\ k \end{array} \right) \text{ is indecomposable, where } (\mathbf{A}^{-1})^k \text{ is the } k\text{-th column of } \mathbf{A}^{-1}.\]

\(^1\) Since \(\mathbf{A}\) is an M-matrix, \(\mathbf{A}^{-1} \succeq [0]\). Hence, for any given \(\mathbf{b} > 0\), define \(\mathbf{x}\) by \(\mathbf{A}^{-1} \mathbf{b}\). Then, \(\mathbf{Ax} > 0\) and \(\mathbf{x} > 0\).
Proof

\((i-1)\) Letting \(c = 1, j = h\) and \(\nu = i\) in (2), then

\[
a^\mu - a^h = (\det A)^{i+1} \cdot \det \left( A \cdot R_{ih}(1) \left( \begin{array}{c} i \\ i \end{array} \right) \right)
\]

\[
= (\det A)^{-1} \cdot \det \left( A \cdot R_{ih}(1) \left( \begin{array}{c} i \\ i \end{array} \right) \right)
\]

Thus, the inequality

\[
\left( A \cdot R_{ih}(1) \left( \begin{array}{c} i \\ i \end{array} \right) \right) \cdot x(i) = \sum_{j \neq i} a^j(i) x_j + \left(a^i(i) + a^h(i)\right) x_h
\]

\[
\geq \sum_{j \neq i} a^j(i) x_j + a^h(i) x_h + a^i(i) x_i
\]

\[
[\cdots \geq x_h \text{ and } a^i(i) \leq 0]
\]

= a subvector obtained from \(Ax\) by deleting the \(i\)-th element.

Thus, it is seen that the inequality

\[
\left( A \cdot R_{ih}(1) \left( \begin{array}{c} i \\ i \end{array} \right) \right) \cdot x(i) > 0
\]

has a positive solution \(x(i)\). Moreover, \(\left( A \cdot R_{ih}(1) \left( \begin{array}{c} i \\ i \end{array} \right) \right)\) is a matrix with offdiagonal elements nonpositive. Therefore, \(\det \left( \left( A \cdot R_{ih}(1) \left( \begin{array}{c} i \\ i \end{array} \right) \right) \right) > 0\). This, together with (5), establishes that

\(a^\mu > a^h\).

\((i-2)\) Define \(x_i = \max_{i \in \mathbb{N}} x_i\). Then, for all \(i \in \mathbb{N}\), \(x_i \geq x_i\). Hence, \((i-1)\) just proved ensures the assertion.

\((i-3)\) By assumption, there is a scalar \(c\) such that \(x_i = c = x_j\) for all \(i\) and \(j\) of \(\mathbb{N}\). Consequently, it follows from \((i-1)\) that

\[
a^\mu > a^h \quad \text{for all } i \text{ and } j \text{ of } \mathbb{N}
\]

\((ii-1)\) Premultiplying the both sides of (3) by \(d^{-1} \cdot A^{-1}\), we obtain the assertion.

\((ii-2)\) When \(i = k\), the assertion is obvious. Hence, in what follows, we assume that \(i \neq k\). Putting \(v = j, j = k\) and \(c = 1\) in (2) and noting that \(\det A > 0\), we obtain

\[
\text{sgn}(a^\mu - a^h) = \text{sgn} \left\{ (-1)^{i+j} \cdot \det \left( A \cdot R_{ik}(1) \left( \begin{array}{c} j \\ i \end{array} \right) \right) \right\}
\]
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Therefore, it suffices to prove that

\[ (-1)^{i+j} \cdot \det \left( A \cdot R_{ik} \right) \left( \begin{array}{c} j \\ i \end{array} \right) \geq 0 \quad \text{for all} \ i \in \mathbb{N}(k) \ \text{and} \ j \in \mathbb{N}(k) \]  

(8)

The assumption that \( A^l = de^k \) implies that

\[ a^k(j) + a^l(j) = de^k(j) - \sum_{r \neq i, k} a^r(j). \]  

(9)

Moreover, \( \left( A \cdot R_{ik} \right) \left( \begin{array}{c} j \\ i \end{array} \right) \) is the matrix obtained from \( A \left( \begin{array}{c} j \\ i \end{array} \right) \) by replacing \( a^k(j) \) with \( a^k(j) + a^l(j) \). This fact, in conjunction with (7), yields

\[
\begin{align*}
\det \left( \left( A \cdot R_{ik} \right) \left( \begin{array}{c} j \\ i \end{array} \right) \right) &= \\
\left\{ \begin{array}{l}
\det \left( a^1(j), \cdots, a^{i-1}(j), a^{i+1}(j), \cdots, a^{k-1}(j), de^k(j), a^{k+1}(j), \cdots, a^n(j), \right)^{i < k} \\
\det \left( a^1(j), \cdots, a^{k-1}(j), da^k(j), \cdots, a^{i-1}(j), a^i(j), a^{i+1}(j), \cdots, a^n(j), \right)^{i > k}
\end{array} \right.
\end{align*}
\]

The cofactor expansion of the determinants of the right hand side of the above equation with respect to the column in which \( de^k(j) \) lies and the Cramer’s rule lead to

\[ (-1)^{i+j} \det \left( \left( A \cdot R_{ik} \right) \left( \begin{array}{c} j \\ i \end{array} \right) \right) = \\
\left\{ (-1)^{i+j} \det \left( A \left( \begin{array}{c} k \\ k \end{array} \right) \left( \begin{array}{c} j \\ i \end{array} \right) \right) = \det \left( A \left( \begin{array}{c} k \\ k \end{array} \right) \right) \cdot \det \left( A \left( \begin{array}{c} k \\ k \end{array} \right)^{-1} \right) \right\} \cdot d \quad (i-k)(j-k) > 0 \]

\[ (-1)^{i+j-1} \det \left( A \left( \begin{array}{c} k \\ k \end{array} \right) \left( \begin{array}{c} j \\ i \end{array} \right) \right) = \det \left( A \left( \begin{array}{c} k \\ k \end{array} \right) \right) \cdot \det \left( A \left( \begin{array}{c} k \\ k \end{array} \right)^{-1} \right) \right\} \cdot d \quad (i-k)(j-k) < 0 \]

(10)

Since \( A \) is assumed to be an M-matrix, so is \( A \left( \begin{array}{c} k \\ k \end{array} \right) \). Hence, \( \det \left( A \left( \begin{array}{c} k \\ k \end{array} \right) \right) > 0 \) and \( \left( A \left( \begin{array}{c} k \\ k \end{array} \right)^{-1} \right) \geq [0] \), where \([0]\) is a zero matrix. Furthermore, \( d > 0 \). Thus (8) is seen to hold, which in turn guarantees (4).
(ii-3) Noticing that the inverse of an M-matrix $A \begin{pmatrix} k \\ k \end{pmatrix}$ is positive if and only if $A \begin{pmatrix} k \\ k \end{pmatrix}$ is indecomposable, it follows from (7) and (10) that the indecomposability of $A \begin{pmatrix} k \\ k \end{pmatrix}$ is equivalent to the strict inequalities

$$a^{ij} > a^{kj} \quad \text{for any } i \in \mathbb{N}(k) \text{ and any } j \in \mathbb{N}(k)$$

This completes the proof.

**Remark** Replacing equation (2) by

$$\left( \left( R_y (c) \cdot A \right)^{-1} \right)_{ij} = a^{ij} - ca^{vi} = \left( \det A \right)^{-1} (-1)^{i+j} \cdot \det \left( R_y (c) \cdot A \right)^{jv}$$

(2')

and noticing that the following assumptions

$$\exists y' > 0 \mid y' A > 0 \quad \text{and} \quad \exists k \in \mathbb{N} \mid A' = e_k$$

are respectively equivalent to

$$\exists y > 0 \mid A' y > 0 \quad \text{and} \quad \exists k \in \mathbb{N} \mid A' 1 = e_k$$

and that $A'$ is also an M-matrix, the theorem just verified is valid for the comparison of two columns of the inverse of an M-matrix, where the prime "'" denotes the transposition of a matrix and/or vector.

4. **Concluding Remarks**

Comparing Atsumi’s theorem with our’s, it is most remarkable that (ii-3) sharpens a part of the third assertion of Atsumi’s theorem (“In addition, if all of the principal minor matrices of order $(n-1)$ of $(I - \alpha)$ are indecomposable, then for any column other than $A_k^2$ the $k$-th element is smaller than any other element in the same column). In fact, (ii-3) clarifies that it is the indecomposability of $A \begin{pmatrix} k \\ k \end{pmatrix}$ that is equivalent to inequality (11). Contrary to this, the quoted part requires the indecomposability of all principal submatrices of order $(n-1)$ as a sufficient condition for (11) to hold. However, unfortunately the remaining part of his third assertion is kept to be verified.

Assertions (ii-1) and (ii-2) are completely same as the second assertion of Atsumi’s theorem.

Assertion (i-3) is parallel to the first assertion of Arsumi’s theorem, though the former seemingly looks slightly stronger than the latter. Nevertheless, this difference is more apparent than real, because we has made use of a positive vector $x$ such that $Ax > 0$ but Atsumi supposed loose inequal-

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2 The $k$-th column of $(I - \alpha)^{-1}$. 
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ity (in his notation, \((x - \mathbf{\alpha}) \mathbf{1} \geq 0\)). Thus, we may well conclude that to consider M-matrix in general in stead of Leontief matrix loses little and gains a little.

References

