Associative Memories Using Dynamical Neural Networks with Limit Cycles

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Abstract This paper studies associative memories using a dynamical neural network of the McCullough-Pitts model in which each class of given standard information patterns is assigned to a limit cycle. In particular, a method to construct a connection matrix of the neural network which assigns limit cycles as required is studied. Further, some examples are given to statistically evaluate the performance of the proposed associative memories.

Key words McCullough and Pitts Model, Domain of Attraction, Limit Cycle

1 Introduction

The problem realizing associative memories using dynamical neural networks has been studied [1]-[6]. The main issues to be studied include:

1. How to store the standard pattern vectors as asymptotically stable equilibrium points.
2. How to adjust the Domain of Attractons of each standard pattern vector.
3. How to decrease the number of spurious asymptotically stable equilibrium points.

This paper studies associative memories using the McCullough-Pitts model of dynamical neural networks in which each class of given standard information patterns is assigned to a limit cycle. In particular, a method to construct a connection matrix of the neural network which assigns limit cycles as required is studied. Further, some computer simulation results are given to evaluate the performance statistically.

2 Dynamical Neural Networks

In the McCullough-Pitts model of dynamical neural networks(DNN's), each neuron assumes discrete values in the state space $B := \{-1, 1\}$. Consider a DNN consisting of $n$ neurons, and define $B^n := \{[x_1, \ldots, x_n]^T | x_i \in B\}$. Then McCullough-Pitts model is given as follows.

Definition 2.1 (McCulloch-Pitts Model)

$$(MP) \begin{cases} y(k + 1) = \text{Sgn}(Wy(k) + h)(k = 0, 1, \cdots) \\ y(0) = z \in B^n \end{cases}$$

where $z \in B^n$ is an input pattern, $y(k) \in B^n(k \geq 0)$ is the output at time $k$, $W \in \mathbb{R}^{n \times n}$ is the connection matrix, $h \in \mathbb{R}^n$ is the threshold vector and Sgn is an $n$-dimensional vector valued signum function, defined by

$$\text{Sgn} := \begin{bmatrix} \text{sgn} \\ \vdots \\ \text{sgn} \end{bmatrix} \in \mathbb{R}^n$$

$$\text{sgn}\{(Wy(k) + h)_i\} := \begin{cases} 1 & (Wy(k) + h)_i > 0 \\ -1 & (Wy(k) + h)_i < 0 \\ y_i(k) & (Wy(k) + h)_i = 0 \end{cases}$$

with $(Wy(k) + h)_i$ being the $i$-th component of $Wy(k) + h$ ($i = 1, \ldots, n$). □

For the MP model, the following definition is introduced.
Definition 2.2

(i) \( y^* \in \mathcal{B}^n \) is called an equilibrium point if
\[
\text{Sgn}(W_y^* + h) = y^*
\]

(ii) For an equilibrium point \( y^* \), the set defined by
\[
D(y^*) := \{ z \in \mathcal{B}^n \mid y(0) = z, \lim_{k \to \infty} y(k) = y^* \}
\]
is called the domain of attraction for \( y^* \).

(iii) If \( D(y^*) = \{ y^* \} \), \( y^* \) is said to be an isolated equilibrium point, and if \( D(y^*) \neq \{ y^* \} \), \( y^* \) is said to be an asymptotically stable equilibrium point. \( \Box \)

The following lemma is well known.

Lemma 2.4 \( y^* \in \mathcal{B}^n \) is an equilibrium point of (MP) if and only if
\[
(W y^* + h)_i \geq 0 \quad (i = 1, \ldots, n)
\]

Further, the following lemma can be easily proved, and its proof is omitted.

Theorem 2.3 Assume that in the MP model, the connection matrix \( W \) is symmetric and non-negative definite (i.e., \( W \) satisfies \( W^T = W \) and \( \lambda^T W x \geq 0, \forall x \in \mathcal{B}^n \)), and let the energy function \( E \) be defined by
\[
E(y) := -1/2 y^T W y - y^T h \quad (\forall y \in \mathcal{B}^n).
\]

Then for any input \( y(0) := z \in \mathcal{B}^n \), the sequence \( \{ y(k) \mid k \geq 0 \} \) of outputs defined by (MP) satisfies
\[
E(y(k+1)) < E(y(k)), \quad y(k+1) \neq y(k).
\]

Moreover, there exist some \( k^* \geq 0 \) and some equilibrium point \( y^* \in \mathcal{B}^n \) such that
\[
y(k^*) = y(k^* + 1) = \cdots = y^*.
\]

(proof) For \( k \geq 0 \), define \( \Delta y(k) := y(k+1) - y(k) \). Noting that \( y(k+1) = y(k) + \Delta y(k) \), it easily follows that
\[
\Delta E(k) := E(y(k+1)) - E(y(k)) = -\Delta y^T(k)(W y(k) + h) -1/2\Delta y^T(k) W \Delta y(k).
\]

Since \( W \) is a non-negative definite matrix, the second term is less than or equal to 0. Noting that if \( (W y(k) + h)_i > 0 \) then \( \Delta y_i(k) = 0 \) or \( 2 \) and if \( (W y(k) + h)_i < 0 \) then \( \Delta y_i(k) = 0 \) or \( -2 \), it is easily seen that the first term is less than or equal to 0. Hence, if \( \Delta y(k) \neq 0 \) the first term is less than 0. Therefore \( \Delta y(k) = y(k+1) - y(k) \neq 0 \) implies \( E(y(k+1)) < E(y(k)) \).

Next, noticing that \( \mathcal{B}^n \) is a finite set, it is easily seen that there is \( k^* \geq 0 \) such that \( y(k^*) = y(k^* + 1) = \cdots = y^* \), implying that \( y^* \) is an equilibrium point. \( \Box \)

Theorem 2.5 Let \( y^{(1)}, y^{(2)}, \ldots, y^{(r)} \in \mathcal{B}^n \) be the standard pattern vectors to be stored in (MP), and the connection matrix \( W \in \mathcal{R}^{n \times n} \) and the threshold vector \( h \in \mathcal{R}^n \) are defined by
\[
\begin{align*}
W := Y Y^T, \quad Y := [y^{(1)} \ y^{(2)} \ \cdots \ y^{(r)}] & \in \mathcal{B}^{n \times r} \\
h := [h_1 \ \cdots \ h_n]^T & \in \mathcal{R}^n, \quad h_i \leq 1
\end{align*}
\]
where \( Y \) is the Moore-Penrose Inverse matrix of \( Y \).

Then following properties hold:

(i) \( W \in \mathcal{R}^{n \times n} \) is symmetric and non-negative definite. Further, each \( y^{(q)} \) is an eigenvector of \( W \) and its eigenvalue is equal to 1, i.e.,
\[
W y^{(q)} = y^{(q)} \quad (q = 1, \ldots, r).
\]

(ii) Each \( y^{(q)} \) is an equilibrium point of (MP), i.e.,
\[
(W y^{(q)} + h)_i y^{(q)}_i \geq 0 \quad (i = 1, \ldots, n)
\]

(iii) The energy function at each \( y^{(q)} \) is given by
\[
E(y^{(q)}) = -n/2 - (y^{(q)})^T h \quad (q = 1, \ldots, r).
\]

(iv) Consider \( W, Y \) to be the linear operators \( W : \mathcal{R}^n \to \mathcal{R}^n, Y : \mathcal{R}^n \to \mathcal{R}^r \), and denote the range of \( Y \) by \( \text{Im} Y \). Then \( W \) is the orthogonal projection from \( \mathcal{R}^n \) onto \( \text{Im} Y \subset \mathcal{R}^n \). \( \Box \)
Based on the above theorem, the choice of the connection matrix \( W \in \mathbb{R}^{n \times n} \) and threshold vector \( h \in \mathbb{R}^n \) given by (2) for (MP) is called the Orthogonal Projection Method (OPM).

In order for the OPM to be meaningful for associative memories, it is naturally required that \( \text{rank}(Y) < n \) because if \( \text{rank}(Y) = n \) then \( W = Y^TY = I \) and hence every \( x \in \mathbb{B}^n \) becomes an equilibrium point. Therefore, in this investigation we assume \( r < n \) to ensure this condition. However, to include the case \( r \geq n \) we can introduce a generalized McCulloch-Pitts model of the form

\[
\begin{align*}
\text{GMP} \quad \begin{cases}
\xi(k + 1) &= \text{Sgn}(W \xi(k) + h) \\
\xi(0) &= \text{Sgn}(V_1 x + g_1) \\
y(k + 1) &= \text{Sgn}(V_2 \xi(k) + g_2) \\
y(0) &= x = z \\
\end{cases}
\end{align*}
\]

where \( \xi(k) \in \mathbb{B}^m \) with \( m > n \), \( W \in \mathbb{R}^{m \times m} \), \( V_1 \in \mathbb{R}^{m \times n} \), \( V_2 \in \mathbb{R}^{n \times m} \), \( h \in \mathbb{R}^m \) and \( g_1 \in \mathbb{R}^m \), \( g_2 \in \mathbb{R}^n \). Therefore, the MP model can be considered as a special case of the GMP model by choosing \( m = n \), \( V_1 = I_n \), \( V_2 = I_m \) and \( g_1 = 0_m \), \( g_2 = 0_n \).

3 The Modified OPM and its Computer Simulations

It is seen from Theorem 2.5 that if the connection matrix \( W \) is given by the orthogonal projection method, each standard pattern vector is assigned as eigenvector of \( W \) corresponding to eigenvalue 1. In this section, we propose a method to construct connection matrix \( W \) in which given standard pattern vectors are stored as equilibrium points in a more general fashion than the OPM. Further, we give some computer simulation which indicates the effectiveness of the proposed method.

**Theorem 3.1** Given standard pattern vectors \( y^{(1)}, y^{(2)}, \ldots, y^{(r)} \in \mathbb{B}^n \) are assigned as equilibrium points in (MP) if and only if the connection matrix \( W \in \mathbb{R}^{n \times n} \) and the threshold vector \( h = [h_1, \ldots, h_n]^T \in \mathbb{R}^n \) satisfy

\[
WY + H = [\Gamma^{(1)}y^{(1)}, \ldots, \Gamma^{(r)}y^{(r)}] = Z \in \mathbb{R}^{n \times r}
\]

for some non-negative definite diagonal matrices \( \Gamma^{(i)} := \text{diag}(\gamma^{(i)}_1, \ldots, \gamma^{(i)}_h) \), \( \gamma^{(i)}_j \geq 0 \), where

\[
Y := [y^{(1)}, \ldots, y^{(r)}] \in \mathbb{B}^{n \times r} \\
H := [h_1, \ldots, h_n] \in \mathbb{R}^{n \times r}.
\]

**Proof** \( \text{Sgn}(W y^{(i)} + h) = y^{(i)} \) \( (i = 1, \ldots, r) \)

\[
\iff (W y^{(i)} + h)_j \geq 0 \\
(\forall i \in 1, \ldots, r; j = 1, \ldots, n)
\]

\[
\iff \text{there is } \gamma^{(i)}_j \geq 0 \text{ such that} \\
(W y^{(i)} + h)_j = \gamma^{(i)}_j y^{(i)}_j \\
(\forall i \in 1, \ldots, r; j = 1, \ldots, n)
\]

\[
\iff \text{there exist non-negative definite diagonal matrices } \Gamma^{(i)}(i = 1, \ldots, r), \text{ such that} \\
WY + H = [\Gamma^{(1)}y^{(1)}, \ldots, \Gamma^{(r)}y^{(r)}].
\]

Based on Theorem 3.1, the choice of connection matrix \( W \in \mathbb{R}^{n \times n} \) and threshold vector \( h \in \mathbb{R}^n \) given by

\[
\begin{align*}
W &:= ZY^T \\
h &:= [h_1, \ldots, h_n]^T, \\
| h_k | &\leq \min\{\gamma^{(i)}_k : i = 1, \ldots, r\}
\end{align*}
\]

is called the Modified Orthogonal Projection Method (MOPM).

**Example 3.2** We performed some computer simulation to illustrate effectiveness of the MOPM. The standard pattern vectors \( y^{(i)}(i = 1, \ldots, 12) \) are taken to be twelve Japanese characters \{ 東, 京, 電, 機, 大, 学, 情, 報, 科, 程, 業, 博 \} expressed in the form of vectors in \( \mathbb{B}^{576} \), and the input \( z = y(0) \) is taken to be various noisy patterns of each \( y^{(i)} \).

In this computer simulation, the non-negative definite diagonal matrices \( \{\Gamma^{(i)} : i = 1, \ldots, 12\} \) are chosen to have the form \( \Gamma^{(i)} := \gamma I_{576}(i = 1, \ldots, 4) \) and \( \Gamma^{(i)} := I_{576}(i = 5, \ldots, 12) \) where \( \gamma = 1.2 \text{(Table 1)} \). A noisy pattern \( z \in \mathbb{B}^{576} \) is generated by converting each component \(-1 \text{ or } 1\) of \( y^{(i)} = [y^{(i)}_1, \ldots, y^{(i)}_{576}]^T \) to \( 0 \) or \(-1 \text{ with a constant probability, and for each standard pattern} y^{(i)} 200 \text{ noisy patterns are generated.} \)

This simulation result shown in Table 1 indicates that (1) no limit cycle is found, (2) each \( y^{(i)} \) is asymptotically stable equilibrium point and (3) the domain of attraction for each \( y^{(i)} \) can be effectively adjusted by the choice of non-negative definite diagonal matrices \( \Gamma^{(i)} \).

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4 DNN’s with Limit Cycles

In this section, we consider how to construct a DNN with limit cycles. First, we give the definition of limit cycles for (MP).

**Definition 4.1** Consider a DNN given by (MP). Then:

(i) An ordered set \( \mathbf{y}^{*} = (y^{*}_1, y^{*}_2, \ldots, y^{*}_m) \), \( y^{*}_i \in B^n \) \((i = 1, \ldots, m)\), is called a limit cycle of prime period \( m \), if for each \( i = 1, \ldots, m \), the output sequence \( \{y(k) \mid k \geq 0\} \) with \( y(0) = y^{*}_i \) satisfies

\[ y(k) \neq y^{*}_i(k = 1, \ldots, m - 1) \]

\[ y(m) = y^{*}_i. \]

(ii) For a limit cycle \( \mathbf{y}^{*} = (y^{*}_1, y^{*}_2, \ldots, y^{*}_m) \) of prime period \( m \), the set defined by

\[ D(\mathbf{y}^{*}) := \{x \in B^n \mid y(0) = x \Rightarrow \exists k \geq 0, y(k) = y^{*}_i\} \]

is called the domain of Attraction of \( \mathbf{y}^{*} \).

(iii) Let \( \mathbf{y}^{*} = (y^{*}_1, y^{*}_2, \ldots, y^{*}_m) \) be a limit cycle of prime period \( m \). If \( D(\mathbf{y}^{*}) \neq \{y^{*}_1, y^{*}_2, \ldots, y^{*}_m\} \), \( \mathbf{y}^{*} \) is called an isolated limit cycle, and if \( D(\mathbf{y}^{*}) \neq \{y^{*}_1, y^{*}_2, \ldots, y^{*}_m\} \), \( \mathbf{y}^{*} \) is called an asymptotically stable limit cycle. □

First of all, the following theorem concerning limit cycles is stated without proof.

**Theorem 4.2** Let \( \{y^{(1)}, y^{(2)}, \ldots, y^{(r)}\} \) be given where each \( y^{(i)} \) has the form

\[ y^{(i)} := (y^{(i)}_1, y^{(i)}_2, \ldots, y^{(i)}_m), \quad y^{(i)}_j \in B^n. \]

Then, each \( y^{(i)} \) is assigned as a limit cycle with prime period \( m_i \) \((i = 1, \ldots, r)\) in (MP) if and only if the connection matrix \( W \in R^{nxn} \) and the threshold vector \( h = [h_1, \ldots, h_n]^T \in R^n \) satisfy

\[ WY + H = U \]

where

\[ Y := [y^{(1)}, y^{(2)}, \ldots, y^{(r)}] \]

\[ U := [u^{(1)}, u^{(2)}, \ldots, u^{(r)}], \]

\[ u^{(i)} := \Gamma_{y_2} y^{(i)}_2, \Gamma^{(i)}_{y_3} y^{(i)}_3, \ldots, \Gamma^{(i)}_{y_m} y^{(i)}_m, \Gamma^{(i)}_1 y^{(i)}_1, \]

\[ \Gamma^{(i)}_j := \text{diag}(\gamma^{(i)}_{j1}, \ldots, \gamma^{(i)}_{jm}), \gamma^{(i)}_{jk} > 0 \]

\[ H := [h, \ldots, h]. \]

Then based on Theorem 4.2, the choice of connection matrix \( W \) and threshold vector \( h \in R^n \) given by

\[ W := U^T \]

\[ h := [h_1, \ldots, h_n]^T, \]

\[ |h_k| \leq \min\{\gamma^{(i)}_{jk} : i = 1, \ldots, r; j = 1, \ldots, m_i\} \]

is called the Limit Cycles Method (LCM).
Example 4.3 We performed computer simulations to illustrate effectiveness of the LCM. The standard limit cycles $y^{(i)}(i = 1, \ldots, 6)$ are constructed using six Japanese characters \{ 東, 京, 電, 機, 大, 学 \} expressed as vectors in $B^{576}$. Each standard limit cycle $y^{(i)}$ has six characters consisting of the standard character, the characters moved to upwards, downwards, left and right, respectively, and an expanded character. As an example, the standard limit cycle for 東 is shown in Fig.2. The positive definite diagonal matrices $\{\Gamma_j^{(i)} : i = 1, \ldots, 6; j = 1, \ldots, 6\}$ are chosen to be $\Gamma_j^{(i)} := \gamma I_{576}$ ($i = 1, 3, 5$) and $\Gamma_j^{(i)} := I_{576}$ ($i = 2, 4, 6$) for each $j = 1, \ldots, 6$ and the threshold vector $h = 0$. The inputs $z = y(0)$ are taken to be various noisy patterns of the standard character in each limit cycle $y^{(i)}$. A noisy pattern $z \in B^{576}$ is generated by changing each component -1 or 1 of the standard character in $y^{(i)}$ to -1 or 1 with a constant probability, and for each $y^{(i)}$ 200 noisy patterns are generated. Fig.3 shows some typical examples of the outputs.

This simulation results for $\gamma = 1$ and $\gamma = 1.2$ are given in Table 2 and Table 3, respectively, and indicate that each standard limit cycle is stored as an asymptotically stable limit cycle. Further, they indicate that, for $\gamma = 1$, each limit cycle has strong stability against the noise, and for $\gamma = 1.2$, the total number of reconstructed patterns is increased for $i = 1, 3, 5$ and decreased for $i = 2, 4, 6$ from the case $\gamma = 1$. Therefore, it is possible to conclude that domains of attraction can be effectively adjusted by the choice of $\Gamma^{(i)}$ as in the case of the MOPM.

Table 2 ($\gamma = 1$)

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<tr>
<td>1 東</td>
<td>198(198)</td>
<td>166(160)</td>
<td>106(83)</td>
<td>45(3)</td>
</tr>
<tr>
<td>2 京</td>
<td>197(197)</td>
<td>162(149)</td>
<td>113(63)</td>
<td>44(11)</td>
</tr>
<tr>
<td>3 電</td>
<td>196(193)</td>
<td>168(161)</td>
<td>118(80)</td>
<td>68(6)</td>
</tr>
<tr>
<td>4 機</td>
<td>196(200)</td>
<td>156(175)</td>
<td>92(100)</td>
<td>26(7)</td>
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<td>5 大</td>
<td>189(190)</td>
<td>161(159)</td>
<td>84(52)</td>
<td>55(3)</td>
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<tr>
<td>6 学</td>
<td>193(193)</td>
<td>158(166)</td>
<td>93(76)</td>
<td>50(8)</td>
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$\Gamma_j^{(i)} = I_{576}(i = 1, \ldots, 6)$

( ) indicates the result of the MOPM.

Table 3 ($\gamma = 1.2$)

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<td>6 学</td>
<td>152(193)</td>
<td>69(166)</td>
<td>18(76)</td>
<td>1(12)</td>
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</tbody>
</table>

$\Gamma_j^{(i)} = 1.2I_{576}(i = 1, 3, 5)$

$\Gamma_j^{(i)} = I_{576}(i = 2, 4, 6)$

( ) indicates the result of the MOPM.

Fig.2 The standard limit cycle for 東.
5 Concluding Remarks

When the connection matrix $W$ is symmetric and non-negative definite, the MP model has no limit cycles. However, in our computer simulations, no limit cycles are found in the MP model even if the connection matrix $W$ is nonsymmetric as given by (4). Therefore, it is strongly expected that there may exist an energy function for the MOPM. So this is an important future problem to be studied.

Further, the computer simulation results for the LCM strongly indicated that each limit cycle is asymptotically stable. Again, this should be investigated as an essential problem for the LCM.

References


