On the LQG Control for a Class of Nonlinear Stochastic Systems—Simulation Studies

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In this paper, under the LQG control performance criterion, a method for realizing the suboptimal control for one-dimensional nonlinear stochastic systems with finite horizon is developed by exhibiting a wide variety of sample runs.

As preliminaries, the equation of determining state trajectory satisfying the performance criterion and its linearization scheme are reviewed. In order to realize finite horizon control for nonlinear systems, the suboptimal state trajectory emanating from the initial state is switched at time t_s into the one hitting the target. Decision of switching time is illustrated by results of simulation experiments.

Finally, relative relation between the initial state level and the target state one is examined.

1. INTRODUCTION

Mathematical aspects of the infinite horizon optimal control were studied in Refs. [1] and [2]. However, in the finite horizon optimal control problem of nonlinear stochastic systems, no feasible methods may be found because many difficulties arise in both theoretical and computational aspects for realizing the suboptimal control strategy.

The key idea of the method presented is to revive a diverging process caused by the unstable strategy and to hit the fixed target in the stochastic sense. Throughout the paper, a class of the one-dimensional nonlinear stochastic systems is considered under the LQG performance criterion.

In Sec. 2, necessary formulas are exhibited for explaining simulation studies where some of them is briefly illustrated with respect to their derivations, in Appendices.

In Sec. 3, introducing a linearization technique, the one-dimensional difference formula implemented by a recurrent computation is derived for determining the suboptimal state trajectory.

In Sec. 4, detailed aspects of performing simulation experiments and of realizing the suboptimal state trajectories are explained by showing representative sample runs of the state trajectories.

Section 5 concludes the study presented here.

2. PRELIMINARIES

The system of concern is given by the one-dimensional stochastic nonlinear system,

$$\begin{aligned} dx &= \{f(x) + bu\}dt + \sigma_0 x dw, \\ x(0) &= x_0, \quad (0 \leq t < T), \end{aligned} \quad (1)$$

where x is the system state, u is the control, $f(x)$ is bounded and $f(0) = 0$, b and σ_0 are constant parameters, and w is the standard Wiener process, respectively. We assume that there exists a continuous solution to Eq. (1).

Given x_0 , the problem is to find the optimal control so as to satisfy the performance criterion,

$$J(u) = \min_{u \in \mathcal{U}^0} \left[E_{x_0} \left\{ \int_0^T (mx_t^2 + nu_t^2) dt \right\} \right], \quad (2)$$

and $E_{x_0}\{x(T)\} = x_T$, where $E_{x_0}\{*\}$ represents the mathematical expectation conditioned by x_0 , $\mathcal{U}^0$ is a family of admissible controls, m and n are constants where $n \neq 0$.

Let the value function V be

$$V(t, x_t) = \min_{u \in \mathcal{U}^0} \left[E_{x_t} \left\{ \int_t^T (mx_s^2 + nu_s^2) ds \right\} \right], \quad (3)$$

Then, the Hamilton-Jacobi-Bellman equation is

$$\min_{u \in \mathcal{U}^0} [mx_t^2 + nu_t^2 + \{f(x_t) + bu_t\}V_{x_t}]$$

$$+ \frac{1}{2} \sigma_0^2 x_t^2 V_{x_t x_t} \Big] = -V_t, \quad (4)$$

where V_* and V_{**} denote partial derivatives of the function V , respectively. We may define formally the function K by

$$K(x_t, u_t, V_{x_t}, V_{x_t x_t}) = m x_t^2 + n u_t^2 + \{f(x_t) + b u_t\} V_{x_t} + \frac{1}{2} \sigma_0^2 x_t^2 V_{x_t x_t}. \quad (5)$$

In the LQG control scheme, the equation for determining the steady state of the optimally controlled system is

$$\frac{n}{b^2} f(\bar{x}) \frac{\partial f(\bar{x})}{\partial \bar{x}} + \sigma_0^2 p_s \bar{x} + m \bar{x} = 0, \quad (6)$$

where p_s is the stationary solution of Riccati equation which is shown and $\bar{x}$ is the steady state of the system (1) (see Appendix-A). The state trajectory of the system (1) satisfying the performance criterion (2) is determined by (see Appendix-B)

$$\begin{aligned} & (dx_t^* - \sigma_0 x_t^* dw)^2 \\ &= \frac{b^2}{n} \{m(x_t^{*-} - \bar{x}^2) \\ &+ \frac{n}{b^2} (f^2(x_t^*) - f^2(\bar{x})) + \sigma_0^2 p_s (x_t^{*-} - \bar{x}^2)\} dt^2. \end{aligned} \quad (7)$$

3. APPROXIMATION AND SUBOPTIMAL STATE TRAJECTORY

In order to solve Eqs. (6) and (7) simultaneously, we consider the time partition such that $t = k\Delta t$ ($k\Delta t \leq t < \infty$), where $k = 0, 1, 2, \dots$, and expanding the nonlinear function $f(x)$ into

$$f(x_t^*) = a_k^* x_t^* + c_k^* \quad (k\Delta t \leq t < \infty), \quad (8)$$

where

$$\begin{aligned} a_k^* &= f_{x_k^*}(x_k^*), \quad c_k^* = f(x_k^*) - f_{x_k^*}(x_k^*) x_k^*, \\ f_{x_k^*}(x_k^*) &= \frac{\partial f(x_k^*)}{\partial x_k^*}. \end{aligned} \quad (9)$$

By using Eq. (8), Eqs. (6) and (7) are linearized as

$$\bar{x}_{sk} = \frac{-n a_k^* c_k^*}{b^2 m + a_k^{*2} n + b^2 \sigma_0^2 p_{sk}}, \quad (10)$$

and

$$\begin{aligned} & (dx_t^* - \sigma_0 x_t^* dw)^2 \\ &= \frac{b^2}{n} \left\{ m + \frac{n}{b^2} a_k^* + \sigma_0^2 p_{sk} \right\} \{(x_t^* - \bar{x}_{sk}) dt\}^2 \\ & \quad \text{for } k\Delta t \leq t < \infty, \end{aligned} \quad (11)$$

respectively, where in Eq. (3), as $T \rightarrow \infty$, $V_{x_t^* x_t^*} \simeq V_{\bar{x} \bar{x}} = 2p_{sk}$ (Ref. [2]), and p_{sk} is the solution to the linearized algebraic Riccati equation,

$$\frac{b^2}{n} p_{sk}^2 - (2a_k^* + \sigma_0^2) p_{sk} - m = 0. \quad (12)$$

The solutions to Eq. (12) are

$$\begin{aligned} & p_{sk}^+, p_{sk}^- \\ &= \frac{n(2a_k^* + \sigma_0^2) \pm \sqrt{n^2(2a_k^* + \sigma_0^2)^2 + 4b^2 m n}}{2b^2}. \end{aligned} \quad (13)$$

Equation (11) is decomposed into

$$\begin{aligned} & dx_t^{*-} = -\sqrt{\phi_k} (x_t^{*-} - \bar{x}_{sk}) dt + \sigma_0 x_t^{*-} dw_t \\ & \quad \text{for } k\Delta t \leq t < \infty, \end{aligned} \quad (14)$$

and

$$\begin{aligned} & dx_t^{*+} = +\sqrt{\phi_k} (x_t^{*+} - \bar{x}_{sk}) dt + \sigma_0 x_t^{*+} dw_t \\ & \quad \text{for } k\Delta t \leq t < \infty, \end{aligned} \quad (15)$$

where

$$\phi_k = n^{-1} [b^2(m + \sigma_0^2 p_{sk}) + n a_k^{*2}]. \quad (16)$$

First, suppose that $\bar{x} = 0$ and as shown in Fig. 1, determine the switching time t_s by $\bar{x}_d^{*-} = \bar{x}_d^{*+}$, where $\bar{x}_d^{*-}$ and $\bar{x}_d^{*+}$ are the ensemble averaged processes and, without destination of $x_0 > 0$ or $x_0 < 0$, these are the solutions to

Fig. 1: Schematic illustration of the switching time t_s .

$$\begin{aligned} \dot{\bar{x}}_d^{*-} &= -\sqrt{\phi_k} (\bar{x}_d^{*-} - \bar{x}_{sk}^-) \\ \text{for } k\Delta t \leq t < \infty, \end{aligned} \quad (17)$$

and

$$\begin{aligned} \dot{\bar{x}}_d^{*+} &= +\sqrt{\phi_k} (\bar{x}_d^{*+} - \bar{x}_{sk}^+) \\ \text{for } k\Delta t \leq t < \infty. \end{aligned} \quad (18)$$

where from (10), we write $\bar{x}_{sk}^-$, $\bar{x}_{sk}^+$ for $\bar{x}_{sk}^- = \bar{x}_{sk}|_{p_{sk}=p_{sk}^-}$, $\bar{x}_{sk}^+ = \bar{x}_{sk}|_{p_{sk}=p_{sk}^+}$, respectively.

Secondly, Eq. (11) is decomposed into

$$\begin{aligned} dx_t^{*-} &= -\sqrt{\phi_k^-} (x_t^{*-} - \bar{x}_{sk}^-)dt + \sigma_0 x_t^{*-} dw_t \\ \text{for } k\Delta t \leq t < t_s = k_s \Delta t, \end{aligned} \quad (19)$$

and

$$\begin{aligned} dx_t^{*+} &= +\sqrt{\phi_k^+} (x_t^{*+} - \bar{x}_{sk}^+)dt + \sigma_0 x_t^{*+} dw_t \\ \text{for } t_s = k_s \Delta t < t \leq T. \end{aligned} \quad (20)$$

where we write ϕ_k^- , ϕ_k^+ for $\phi_k^- = \phi_k|_{p_{sk}=p_{sk}^-}$, $\phi_k^+ = \phi_k|_{p_{sk}=p_{sk}^+}$. Start with the given initial state x_0 and the sample process x_t^{*-} runs along Eq. (19) to the switching time t_s .

At time t_s the sample process x_t^{*-} is switch into the x_t^{*+} -process and this runs along Eq. (20) to the terminal time T .

4. SIMULATED RUNS

The nonlinear function of concern is

$$f(x) = \alpha x - \beta x^3, \quad (\alpha, \beta > 0). \quad (21)$$

In Eqs. (1) and (21), the values of the parameters were

$$\alpha = 1, \quad \beta = 1, \quad b = 4, \quad \sigma_0 = 0.3 \quad (22)$$

and setting as $m = 1, n = 1$ and $T = 12$, the performance criterion given by (2) was written by

$$J(u) = \min_{u \in \mathcal{U}^0} \left[E_{x_0} \left\{ \int_0^{12} (x_t^2 + u_t^2) dt \right\} \right]. \quad (23)$$

It is natural that there exist many values of $\bar{x}$ satisfying Eq. (6). However, in the light of Eq. (21), $\bar{x} = 0$ is chosen among five values of $\bar{x}$ obtained by Eq. (6). It should be noted that the application of this choice is limited to the preassigned

range of initial states which will be explained in the subsequent paper. Thus, omitting to explain the situation of initial states, we set as $x_0 = 1$ which is allowed to choose the value of $\bar{x} = 0$.

First, Eqs. (17) and (18) were simulated a digital computer. Along the line of procedures described in the previous section, the switching time t_s may be determined. Table 1 and Fig. 2 show the values of t_s depending on the target level x_T . Figure 3 shows the x_d^{*+} -run connected with the x_d^{*-} -run at time t_s . Representative runs of the x_t^* -process determined by Eqs. (19) and (20) emanating from $x_0 = 1$ and $x_0 = 0.5$ to the target states $x_T = 0.5$ and $x_T = 1$ are shown in Figs. 4 and 5, respectively. Table 2 gives numerical versions of the time evolution of x_t^* , $\bar{x}_{sk}$, p_{sk} corresponding to the experiments shown in Fig. 4.

5. CONCLUSION

A method for finding the finite horizon suboptimal control strategy under the LQG performance criterion has been presented for a class of nonlinear stochastic systems. It is well-known that two solutions are found to the optimal control problem within the framework of dynamic programming approach; one gives the stable optimal state trajectory and the other the unstable one which is turned down the advent.

In this paper, reviving the unstable state trajectory so as to hit the target state and connecting this with the stable optimal state trajectory emanating from the initial state at time t_s , the suboptimal state trajectory is completed for the control interval $[0, T]$. An inescapable fact is that a discontinuity at time t_s occurs on the suboptimal stochastic state trajectories. This fact and a modification of the definition of admissible control will be described in the subsequent paper, including extended versions to the n -dimensional stochastic nonlinear system.

6. REFERENCES

- [1] D. A. Carlson, A. B. Haurie and A. Leizarowitz: Infinite Horizon Optimal Control Deterministic and Stochastic System, Springer, N. Y. 1991.
- [2] Y. Sunahara, H. Ohtagaki, T. Tsutsumi: On the control performance and optimal stationary

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Appendix-A Derivation of Eq. (6).

Consider the averaged process, $E_{x_0}\{x_t\} = \tilde{x}_t$, determined by

$$\frac{d\tilde{x}_t}{dt} = f(\tilde{x}_t) + b\tilde{u}_t, \quad \tilde{x}(0) = \tilde{x}_0$$

$$\text{for } 0 \leq t < \infty, \quad (\text{A.1})$$

where $\tilde{u}_t = E_{x_0}\{u_t\}$ and an approximation $\tilde{f}(x_t) = f(\tilde{x}_t)$ has been made. The performance criterion is

$$\min_{\tilde{u}_t \in \mathcal{U}^0} \left[\tilde{J}_T(\tilde{u}_t) = \int_0^\infty \{L_s(\tilde{x}_t) + n\tilde{u}_t^2\} dt \right], \quad (\text{A.2})$$

where

$$L_s(\tilde{x}_t) = m\tilde{x}_t^2 + \frac{1}{2}\sigma_0\tilde{x}_t^2 V_{\tilde{x}_t\tilde{x}_t}. \quad (\text{A.3})$$

The Hamilton-like function associated with (A.1) and its stationary version are

$$\tilde{K}(\tilde{x}_t, \tilde{u}_t, \tilde{V}_{\tilde{x}_t}) = m\tilde{x}_t^2 + n\tilde{u}_t^2$$

$$+ \{f(\tilde{x}_t) + b\tilde{u}_t\}V_{\tilde{x}_t} + \frac{1}{2}\sigma_0\tilde{x}_t^2 V_{\tilde{x}_t\tilde{x}_t}, \quad (\text{A.4})$$

and

$$\tilde{K}_{st}(\tilde{x}_{st}, \tilde{u}_{st}) = m\tilde{x}_{st}^2 + n\tilde{u}_{st}^2 + \frac{1}{2}\sigma_0\tilde{x}_{st}^2 V_{\tilde{x}_{st}\tilde{x}_{st}}, \quad (\text{A.5})$$

where as far as the control interval of concern is for $t \in [0, \infty)$, $\lim_{t \rightarrow \infty} V_{\tilde{x}_t\tilde{x}_t} = V_{\tilde{x}_{st}\tilde{x}_{st}}$ is allowed to introduce. Differentiating the RHS of (A.5) with respect to $\tilde{u}_{st}$ and setting the result as zero obtains

$$\frac{n}{b^2}f(\bar{x})\frac{\partial f(\bar{x})}{\partial \bar{x}} + \frac{1}{2}\sigma_0^2 \bar{x} V_{\bar{x}\bar{x}} + m\bar{x} = 0, \quad (\text{A.6})$$

where through $\tilde{x}_{st}^*$ satisfying the condition, $d\tilde{K}_{st}/d\tilde{u}_{st} = 0$, we define

$$\lim_{t \rightarrow \infty} \tilde{x}^*(t) = \tilde{x}_{st}^* = \bar{x}. \quad (\text{A.7})$$

In the LQG performance criterion, it follows that [2]

$$V_{\bar{x}\bar{x}} = 2p_s. \quad (\text{A.8})$$

Substitution of (A.8) into (A.6) brings (6)

Appendix-B Derivation of Eq. (7).

The condition $dK/du_t = 0$ in Eq. (5) is

$$u_t^* = -\frac{b}{2n}V_{x_t^*}, \quad (\text{B.1})$$

where we write u_t^*, x_t^* satisfying the condition for u_t, x_t , respectively. Then,

$$K^*(x_t^*, V_{x_t^*}, V_{x_t^*x_t^*}) = mx_t^{*2} - \frac{1}{4}\frac{b^2}{n}V_{x_t^*}^2$$

$$+ f(x_t^*)V_{x_t^*} + \frac{1}{2}\sigma_0^2 x_t^{*2} V_{x_t^*x_t^*}. \quad (\text{B.2})$$

Applying (B.1) to Eq. (1) obtains,

$$V_{x_t^*} = \frac{2n}{b^2} \frac{d}{dt} [f(x_t^*)dt + \sigma_0 x_t^* dw_t - dx_t^*]. \quad (\text{B.3})$$

On the other hand, noting that $f(\tilde{x}_{st}) + b\tilde{u}_{st}^* = 0$,

$$\tilde{K}_{st}^*(\tilde{x}_{st}^*, \tilde{u}_{st}^*)$$

$$= m\tilde{x}_{st}^{*2} + \frac{1}{2}\sigma_0^2 \tilde{x}_{st}^{*2} V_{\tilde{x}_{st}^*} + \frac{n}{b^2} f^2(\tilde{x}_{st}^*). \quad (\text{B.4})$$

Based on the assumption that

$$K^*(x_t^*, V_{x_t^*}, V_{x_t^*x_t^*}) = \tilde{K}_{st}(\tilde{x}_{st}^*, \tilde{u}_{st}^*), \quad (\text{B.5})$$

from (B.2), (B.3), (B.4), (A.7) and (A.8), after somewhat tedious calculations, Eq. (7) can be derived.

Table 1: The switching time t_s vs. $\bar{x}_d^*(T)$ ($x_0 = 1, T = 12$)

x_T	t_s	$\bar{x}_d^*(T)$
1.00	10.810	0.9989
0.75	10.953	0.7495
0.50	11.093	0.4998
0.25	11.308	0.2492

Table 2: The numerical versions of time evolution of suboptimal state, linearized steady state and Riccati solution

t	$\bar{x}_{sk}$	x_t^*	p_{sk}
0		1.0	
1.0	-2.167×10^{-4}	3.740×10^{-2}	-1.607×10^{-1}
2.0	-2.771×10^{-6}	1.534×10^{-2}	-1.600×10^{-1}
3.0	-1.773×10^{-6}	1.178×10^{-2}	-1.600×10^{-1}
4.0	-1.737×10^{-6}	1.795×10^{-2}	-1.600×10^{-1}
5.0	-1.736×10^{-6}	1.354×10^{-2}	-1.600×10^{-1}
6.0	-1.736×10^{-6}	1.530×10^{-2}	-1.600×10^{-1}
7.0	-1.736×10^{-6}	1.439×10^{-2}	-1.600×10^{-1}
8.0	-1.736×10^{-6}	1.861×10^{-2}	-1.600×10^{-1}
9.0	-1.736×10^{-6}	1.367×10^{-2}	-1.600×10^{-1}
10.0	-1.736×10^{-6}	1.651×10^{-2}	-1.600×10^{-1}
11.0	-1.736×10^{-6}	1.448×10^{-2}	-1.600×10^{-1}
11.092	-1.736×10^{-6}	1.307×10^{-2}	-1.600×10^{-1}
11.093	-1.641×10^{-6}	1.308×10^{-2}	6.249
12.0	-8.650×10^{-2}	4.749×10^{-1}	4.799

Fig. 2: Deterministic runs hitting various target x_T

Fig. 3: Deterministic run with $x_0 = 1, x_T = 0.5$

Fig. 4: Stochastic run with $x_0 = 1, x_T = 0.5$

Fig. 5: Stochastic run with $x_0 = 0.5, x_T = 1$