Solution Structure of Algebraic Matrix Riccati Equations with Nonnegative-Definite Quadratic and Constant Terms

Hiroiyuki Kano¹ and Toshimitsu Nishimura²

1) College of Science and Engineering, Tokyo Denki University

2) Institute of Space and Astronautical Sciences

Abstract In this paper, algebraic matrix Riccati equations are analyzed that have nonnegative-definite quadratic as well as constant terms, employing the so-called algebraic method. Necessary and sufficient conditions are established for the existence of stabilizing solution, which is not necessarily nonnegative-definite unlike the case of standard equations arising in LQG problems. Nonnegative-definite stabilizing solution is shown to exist if and only if the system is asymptotically stable and the H_∞ norm of transfer matrix be less than one. For positive-definite solution, we need additional condition that the system be controllable. Next, necessary and sufficient conditions are established for the existence of the so-called anti-stabilizing solution for both general and positive-definite cases. It is shown that stabilizing solution is minimal and anti-stabilizing solution is maximal of all the real symmetric solutions. Then, assuming the existence of nonnegative-definite stabilizing solution, all the other (hence nonnegative-definite) solutions with lattice structure are derived, together with the number of nonnegative-definite solutions.

1. Introduction

Matrix Riccati equations have played key roles in the optimal control and optimal filtering problems, and have been studied extensively. Especially the algebraic equation of the following form has been analyzed in detail

$$FP + PF^T - PH^T HP + GG^T = 0 \quad (1)$$

together with the stability of the closed-loop system $F_c = F - PH^T H$. The results include (i) there exists a unique $P \geq 0$ and F_c is asymptotically stable iff (H, F) is detectable and (F, G) is stabilizable [6], and (ii) there exists a stabilizing solution $P > 0$ iff (H, F) is detectable and $(-F, G)$ is stabilizable [3]. Kucera [7] also studied all the nonnegative-definite solutions and showed that they constitute lattice.

On the other hand, eq.(1) with the term $H^T H$ replaced by a matrix of the form $H_1^T H_1 - H_2^T H_2$ has been shown to play an important role in the solution of H_∞ -control theory. Thus the analysis of such equations is of great interest. In this paper, we investigate an algebraic matrix Riccati equation which has the quadratic term $+PH^T HP$ instead of $-PH^T HP$ in eq.(1). This together with the case of eq.(1) provides the two extreme cases as well as useful tools for the analysis of Riccati equations arising in H_∞ problems [2].

Here the purpose of this study is to develop a comprehensive theory of such an equation along the

line of Kucera [6,7]. More specifically, we establish the existence conditions of stabilizing solution, anti-stabilizing solution, and other nonnegative-definite solutions as well as to clarify the relationship among the solutions [5]. For the analysis, the so-called algebraic method is employed which is based on an eigenvalue problem of a Hamiltonian matrix.

Principal symbols used throughout the paper are introduced. 'T', '~', and '*' respectively denote transposition, complex conjugate, and complex conjugate transposition of matrices or vectors. '-T' is used to denote inverse transposition of nonsingular matrices. $|\cdot|$ represents matrix determinant or magnitude of scalars, and $\|\cdot\|_\infty$ the H_∞ norm. $\Re[\cdot]$ stands for real part of complex values. For Hermitian matrices, say M and N , $M > N$ and $M \geq N$ show that $M - N$ is positive-definite and nonnegative-definite respectively.

2. Mathematical Preliminaries

We consider the algebraic matrix Riccati equations of the following form,

$$FP + PF^T + PH^T HP + GG^T = 0. \quad (2)$$

Here, F , GG^T and $H^T H$ are $n \times n$ real matrices.

Real, symmetric solutions $P(n \times n)$ are studied with reference to the stability of the following closed-loop system matrix,

$$F_c = F + PH^T H. \quad (3)$$

As in the case of eq.(1), eq.(2) can be analyzed by introducing a $2n \times 2n$ matrix A defined as

$$A = \begin{bmatrix} F & GG^T \\ -H^T H & -F^T \end{bmatrix} \quad (4)$$

Then A is a Hamiltonian matrix and its eigenvalues appear symmetrically with respect to the imaginary axis in the complex plane.

Now let λ_i be an eigenvalue of A and let the corresponding chain of generalized eigenvectors be $z_i, z_{i+1}, \dots, z_{i+p_i-1}$, namely

$$(A - \lambda_i I)z_i = 0, \quad (5a)$$

$$(A - \lambda_i I)z_{i+j} = z_{i+j-1}, \quad j = 1, \dots, p_i - 1, \quad (5b)$$

for $p_i > 1$. In this paper, generalized eigenvectors are simply called eigenvectors whenever it causes no confusion. Equation (5) may be written in matrix form as

$$AZ_i = Z_i J_i \quad (6)$$

where $Z_i = [z_i, z_{i+1}, \dots, z_{i+p_i-1}]$ and J_i is the Jordan block of size p_i given as

$$J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix} \quad (7)$$

Collecting eq.(6) for all the $i = 1, 2, \dots$ yields

$$AT = TJ_A \quad (8)$$

In the algebraic approach, $2n$ -dimensional eigenvectors z_i are partitioned into two n -dimensional vectors y_i and x_i as $z_i = \begin{bmatrix} y_i \\ x_i \end{bmatrix}$. Now choose any n eigenvectors and the corresponding eigenvalues expressed in matrix form as

$$A \begin{bmatrix} Y \\ X \end{bmatrix} = \begin{bmatrix} Y \\ X \end{bmatrix} J, \quad (9)$$

where $Y = [y_1 \ y_2 \ \dots \ y_n]$ ($n \times n$), $X = [x_1 \ x_2 \ \dots \ x_n]$ ($n \times n$), and J is the Jordan matrix of size n .

Then every solution P for eq.(2) can be obtained, for nonsingular X , as

$$P = YX^{-1} \quad (10)$$

and the corresponding closed-loop system matrix is given by

$$F_c = -X^{-T} J^T X^T \quad (11)$$

Moreover, it is known that the eigenvectors satisfy

$$x_i^T y_j - y_i^T x_j \neq 0 \quad \text{if} \quad \lambda_i + \lambda_j = 0 \quad (12a)$$

$$x_i^T y_j - y_i^T x_j = 0 \quad \text{otherwise} \quad (12b)$$

and for matrices Y and X we have

$$X^* Y = Y^* X \quad \text{if} \quad \lambda_i + \lambda_j^* \neq 0, \quad 1 \leq i, j \leq n \quad (13a)$$

$$X^T Y = Y^T X \quad \text{if} \quad \lambda_i + \lambda_j \neq 0, \quad 1 \leq i, j \leq n \quad (13b)$$

Now notice that eq.(6) may be rewritten as

$$A \bar{Z}_i = \bar{Z}_i \bar{J}_i \quad (14)$$

where $\bar{Z}_i = [\bar{z}_{i+p_i-1}, \dots, \bar{z}_i]$ with $\bar{z}_{i+j} = (-1)^j z_{i+j}$ and

$$\bar{J}_i = \begin{bmatrix} \lambda_i & & & \\ -1 & \ddots & & \\ & \ddots & \ddots & \\ & & -1 & \lambda_i \end{bmatrix} \quad (15)$$

Such vectors $\bar{z}_{i+j}$ are also called generalized eigenvectors (or simply eigenvectors) in this paper.

For later discussions, it is convenient to consider all the eigenvalues and eigenvectors of A arranged in a specific manner. Namely, assuming that A has no eigenvalues on the imaginary axis, T and J_A in eq.(8) may be written as

$$T = \begin{bmatrix} Y & V \\ X & U \end{bmatrix}, \quad J_A = \begin{bmatrix} J & 0 \\ 0 & -J^T \end{bmatrix}, \quad \Re[\lambda_i(J)] > 0 \quad (16)$$

Here J is an $n \times n$ Jordan matrix consisting of blocks of the form in eq.(7), and Y, X, V and U are all

$n \times n$ matrices. In words, $\begin{bmatrix} Y \\ X \end{bmatrix}$ contains eigenvectors in eq.(5) corresponding to the eigenvalues with positive real parts, and $\begin{bmatrix} V \\ U \end{bmatrix}$ contains eigenvectors $\bar{z}_{i+j}$ in eq.(14) corresponding to the eigenvalues with negative real parts. It is noted that eqs.(10) and (11) hold when eq.(9) contain blocks in eq.(14). In this case, by employing the formulation in eqs.(14) and (15), we may normalize eigenvectors of A in such a way that $T^{-1} = \begin{bmatrix} U^T & -V^T \\ -X^T & Y^T \end{bmatrix}$.

Then $T^{-1}T = I$ yields

$$U^T Y - V^T X = I, \quad Y^T X = X^T Y, \quad V^T U = U^T V, \quad (17a)$$

or by denoting i -th columns of Y, X, V and U by y_i, x_i, v_i and u_i respectively, we have

$$u_i^T y_j - v_i^T x_j = \delta_{ij}, \quad y_i^T x_j - x_i^T y_j = 0, \quad v_i^T u_j - u_i^T v_j = 0 \quad (17b)$$

for $i, j = 1, 2, \dots, n$. Here δ_{ij} stands for Kronecker delta.

3. Existence Conditions of Stabilizing Solutions

Stabilizing solution of eq.(2) is such a solution that yields asymptotically stable closed-loop system matrix in eq.(3), and will be denoted as P_s .

The following theorem establishes an existence condition of stabilizing solution.

Theorem 1. *There exists a stabilizing solution P_s of eq.(2) if and only if A has no eigenvalue on the imaginary axis and the pair (H, F) is detectable. Moreover, such a stabilizing solution is unique.*

(Proof) ('if' part) Since A has no eigenvalue on the imaginary axis, we can set $\Re[\lambda_i] > 0$ for all $i = 1, 2, \dots, n$ in eq.(9) which is written as

$$FY + GG^T X = YJ, \quad -H^T HY - F^T X = XJ \quad (18)$$

If $|X| \neq 0$, the solution $P = YX^{-1}$ is stabilizing, and real, symmetric from eq.(13). In order to prove $|X| \neq 0$, we assume the contrary, i.e. $|X| = 0$. Then there exists an n -vector $b \neq 0$ such that $Xb = 0$. Postmultiplying eq.(18) by b yields

$$FYb = YJb, \quad -H^T HYb = XJb \quad (19)$$

Then we get $-b^* Y^* H^T HYb = b^* Y^* XJb = b^* X^* YJb = 0$, implying $HYb = 0$. Thus $XJb = 0$ by eq.(19), or $Jb \in \mathcal{N}(X)$, the null space of X . Hence there exists an n -vector $\hat{b} \neq 0$ such that $X\hat{b} = 0$ and $J\hat{b} = \mu\hat{b}$. From eq.(19) with b replaced by $\hat{b}$, we then have

$$FY\hat{b} = YJ\hat{b} = \mu Y\hat{b}, \quad HY\hat{b} = 0 \quad (20)$$

Noting that $Y\hat{b} \neq 0$, this shows that μ is an unobservable eigenvalue of F . Since $\Re[\mu] > 0$, the pair is undetectable, a contradiction. Thus $|X| \neq 0$.

('only if' part) Since stabilizing solution can be derived only from the n eigenvalues λ of A with $\Re[\lambda] > 0$, the other n eigenvalues have negative real parts. Thus $\Re[\lambda] \neq 0$ for all the eigenvalues λ of A . Now if the pair (H, F) is undetectable, there exists an unobservable eigenvalue λ of F with $\Re[\lambda] \geq 0$. In this case, A has the same eigenvalue with the eigenvector of the form $\begin{bmatrix} u \\ 0 \end{bmatrix}$. Thus, if $\Re[\lambda] = 0$, then A has an eigenvalue on the imaginary axis, a contradiction. On the other hand, $\Re[\lambda] > 0$ implies $|X| = 0$, contradicting the existence of P_s . Therefore (H, F) must be detectable. (Q.E.D.)

Remark 1. *It is noted that the condition in this theorem is the same as that for eq.(1) (see [6]) except that P_s is not necessarily nonnegative-definite in this case. An obvious example is a scalar equation $p^2 + 3p + 2 = 0$.*

Based on Theorem 1, we can now prove the existence conditions for nonnegative-definite and positive-definite P_s [8].

Lemma 1. *There exists a nonnegative-definite, stabilizing solution P_s of eq.(2) if and only if A has no eigenvalue on the imaginary axis and F is asymptotically stable.*

(Proof) ('if' part) Obviously the pair (H, F) is detectable in this case, and the existence of P_s is guaranteed by Theorem 1. Now from eq.(2), we have $FP_s + P_s F^T = -P_s H^T H P_s - GG^T$, and since the matrix F is asymptotically stable,

$$P_s = \int_0^\infty e^{Ft} (P_s H^T H P_s + GG^T) e^{F^T t} dt \geq 0 \quad (21)$$

('only if' part) From Theorem 1, $\Re[\lambda(A)] \neq 0$ is necessary. Now suppose that F has an eigenvalue λ with $\Re[\lambda] \geq 0$, and let $F^T x = \lambda x, x \neq 0$. Then from eq.(2), we get

$$\begin{aligned} x^* (FP_s + P_s F^T + P_s H^T H P_s + GG^T) x \\ = (\lambda + \lambda^*) x^* P_s x + x^* P_s H^T H P_s x + x^* GG^T x = 0 \end{aligned} \quad (22)$$

Thus $HP_s x = 0$, and for the closed-loop system in eq.(3) we get

$$F_c^T x = (F^T + H^T H P_s) x = F^T x = \lambda x \quad (23)$$

Since $\Re[\lambda] \geq 0$, this shows that F_c is not asymptotically stable, and hence contradicts that P_s is the stabilizing solution. (Q.E.D.)

Lemma 2. *There exists a positive-definite, stabilizing solution P_s of eq.(2) if and only if A has no eigenvalue on the imaginary axis, F is asymptotically stable, and (F, G) is controllable.*

(Proof) ('if' part) By Lemma 1, since $P_s \geq 0$ exists, we only need to prove $|P_s| \neq 0$. Assume, on the contrary, that $|P_s| = 0$. Then there exists an n -vector $b \neq 0$ such that $Pb = 0$. Equation (2) premultiplied by b^* and postmultiplied by b yields $G^T b = 0$. Then, postmultiplying eq.(2) by b , we get $P F^T b = 0$. Hence $F^T b \in \mathcal{N}[P]$ and there exists an n -vector $\hat{b} \neq 0$ such that $P\hat{b} = 0$ and $F^T \hat{b} = \mu\hat{b}$ for some scalar μ . Since we also have $G^T \hat{b} = 0$, this shows that an eigenvalue μ of F is uncontrollable, a contradiction. Thus $|P| \neq 0$.

('only if' part) From Lemma 1, $\Re[\lambda(A)] \neq 0$ and F being asymptotically stable are necessary. Now suppose that an eigenvalue, say λ_i , of F is uncontrollable. Hence $\Re[\lambda_i] < 0$. In this case we see that A has the eigenvalue $-\lambda_i$ with $\Re[-\lambda_i] > 0$ and the corresponding eigenvector is of the form $\begin{bmatrix} 0 \\ x_i \end{bmatrix}$. Thus one column of Y is the zero vector and P cannot be positive-definite. (Q.E.D.)

In Theorem 1 and Lemmas 1 and 2, we used the condition that A has no eigenvalues on the imaginary axis, for which the following result is known [1].

Lemma 3. Assume that F is asymptotically stable. Then A has no eigenvalue on the imaginary axis if and only if $\|H(sI - F)^{-1}G\|_\infty < 1$.

Using Lemma 3 in Lemmas 1 and 2, we readily have the following existence conditions. Theorem 2 is known as strict bounded real lemma (see [2]).

Theorem 2. There exists a nonnegative-definite, stabilizing solution P_s of eq.(2) if and only if F is asymptotically stable and $\|H(sI - F)^{-1}G\|_\infty < 1$.

Theorem 3. There exists a positive-definite, stabilizing solution P_s of eq.(2) if and only if F is asymptotically stable, $\|H(sI - F)^{-1}G\|_\infty < 1$, and (F, G) is controllable.

4. Other Nonnegative-Definite Solutions

Let P_m be the solution derived from n eigenvalues with negative real parts, namely $P_m = VU^{-1}$ where V and U are specified in eqs.(8) and (16). P_m is called anti-stabilizing solution since all the eigenvalues of the corresponding closed-loop system have positive real parts. In this case, we have

$$FV + GG^T U = -VJ^T, \quad -H^T H V - F^T U = -UJ^T \quad (24)$$

and using analogous arguments as in the proof of Theorem 1, we can get

Theorem 4. There exists an anti-stabilizing solution P_m of eq.(2) if and only if A has no eigenvalue on the imaginary axis and the pair $(H, -F)$ is detectable. Moreover such a solution is unique.

Remark 2. The condition in this theorem is the same as that for eq.(1) (see [4]) except that P_m is not necessarily nonpositive-definite in this case as will be seen shortly.

In the case of eq.(1), the stabilizing solution P_s , if it exists, is the maximal solution, and P_m if it exists is the minimal solution. In contrast, we have the following property.

Lemma 4. If P_s and P_m exist, then $P_s \leq P$ and $P \leq P_m$ for any real, symmetric solution P .

(Proof) Since both P_s and P satisfy eq.(2), we can derive the following equation,

$$F_c Q + Q F_c^T + Q H^T H Q = 0 \quad (25)$$

where $Q = P - P_s$ and $F_c = F + P_s H^T H$. Since F_c is asymptotically stable, if Q (or equivalently P) exists, we have

$$Q = \int_0^\infty e^{F_c t} Q H^T H Q e^{F_c^T t} dt \geq 0 \quad (26)$$

Hence $P \geq P_s$.

Now letting $Q = P - P_m$ and $F_u = F + P_m H^T H$, we have the similar equation as (25) with F_c replaced by F_u . Then noting that all the eigenvalues of F_u have positive real parts, we get

$$Q = - \int_0^\infty e^{-F_u t} Q H^T H Q e^{-F_u^T t} dt \leq 0 \quad (27)$$

Hence $P \leq P_m$. (Q.E.D.)

Thus, clearly, all the solutions P are nonnegative-definite if $P_s \geq 0$ exists. In the subsequent discussions, assuming the existence of $P_s \geq 0$, we investigate such nonnegative-definite solutions starting with P_s . Here, by "a solution $\hat{P}$ is derived from solution P ", we mean that the set of eigenvalues used to compute $\hat{P}$ is obtained from that for P by replacing an eigenvalue λ with positive real part by its negative counterpart $-\lambda$ if $-\lambda$ is not an unobservable eigenvalue of F . In case λ is complex, the conjugate λ^* is simultaneously replaced. Moreover, in the following theorem, by solution, we mean real, symmetric (hence by Lemma 4, nonnegative-definite) solution.

Theorem 5. Assume that there exists $P_s \geq 0$. Then (i) If F has no observable eigenvalue, P_s is the unique solution. (ii) If F has observable eigenvalue, it yields a solution $\hat{P}$ ($\neq P_s$). Moreover, let $\hat{P}$ be a solution derived from the other solution P , then $\hat{P} \geq P$. (iii) If (H, F) is observable, then the maximal solution P_m exists and $P_m > 0$.

(Proof of (i)) Since all the eigenvalues of F are unobservable, $u_i = 0$ for $i = 1, 2, \dots, n$. Hence only the eigenvalues of A with positive real parts can be employed to constitute solution, which is P_s .

(Proof of (ii)) Let a solution P exist and let it be derived from eigenvalues $\lambda_i, i = 1, 2, \dots, m$ and $-\lambda_i, i = m + 1, m + 2, \dots, n$, with $\Re[\lambda_i] > 0, i = 1, 2, \dots, n$. Notice that, as P , at least P_s exists which corresponds to the case $m = n$.

Now let $\begin{bmatrix} y_i \\ x_i \end{bmatrix}$ and $\begin{bmatrix} v_i \\ u_i \end{bmatrix}$ be the eigenvectors of A corresponding to eigenvalues λ_i and $-\lambda_i$ with $\Re[\lambda_i] > 0$ respectively. Then

$$P \equiv Y_0 X_0^{-1} = \begin{bmatrix} y_1 & y_2 & \cdots & y_m & v_{m+1} & \cdots & v_n \\ x_1 & x_2 & \cdots & x_m & u_{m+1} & \cdots & u_n \end{bmatrix}^{-1} \geq 0 \quad (28)$$

With the obvious definitions of matrices, rewrite eq.(28) more compactly as

$$P = \begin{bmatrix} y_1 & Y_2 & V_3 \end{bmatrix} \begin{bmatrix} x_1 & X_2 & U_3 \end{bmatrix}^{-1} = \begin{bmatrix} y_1 & \hat{Y}_2 \end{bmatrix} \begin{bmatrix} x_1 & \hat{X}_2 \end{bmatrix}^{-1} \quad (29)$$

Then, let $\Omega = X_0^* Y_0$, and since $\Omega = \Omega^*$ and $P \geq 0$, we have

$$\Omega = \begin{bmatrix} x_1^* y_1 & x_1^* \hat{Y}_2 \\ \hat{X}_2^* y_1 & \hat{X}_2^* \hat{Y}_2 \end{bmatrix} = \begin{bmatrix} y_1^* x_1 & y_1^* \hat{X}_2 \\ \hat{Y}_2^* x_1 & \hat{Y}_2^* \hat{X}_2 \end{bmatrix} \geq 0 \quad (30)$$

We now consider to derive another solution $\hat{P}$ from P . Without loss of generality, it is assumed that $-\lambda_1$ is not an unobservable eigenvalue of F and moreover it is assumed real for simplicity. Note that in the case when $-\lambda_1$ is complex, we only need to treat $-\lambda_1$ and $-\lambda_2 = \lambda_1^*$ simultaneously. Assuming $|[u_1 \ \hat{X}_2]| \neq 0$, we constitute $\hat{P}$ by

$$\hat{P} \equiv \hat{Y} \hat{X}^{-1} = [v_1 \ \hat{Y}_2][u_1 \ \hat{X}_2]^{-1} \quad (31)$$

It is noted that, since λ_1 is assumed real, $n \times 1$ vectors y, x, v and u are all real. This $\hat{P}$ is real, symmetric, since by using eq.(17) we have $\hat{\Omega} \equiv \hat{X}^* \hat{Y} = \hat{Y}^* \hat{X} = \hat{\Omega}^*$ and $\hat{\Omega}' \equiv \hat{X}^T \hat{Y} = \hat{Y}^T \hat{X} = \hat{\Omega}'^T$. In particular, since $\hat{P} \geq 0$, it holds that

$$\hat{\Omega} = \begin{bmatrix} u_1^* v_1 & u_1^* \hat{Y}_2 \\ \hat{X}_2^* v_1 & \hat{X}_2^* \hat{Y}_2 \end{bmatrix} = \begin{bmatrix} v_1^* u_1 & v_1^* \hat{X}_2 \\ \hat{Y}_2^* u_1 & \hat{Y}_2^* \hat{X}_2 \end{bmatrix} \geq 0 \quad (32)$$

Now we prove that in fact $|\hat{X}| = |[u_1 \ \hat{X}_2]| \neq 0$. Let

$$A \begin{bmatrix} v_1 & \hat{Y}_2 \\ u_1 & \hat{X}_2 \end{bmatrix} = \begin{bmatrix} v_1 & \hat{Y}_2 \\ u_1 & \hat{X}_2 \end{bmatrix} \begin{bmatrix} -\lambda_1 & 0 \\ 0 & \hat{J}_2 \end{bmatrix} \quad (33)$$

then we have

$$-H^T H v_1 - F^T u_1 = -\lambda_1 u_1 \quad (34a)$$

$$-H^T H \hat{Y}_2 - F^T \hat{X}_2 = \hat{X}_2 \hat{J}_2 \quad (34b)$$

Suppose, on the contrary, $|[u_1 \ \hat{X}_2]| = 0$. Notice $\text{rank}(\hat{X}_2) = n - 1$ and $u_1 \neq 0$ by the condition that $-\lambda_1$ is not an unobservable eigenvalue of A . Then there exists an $(n - 1)$ -vector $b \neq 0$ such that $u_1 = \hat{X}_2 b$. From eq.(34), we then get

$$-H^T H(\hat{Y}_2 b - v_1) = \hat{X}_2 \hat{J}_2 b + \lambda_1 u_1 = \hat{X}_2(\hat{J}_2 + \lambda_1 I)b \quad (35)$$

Moreover in view of eq.(32), we have

$$(\hat{Y}_2 b - v_1)^* \hat{X}_2 = b^* \hat{X}_2^* \hat{Y}_2 - v_1^* \hat{X}_2 = u_1^* \hat{Y}_2 - v_1^* \hat{X}_2 = 0 \quad (36)$$

Equations (35) and (36) yield

$$-(\hat{Y}_2 b - v_1)^* H^T H(\hat{Y}_2 b - v_1) = 0 \quad (37)$$

or,

$$H(\hat{Y}_2 b - v_1) = 0 \quad (38)$$

From eq.(35), we then get $\hat{X}_2(\hat{J}_2 + \lambda_1 I)b = 0$, and $\text{rank}(\hat{X}_2) = n - 1$ yields

$$(\hat{J}_2 + \lambda_1 I)b = 0 \quad (39)$$

On the other hand, from eq.(33), we have

$$F v_1 + G G^T u_1 = -\lambda_1 v_1, \quad F \hat{Y}_2 + G G^T \hat{X}_2 = \hat{Y}_2 \hat{J}_2 \quad (40)$$

Equations (40) and (39) yield

$$F(\hat{Y}_2 b - v_1) = \hat{Y}_2 \hat{J}_2 b + \lambda_1 v_1 = -\lambda_1(\hat{Y}_2 b - v_1) \quad (41)$$

Here note that $\hat{Y}_2 b - v_1 \neq 0$, since otherwise we have $\begin{bmatrix} v_1 \\ u_1 \end{bmatrix} = \begin{bmatrix} \hat{Y}_2 \\ \hat{X}_2 \end{bmatrix} b$ with $b \neq 0$ contradicting the linear independence of eigenvectors of A . Then eqs.(41) and (38) imply that $-\lambda_1$ is an unobservable eigenvalue of F , a contradiction. Hence $|\hat{X}| \neq 0$ is proved.

It should be clear that $\hat{P} \neq P$, since the set of eigenvalues employed to constitute $\hat{P}$ differs from that for P . Next we have to show that $\hat{P} \geq P$. Since $P = Y_0 X_0^{-1}$ and $\hat{P} = \hat{Y} \hat{X}^{-1}$, we get

$$P - \hat{P} = (X_0^{-1})^* [\Omega - X_0^* (\hat{X}^*)^{-1} \hat{\Omega} \hat{X}^{-1} X_0] X_0^{-1} \\ \equiv (X_0^{-1})^* M X_0^{-1} \quad (42)$$

with $M = \Omega - X_0^* (\hat{X}^*)^{-1} \hat{\Omega} \hat{X}^{-1} X_0$. Now let

$$\hat{X}^{-1} \equiv \begin{bmatrix} \gamma_1^* \\ \Gamma_2^* \end{bmatrix} \quad (43)$$

Then using eqs.(30) and (32) and after some manipulations, we can derive

$$M = \begin{bmatrix} x_1^* \gamma_1 & 0 \\ 0 & 0 \end{bmatrix} \quad (44)$$

where we used $u_1^* y_1 - v_1^* x_1 = u_1^T y_1 - v_1^T x_1 = 1$ by eq.(17b).

We now evaluate $x_1^* \gamma_1$ in eq.(44). Here letting X and U be as given in eq.(16), we have $XU^T = UX^T$ and it can be shown that

$$XU^T = [x_1 \ x_2 \ x_3][u_1 \ u_2 \ u_3]^T = \tilde{X} \tilde{U}^T \leq 0 \quad (45)$$

Moreover let $\Gamma_2^* = \begin{bmatrix} \hat{\Gamma}_2^* \\ \hat{\Gamma}_3^* \end{bmatrix}$ in eq.(43), then

$$\gamma_1^* XU^T \gamma_1 = \gamma_1^* x_1 u_1^T \gamma_1 + \gamma_1^* x_2 u_2^T \gamma_1 + \gamma_1^* x_3 u_3^T \gamma_1 \leq 0 \quad (46)$$

Since $u_1^T \gamma_1 = \gamma_1^T u_1 = \gamma_1^* u_1 = 1$, $\gamma_1^* x_2 = 0$ and $u_3^T \gamma_1 = (\gamma_1^T u_3)^T = (\gamma_1^* u_3)^T = 0$, we get $\gamma_1^* XU^T \gamma_1 = x_1^* \gamma_1 \leq 0$. Thus $M \leq 0$ in eq.(44), or

$$\hat{P} \geq P \quad (47)$$

(Proof of (iii)) Since all the eigenvalues of F are observable, the above process in (ii) can be repeated to reach the solution $P_m = VU^{-1}$. From eq.(17a), we have $U^T Y - V^T X = I$. Moreover since $|X| \neq 0$ and $|U| \neq 0$, $XU^T \leq 0$ implies $XU^T < 0$. Thus,

$$P_s - P_m = YX^{-1} - VU^{-1} = (XU^T)^{-1} < 0 \quad (48)$$

From $P_s \geq 0$, we get $P_m > 0$. (Q.E.D.)

Sufficient condition for the existence of P_m is given in Theorem 5 (iii). We now show that it is necessary as well.

Corollary 1. *There exists a positive-definite solution P_m of eq.(2) if and only if F is asymptotically stable, $\|H(sI - F)^{-1}G\|_\infty < 1$, and (H, F) is observable.*

(Proof) Since 'if' part holds from Theorem 5 (iii), here we prove 'only if' part of the theorem. First notice from Theorem 4 that $\Re[\lambda(A)] \neq 0$ for all the eigenvalues of A and that $(H, -F)$ is detectable. Suppose now that F has an eigenvalue λ with $\Re[\lambda] \geq 0$, and let

$$F^T x = \lambda x, \quad \Re[\lambda] \geq 0 \quad (49)$$

Equation (2) premultiplied by x^* and postmultiplied by x yields

$$(\lambda + \lambda^*)x^*P_m x + x^*P_m H^T H P_m x + x^*G G^T x = 0 \quad (50)$$

Noting that $x^*P_m x > 0$, we get $\Re[\lambda] = 0$ and $G^T x = 0$. But this shows λ is an uncontrollable eigenvalue of F implying that A has an eigenvalue on the imaginary axis, a contradiction. Thus $\Re[\lambda(F)] < 0$ for all the eigenvalues of F . In this case, the condition $\Re[\lambda(A)] \neq 0$ leads to $\|H(sI - F)^{-1}G\|_\infty < 1$, and $(H, -F)$ -detectable is equivalent to (H, F) -observable. (Q.E.D.)

In Theorem 5, we see that if a negative real eigenvalue $-\lambda_i$ of A is not an unobservable eigenvalue of F , both λ_i and $-\lambda_i$ could yield a solution. In case $-\lambda_i$ is complex with negative real part, then the pairs (λ_i, λ_i^*) and $(-\lambda_i, -\lambda_i^*)$ yield a solution. Moreover, both λ_i and $-\lambda_i$ have the associated Jordan block of size p_i resulting in total of $(1+p_i)$ combinations of generalized eigenvectors for real symmetric solution. Thus, let S be an index set defined by $S = \{i | -\lambda_i(A) \text{ with } \Re[-\lambda_i(A)] < 0 \text{ is not an unobservable eigenvalue of } F\}$ with the understanding that, if $-\lambda_i(A)$ is complex and $-\lambda_j(A) = -\lambda_i(A)^*$, $j > i$, then j is excluded from S . Then, we get

Corollary 2. *Assume that there exists $P_s \geq 0$ and that A is a cyclic matrix. Then, the number of solutions (hence nonnegative-definite solutions) is $\prod_{i \in S} (1 + p_i)$.*

5. Conclusions

Nonnegative-definite solutions of algebraic matrix Riccati equations having nonnegative-definite quadratic and constant terms were studied. Employing an analysis method based on the eigenvalue problem of Hamiltonian matrix, a comprehensive theory is developed along the line of Kucera for conventional Riccati equations arising in LQG problems.

Necessary and sufficient conditions were derived for the existence of stabilizing solution P_s (not necessarily $P_s \geq 0$), $P_s \geq 0$, $P_s > 0$, and anti-stabilizing solution P_m and $P_m > 0$. Moreover, we

clarified the solution structure of the equation such as the ordering of all the solutions $P \geq 0$, and the number of solutions $P \geq 0$. The theory developed here not only provides useful results for the development of H_∞ control theory, but also are important from the point of view of mathematical analysis of matrix quadratic equations.

This work was supported in part by a Grant-in-Aid for General Scientific Research of The Ministry of Education, Science and Culture under Grant C-05650417.

References

- [1] S. Boyd, V. Balakrishnan and P. Kabamba, A bisection method for computing the H_∞ norm of a transfer matrix and related problems, *Mathematics of Control, Signals, and Systems* **2** (1989) 207-219.
- [2] G. Hewer, Existence theorems for positive semidefinite and sign indefinite stabilizing solutions of H_∞ Riccati equations, *SIAM J. Control and Optimization* **31** (1993) 16-29.
- [3] H. Kano, Existence condition of positive-definite solutions for algebraic matrix Riccati equations, *Automatica* **23** (1987) 393-397.
- [4] H. Kano and T. Nishimura, Generalized existence conditions for real symmetric solutions of algebraic matrix Riccati equations, in *Systems and Control - Topics in theory and its applications* - (eds. T. Ono and F. Kozin), Mita Press, Tokyo (1991) 175-190.
- [5] H. Kano and T. Nishimura, Nonnegative-definite solutions of algebraic matrix Riccati equations with nonnegative-definite quadratic and constant terms, presented at the 1993 MTNS, Regensburg, Germany.
- [6] V. Kucera, A contribution to matrix quadratic equations, *IEEE Trans. Automatic Control* **AC-17** (1972) 344-347.
- [7] V. Kucera, On nonnegative-definite solutions to matrix quadratic equations, *Automatica* **8** (1972) 413-423.
- [8] T. Nishimura, Nonnegative-definite solutions of matrix Riccati equations having nonpositive-definite weighting matrices, *Proc. 10th SICE Symposium on Adaptive Control* (1990) 69-74.
- [9] T. Nishimura, Generalized existence condition for solutions of matrix Riccati equation in the H_∞ Theory, *Proc. 13th SICE Symposium on Dynamical System Theory* (1991) 19-24.