Optimal Control of Flexible Structures Tipped with Dynamic Actuators Subject to Random Disturbance

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This paper is dedicated to the memory of Dr. Yoshiyumi Sunahara, Professor Emeritus of Kyoto Institute of Technology

Abstract—In this paper an active control problem is investigated for a class of flexible cantilevered beams tipped with dynamic actuators subject to random disturbance. The mathematical models of the system are hybrid, described by an Euler-Bernoulli type partial and a second-order ordinary differential equations. Since the actuator is tipped, the control problem constitutes a kind of boundary control problems. By introducing boundary homogenization approach and the modal representation, it is reduced to the finite dimensional control problem. Simulation studies are provided.

1. Introduction

THE ACTIVE CONTROL PROBLEMS for suppressing vibrations have been investigated by many control engineers for a class of flexible structures (e.g. [1]). From the practical point of view the cantilevered beam type structures have been mainly studied. Conceivably, for such structures it will be the most effective to attach a controller to the free-end of the beam. Such situation leads us to solve the boundary control problems. In most works concerning to these boundary control problems an ideal actuator is assumed to control the vibration of the structure, neglecting or leaving out the actuator dynamics. However, from the practical point of view, it goes without saying that a mathematical theory is expected to be developed for active (boundary) control systems, including the actuator dynamics. Up to the present time a few papers have been appeared. For example, Kubo and Shimemura [2] investigated the optimal control problem for a class of cantilevered beam tipped with a lumped actuator. They did not consider the concrete mechanism of the actuator. The authors [3]-[5] have discussed the modeling, stabilization and control of a class of cantilevered beam tipped with a dynamic actuator. These studies are restricted to the case of no contin-

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Fig.1. A model of flexible structure tipped with a dynamic actuator.

uous random disturbance. In this paper, we develop the study made in [4] and [5] to the flexible structures subject to continuous random disturbance.
2. Model and Problem Formulation

The model considered is illustrated in Fig. 1. The structure consists of two parts. The mechanically flexible part is simply modeled by an Euler-Bernoulli type cantilevered beam, and the dynamic actuator is attached at the free-end of the beam to suppress actively the dynamic vibration due to a random disturbance such as longitudinal wind action on the whole structure. The dynamic actuator consists of two masses called the tip and proof masses, a dashpot, a spring, and a controller.

The beam is assumed to be uniform with length $\ell$, mass density $\rho$, cross-sectional area $A$, Young's modulus $E$, second moment of inertia $I$, and internal damping modulus $c_D$; while the actuator with tip mass $m_T$, proof mass $m_P$, spring constant $k_s$, and damping coefficient $c_a$.

The mathematical models of the system have been derived by the actuators using the Lagrangian formulation: [4], [5]

$$
\rho S \frac{\partial^2 u(t,x)}{\partial t^2} + c_D \frac{\partial^3 u(t,x)}{\partial t \partial x^2} + EI \frac{\partial^4 u(t,x)}{\partial x^4} \\
= g_B \gamma_B(t,x), \quad (t,x) \in (0,T) \times (0,\ell)
$$

(1a)

$$
m_p \frac{\partial^2 y_P(t)}{\partial t^2} + c_a \left\{ \frac{dy_P(t)}{dt} - \frac{\partial u(t,\ell)}{\partial t} \right\} \\
+ k_a \left\{ y_P(t) - u(t,\ell) \right\} = a_0 f(t) + g_P \gamma_P(t)
$$

(1b)

B.C.:

$$
c_D I \frac{\partial^4 u(t,\ell)}{\partial x^4} + EI \frac{\partial^3 u(t,\ell)}{\partial x^3} \\
= m_T \frac{\partial^2 u(t,\ell)}{\partial t^2} - c_a \left\{ \frac{dy_P(t)}{dt} - \frac{\partial u(t,\ell)}{\partial t} \right\} \\
- k_a \left\{ y_P(t) - u(t,\ell) \right\} + a_0 f(t) - g_B \gamma_B(t,\ell)
$$

(2a)

$$
c_D I \frac{\partial^3 u(t,\ell)}{\partial x^3} + EI \frac{\partial^2 u(t,\ell)}{\partial x^2} = 0
$$

(2b)

$$
u(t,0) = \frac{\partial u(t,0)}{\partial x} = 0.
$$

(2c)

In (1) and (2), $u(t,x)$ denotes the transverse displacement of the beam from its equilibrium state; $y_P(t)$ and $y_P(t)$ the displacements of the proof and tip masses, respectively; $\gamma_B(t,x)$ and $\gamma_P(t)$ the random disturbances modeled by white Gaussian noise processes having zero means and covariances, $\mathbb{E}\{\gamma_B(t,x)\gamma_B(s,z)\} = q(x,z)\delta(t-s)$, $\mathbb{E}\{\gamma_P(t)\gamma_P(s)\} = p^2\delta(t-s)$ ($\mathbb{E}(\cdot)$ and $\delta(\cdot)$ denote the expectation operator and Dirac delta function, respectively), where $q(x,z)$ is a positive-definite and symmetric (in $x$ and $z$) function; $f(t)$ the control force acting on the double-mass dashpot-spring system; and $a_0$, $g_B$ and $g_P$ positive constants. The following constraint at the free-end of the beam is supplemented to the boundary conditions:

$$
y_P(t) = u(t,\ell).
$$

(3)

From this constraint and the initial conditions for the beam, $u(0,x) = u_0(x), \partial u(0,\ell)/\partial t = u_0(x), y_P(0) = y_P(\ell)$ and the initial conditions of tip mass are given automatically.

The equation (1a) is familiar and known as the Euler-Bernoulli type partial differential equation; while the dynamics for the proof mass is described by the second-order ordinary differential equation (1b).

The transverse displacement $u(t,\cdot)$, its velocity $u'(t,\cdot)$, the displacement of proof mass $y_P(t)$ and its velocity $\dot{y}_P(t)$ are measured by means of several sensors installed at each preassigned locations in the following manner:

$$
z_0(t) = c_1 y_P(t) + c_2 \ddot{y}_P(t) + h_0 \delta(t)
$$

(4a)

$$
z_j(t) = c_1 u(t,\zeta_j) + c_2 \ddot{u}(t,\zeta_j) + h_j \delta(t),
$$

(4b)

where $\zeta_j$ ($0 \leq \zeta_j \leq \ell; j = 1, 2, \cdots, P$) denotes the location where the $j$th sensor is set along the beam, and the last $P$th sensor is set on the free end of the beam for the displacement $y_T(t) = u(t,\ell)$. Furthermore, a sensor is also set on the proof mass. In (4) $\delta(t)$ and $\delta_j(t)$ ($j = 1, \cdots, P$) are observation noises modeled by white Gaussian noises with zero means and the covariances $\mathbb{E}\{\delta(t)\delta^T(s)\} = [V], \delta(t-s)$ with $\delta(t) = [\delta_0(t), \delta_1(t), \cdots, \delta_P(t)]$ and $V = \text{diag} \{\sigma_0^2, \sigma_1^2, \cdots, \sigma_P^2\}$; and $h_0, h_j$ are positive constants.

Our problem is to construct an optimal control system for suppressing vibrations due to the random disturbances by applying a control signal generated in the actuator. For this, let us consider the quadratic cost functional,

$$
J(f) = \lim_{T \to \infty} \frac{1}{T} \mathbb{E} \left\{ \int_0^T \left[ \int_0^T \left( q_0 u^2(t,x) + q_1 u'^2(t,x) \right) dx \right. \right. \\
\left. \left. + m_0 y_P^2(t) + m_1 \dot{y}_P^2(t) + r f^2(t) \right] dt \right\}
$$

(5)

where $q_0, q_1, m_0$ and $m_1$ are nonnegative and $r$ is a positive constant.
3. State Space Representation

Now we are confronted with treating the inhomogeneous partial differential equation with inhomogeneous boundary conditions. To circumvent the mathematical trouble, we apply the technique of boundary homogenization used in [6] (which is a technique to transform inhomogeneous boundary conditions to homogeneous ones).

Let \( \varphi(t, x) \) be an arbitrary function such that \( \varphi \in X = \{ \varphi | \varphi \in L^2(0, T; H^4(0, \Omega)), \varphi/\partial t, \partial^2\varphi/\partial t^2, \partial^4\varphi/\partial x^4, \partial^5\varphi/\partial x^5 \partial t \in L^2(\Omega); \varphi(t, 0) = \varphi(t, T)/\partial x = \partial^2\varphi(t, 0)/\partial x^2 = \partial^3\varphi(t, x)/\partial x^3 = \varphi(T, x) = \partial_\varphi(T, x)/\partial t = 0 \}, \Omega = (0, T) \times (0, \ell), \) and \( H \) denotes the Sobolev space. Multiplying (1a) by \( \varphi(t, x) \), integrating over \( \Omega \), integrating both sides of the resultant equation by parts and noting the initial and boundary conditions of \( u(t, x) \) and \( \varphi(t, x) \), we have the integral equation: [2]

\[
\int_0^T \int_0^T u(t, x) \left\{ \rho S \frac{\partial^2 u(t, x)}{\partial t^2} - c_D \frac{\partial^5 \varphi(t, x)}{\partial x^5 \partial t} \right\} dx dt + \int_0^T \int_0^T \varphi(t, x) \left\{ \frac{m_T}{\rho S} \frac{\partial u(t, x)}{\partial t} - c_a \left( \frac{dy(t)}{dt} - \frac{\partial u(t, x)}{\partial t} \right) \right\} dx dt - k_a \left\{ y_p(t) - u(t, \ell) \right\} dt + c_0 f(t) + g_B(t, \ell) \] 

\[
\int_0^T \int_0^T \left\{ \frac{\rho S}{2} \frac{\partial u(0, x)}{\partial t} dx + c_D \int_0^T \int_0^T u(T, x) \frac{\partial^4 \varphi(t, x)}{\partial t^4} dx \right\} 
\end{align}

\[
= g_B \int_0^T \int_0^T \gamma_B(t, x) \varphi(x, t) dx dt.
\]  

(6)

Here, define an operator \( A := (EI/\rho S) d^4/dx^4 \) with domain \( \mathcal{D}(A) := \{ \phi | \phi, A\phi \in L^2(0, T); \phi(0) = d\phi(0)/dx = d^2\phi(t)/dx^2 = d^3\phi(t)/dx^3 = 0 \} \). For \( A \), let \( \lambda_k \) and \( \phi_k(x) \) \( (k = 1, 2, \ldots, \) be eigenvalues and corresponding orthonormal eigenfunctions, respectively, and assume that \( 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots, \lim_{k \to \infty} \lambda_k = \infty \).

Using the eigenfunctions for \( A \), construct an approximate solution by

\[
u(t, x) \equiv \sum_{k=1}^{\infty} u_k^{(L)}(t) \phi_k(x).
\]

To determine \( \{ u_k^{(L)}(t) \} \) take the function \( \varphi(x, t) \) in (6) as follows:

\[
\varphi(t, x) = \varphi(t) \delta_k(x), \quad \psi \in C^2(0, T),
\]

\[
\psi(T) = \psi(T) = 0 \quad (1 \leq j \leq L).
\]

Substituting (7) and (8) into (6) and performing the integration by parts, we obtain

\[
\int_0^T \psi(t) \left( \sum_{k=1}^L \left\{ \delta_jk + \frac{m_T}{\rho S} \phi_k(\ell) \phi_j(\ell) \right\} u_k^{(L)}(t) \right) dt + \sum_{k=1}^L \left\{ \frac{c_D}{E} \lambda_k \delta_jk + \frac{c_a}{\rho S} \phi_k(\ell) \phi_j(\ell) \right\} u_k^{(L)}(t) dt + \sum_{k=1}^L \left\{ \lambda_k \delta_jk + \frac{k_a}{\rho S} \phi_k(\ell) \phi_j(\ell) \right\} u_k^{(L)}(t) dt + \int_0^T \int_0^T \gamma_B(t, x) \varphi(x, t) dx dt = 0.
\]

(9)

where \( \delta_jk \) is the Kronecker delta, and \( \gamma_j(t) = \int_0^t \gamma_B(t, x) \phi_j(x) dx \). Here, the control \( f(t) \) is written as \( f_L(t) \) by indicating the suffix \( L \) because, in general, the control force depends on the mode number \( L \).

The solution process \( y_p(t) \) of (1b) depends also on \( L \), so we write it as \( y_p^{(L)}(t) \).

Since the function \( \psi(t) \) is arbitrary, \( \{ u_k^{(L)}(t) \} \) must satisfy the differential equations:

\[
\sum_{k=1}^L \left\{ \delta_jk + \frac{m_T}{\rho S} \phi_k(\ell) \phi_j(\ell) \right\} u_k^{(L)}(t) + \sum_{k=1}^L \left\{ \frac{c_D}{E} \lambda_k \delta_jk + \frac{c_a}{\rho S} \phi_k(\ell) \phi_j(\ell) \right\} u_k^{(L)}(t)
\]

\[
+ \sum_{k=1}^L \left\{ \lambda_k \delta_jk + \frac{k_a}{\rho S} \phi_k(\ell) \phi_j(\ell) \right\} u_k^{(L)}(t)
\]

\[
= \frac{c_a}{\rho S} \phi_j(\ell) y_p^{(L)}(t) \quad (j = 1, 2, \ldots, L).
\]

(10)

with initial conditions:

\[
u_k^{(L)}(0) = \int_0^t u_0(x) \phi_j(x) dx \]

\[a]

(11a)
\[ u_j^{(L)}(0) = \int_0^t u_0(z) \phi_j(z) \, dz. \quad (11b) \]

By using the modal expression (7), the dynamic equation of the proof mass (1b) is simply transformed into the following form:

\[ m_p x_p^{(L)}(t) + c_a y_p^{(L)}(t) + k_a y_p^{(L)}(t) = \sum_{k=1}^L \phi_k(t) u_k^{(L)}(t) + a_0 f_L(t) + g_p \gamma_p(t). \quad (12) \]

As shown in (10) and (12), \( \{u_j^{(L)}(t)\}_{j=1,2,\ldots,L} \) and \( y_p^{(L)}(t) \) are given by the simultaneous equations. Furthermore, these equations depend on \( \{\phi_j(t)\} \), values of the eigenfunctions at the location of the actuator. In this point, the control problem which we treat is quite different from the conventional control problems of the flexible structures (neglecting any actuator dynamics).

By introducing new variables such as
\[
u_{1j}^{(L)}(t) = u_j^{(L)}(t), \quad \nu_{2j}^{(L)}(t) = \dot{u}_j^{(L)}(t), \quad \nu_{10}^{(L)}(t) = y_p^{(L)}(t), \quad \nu_{20}^{(L)}(t) = \ddot{y}_p^{(L)}(t), \quad (13)\]
we have for (10) and (12) the equations:

\[
u_{1j}^{(L)}(t) = \nu_{2j}^{(L)}(t) \quad (0 \leq j \leq L),
\]
\[
u_{10}^{(L)}(t) = \dot{y}_p^{(L)}(t) - \bar{c}_a \nu_{10}^{(L)}(t) + \sum_{k=1}^L \phi_k(t) \nu_{1k}^{(L)}(t)
\]
\[ + \bar{a}_0 f_L(t) + \bar{g}_p \gamma_p(t) + \bar{g}_p \dot{\gamma}_p(t), \quad (14)\]

where \( \bar{c}_a \) denotes \( c_a / \rho S \), \( \bar{\gamma}_p(t) = \gamma_p(t) + \bar{g}_p (t, \psi) \phi_j(t) \), and the parameters \( \mathcal{C} \) and \( \Phi_j \) are set as \( \mathcal{C} = cD/E, \quad \Phi_j = \phi_j(t) \phi_j(t) \).

Here, let us introduce a state vector,
\[
u_L(t) = \begin{bmatrix} \nu_{10}^{(L)}(t), \nu_{11}^{(L)}(t), \cdots, \nu_{1L}^{(L)}(t), \\ \nu_{20}^{(L)}(t), \nu_{21}^{(L)}(t), \cdots, \nu_{2L}^{(L)}(t) \end{bmatrix}^T, \quad (15)\]

where the superscript \( T \) denotes transpose. Then, we have the following state space representation:

\[ T_L \dot{v}_L(t) = A_L v_L(t) + B_L f_L(t) + G_L \gamma_L(t), \quad (16)\]

where
\[ T_L = \begin{bmatrix} I_{L+1} & O_{L+1} \\ O_{L+1} & M_L \end{bmatrix}, \quad A_L = \begin{bmatrix} O_{L+1} I_{L+1} \\ -K_L -C_L \end{bmatrix}, \quad B_L = \begin{bmatrix} \tilde{\gamma}_L \end{bmatrix}_{L+1}^T, \quad G_L = \begin{bmatrix} O_{L+1} \\ \tilde{\gamma}_B \end{bmatrix}. \]

As shown in (10) and (12), \( \{u_j^{(L)}(t)\}_{j=1,2,\ldots,L} \) and \( y_p^{(L)}(t) \) are given by the simultaneous equations. Furthermore, these equations depend on \( \{\phi_j(t)\} \), values of the eigenfunctions at the location of the actuator. In this point, the control problem which we treat is quite different from the conventional control problems of the flexible structures (neglecting any actuator dynamics).

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we have for (10) and (12) the equations:

\[ \nu_{1j}^{(L)}(t) = \nu_{2j}^{(L)}(t) \quad (0 \leq j \leq L), \]
\[ \nu_{10}^{(L)}(t) = \dot{y}_p^{(L)}(t) - \bar{c}_a \nu_{10}^{(L)}(t) + \sum_{k=1}^L \phi_k(t) \nu_{1k}^{(L)}(t) \]
\[ + \bar{a}_0 f_L(t) + \bar{g}_p \gamma_p(t) \]
\[ + \bar{g}_p \dot{\gamma}_p(t) \quad (14)\]

where \( \bar{c}_a \) denotes \( c_a / \rho S \), \( \bar{\gamma}_p(t) = \gamma_p(t) + \bar{g}_p (t, \psi) \phi_j(t) \), and the parameters \( \mathcal{C} \) and \( \Phi_j \) are set as \( \mathcal{C} = cD/E, \quad \Phi_j = \phi_j(t) \phi_j(t) \).

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where
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(17) (1L+1 and O_L+1 imply \((L+1) \times (L+1))-identity and null matrices, respectively) with

\[ M_L = \begin{bmatrix} \bar{m}_p & O_{1 \times L} \\ O_{L \times 1} & \bar{M}_L \end{bmatrix}, \quad C_L = \begin{bmatrix} \bar{c}_a & -\bar{c}_a \phi_{\psi \psi} \\ -\bar{c}_a \phi_{\psi \psi} & \alpha \bar{\Lambda} + \bar{c}_a \Phi_{\psi \psi} \end{bmatrix}, \quad (18)\]

where \( \phi_{\psi \psi} = \begin{bmatrix} \phi_1(t), \phi_2(t), \cdots, \phi_L(t) \end{bmatrix}^T, \Phi_{\psi \psi} = \phi_{\psi \psi}^T \). It can be readily shown that the \((2(L+1) \times 2(L+1))-matrix T_L \) is nonsingular; so that (16)

\[ \dot{v}_L(t) = \tilde{A}_L v_L(t) + \tilde{B}_L f_L(t) + \tilde{G}_L \gamma_L(t), \quad (19)\]

where the new matrices \( \tilde{A}_L \) and \( \tilde{B}_L \) are given by \( \tilde{A}_L = T_L^{-1} A_L, \tilde{B}_L = T_L^{-1} B_L \) and \( \tilde{G}_L = T_L^{-1} G_L \).

Similarly, by using the modal expression (7) and a state vector \( v_L(t) \), the observation data are also transformed as follows

\[ z(t) = \begin{bmatrix} z_0(t), z_1(t), \cdots, z_p(t) \end{bmatrix}^T \]
\[ = \tilde{C}_L v_L(t) + H \theta(t), \quad (20)\]

where
\[ \tilde{C}_L = \begin{bmatrix} c_1 & O_{1 \times L} & c_2 & O_{1 \times L} \\ O_{P \times 1} & c_1 \Phi_{\psi} & O_{P \times 1} & c_2 \Phi_{\psi} \end{bmatrix}, \quad H = \text{diag} \{h_0, h_1, \cdots, h_P\}. \quad (22)\]

with \( \Phi_{\psi} = \{ \phi_j \}_{i=1, \cdots, P, j=1, \cdots, L} = \{ \phi_j \psi_i \} \).
Since the above equations (19) and (20) are quite formal because of white noise processes, we should express them more precisely as the following Itô stochastic differential equations:

\[ d\hat{v}_L(t) = \hat{A}_L\hat{v}_L(t)dt + \hat{B}_Lf_L(t)dt + \hat{G}_Ldw_{L1}(t) \quad (23) \]

\[ d\hat{\varepsilon}(t) = \hat{C}_L\hat{v}_L(t)dt + Hdw_{L2}(t), \quad (24) \]

where \( \hat{\varepsilon}(t) := \int_0^t z(s)ds \), \( w_{L1}(t) \) and \( w_{L2}(t) \) are Wiener processes related to the white Gaussian noise processes \( \gamma_L(t) \) and \( \theta(t) \), respectively.

4. Optimal Control

The resulting reduced-order models (23) and (24) with quadratic cost functional (5) constitute a standard linear stochastic control problem. We employ the stationary Kalman filter to generate the optimal estimate \( \hat{v}_L(t) \) for \( v_L(t) \) as follows:

\[ d\hat{v}_L(t) = \hat{A}_L\hat{v}_L(t)dt + \hat{B}_Lf_L(t)dt + \hat{K}d\hat{\varepsilon}(t) \]

\[ d\hat{\varepsilon}(t) := d\hat{\varepsilon}(t) - \hat{C}_L\hat{v}_L(t), \quad (25) \]

where \( d\hat{\varepsilon}(t) \) is the innovation process and the filter gain matrix is: \( \hat{K} := \hat{P}\hat{C}_L(H\hat{V}H^T)^{-1} \), and \( \hat{P} \) is the estimate error covariance matrix given as the positive-definite solution of the algebraic Riccati equation:

\[ \hat{A}_L\hat{P} + \hat{P}\hat{A}_L^T + \hat{G}_LW_L\hat{G}_L^T \]

\[ + \hat{P}\hat{C}_L^T(H\hat{V}H^T)^{-1}\hat{C}_L\hat{P} = 0, \quad (26) \]

where \( [W_L]_{ij} := \int_0^t \int_0^t \varphi_i(s)\varphi_j(z)d\varepsilon(s)dz. \)

Using the approximation in Parseval equality

\[ \int_0^t u^2(t, z)dz \cong \sum_{k=1}^L \left\{ \hat{u}^{(L)}_k(t) \right\}^2, \quad (27) \]

the cost functional \( J(f) \) is rewritten in terms of \( \hat{v}_L(t) \) as follows:

\[ J(f) = \lim_{T \to \infty} \frac{1}{T} E \left\{ \int_0^T [u_L(t)\hat{M}_L\hat{v}_L(t) + rf_L(t)^2]dt \right\}, \quad (28) \]

where \( \hat{M}_L = \text{diag}\{m_0, q_0, \ldots, q_0, m_1, q_1, \ldots, q_1\} \).

According to the celebrated LQG control theory, the optimal control \( f_L(t) \) minimizing the cost functional (28) is given by the following feedback form:

\[ f_L(t) = -\frac{1}{r}\hat{B}_L^T\Pi_L\hat{v}_L(t) \equiv K_0\hat{v}_L(t), \quad (29) \]

where \( \Pi_L \) is the \((2L + 2) \times (2L + 2)\)-matrix solution of the algebraic Riccati equation:

\[ \Pi_L\hat{A}_L + \hat{A}_L^T\Pi_L + \hat{M}_L - \frac{1}{r}\Pi_L\hat{B}_L\hat{B}_L^T\Pi_L = 0. \quad (30) \]

From the definition of \( v_L(t) \) given by (15), we see that the optimal control is realized by the feedback form of the estimates of displacements and velocities of the proof mass and the state \( \{u_j^{(L)}(t)\}_{j=1, 2, \ldots, \ell}. \)

5. Simulation Studies

In simulation studies, we consider a uniform steel beam with \( \ell = 10 \text{ [m]} \), \( \rho = 7.860 \times 10^3 \text{ [kg \cdot m}^{-3}] \), \( S = 0.1 \times 0.02 \text{ [m}^2], \quad E = 2.058 \times 10^{11} \text{ [N} \cdot \text{m}^{-2}], \quad I = 6.667 \times 10^{-8} \text{ [m}^4], \quad c_D = 1.029 \times 10^6 \text{ [N} \cdot \text{m} \cdot \text{s}^{-1}], \quad m_1 = 0.1 \text{ [kg]}, \quad m_2 = 0.001 \text{ [s}^{-2}], \quad k_1 = 3 \text{ [N} \cdot \text{m}^{-1}]. \]

The sensor's location on the beam are set: \( (0.3f) \), \( (0.5f) \), \( (0.7f) \), and \( (1.0f) \) for the central 2nd-order is adopted with time and spatial partitions: \( \Delta t = 0.001 \text{ [s]}, \quad \Delta x = 0.25 \text{ [m]}. \) The number of mode \( \ell \) is fixed as \( \ell = 7 \). The initial conditions are set

\[ u_0(x) = 0.5 \left( 1 - \cos \frac{\pi x}{\ell} \right), \quad y_p = 0.0 \text{ [L]} \]

\[ w_0(x) = 0.1 \left( 1 - \cos \frac{\pi x}{\ell} \right), \quad y_p = 0.0 \text{ [L]} \quad (32) \]

The sensor's location on the beam are set: \( \zeta_1 = 3.0 \text{ [m]} \), \( \zeta_2 = 5.0 \text{ [m]} \), \( \zeta_3 = 7.0 \text{ [m]} \), \( \zeta_4 = 9.0 \text{ [m]} \), and \( \zeta_5 = 10 \text{ [m]} \). The constant parameters are set: \( g_B = 2, \quad g_p = 1, \quad a_0 = 1, \quad c_1 = 300, \quad c_2 = 100, \quad q_0 = q_1 = m_0 = m_1 = 1 \times 10^3, \quad r = 10. \)
In case of the free motions of the beam and the actuator, the whole structure keeps on vibrating for a long time. The results of the controlled case are illustrated in Figs. 2 and 3. The effectiveness of the control is undoubtedly obvious. Figure 4 illustrates the influence of the ratio $m_p/\rho S l$ upon the minimum cost for a fixed value of $m_T/\rho S l = 0.006$.

6. Conclusions
For the flexible structure tipped with a dynamic actuator subject to random disturbance, a new approach to the optimal boundary control problem is proposed. By taking into consideration the actuator dynamics, our system may offer a useful model reflecting the practical system.

REFERENCES


